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Energy balance and fundamental relations in dynamic anisotropic poro-viscoelasticity

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An energy balance equation and fundamental relations are obtained for wave motion in anisotropic poro-viscoelastic media. The balance of energy identifies the strain and kinetic energy densities, and the dissipated energy densities due to viscoelastic and viscodynamic effects. The relations allow the calculation of these energies in terms of the Umov–Poynting vector and kinematic variables.

The energy balance is obtained for time-harmonic fields, while the following relations are valid for inhomogeneous body waves.

(i) The magnitude of the phase velocity is equal to the projection of the energy velocity vector onto the propagation direction.

(ii) The time-average of the dissipated energy density is obtained from the projection of the average power-flow vector onto the attenuation direction.

(iii) The time-average energy density (kinetic plus strain) is obtained from the projection of the average power-flow vector onto the propagation direction.

(iv) The strain energy equals the kinetic energy when the medium is lossless.

These relations are shown to be valid for anisotropic poro-viscoelasticity at all frequency ranges.

An example of ultrasonic wave propagation in an orthorhombic medium (human femoral bone saturated with water) illustrates the theory. Measurable quantities, like the attenuation factor and the energy velocity, can easily be interpreted in terms of microstructural properties such as tortuosity and permeability.

Keywords: porous media; viscoelasticity; anisotropy; energy balance; wave propagation; eigenstrain

1. Introduction

Anisotropic poroelasticity was introduced by Biot (1955, 1956) and Biot & Willis (1957) in terms of bulk parameters of total stress and strain. To our knowledge, Brown & Korringa (1975) were the first to obtain the material coefficients in terms of the properties of the grain, pore-fluid and frame. Later, Carroll (1980) and Thompson & Willis (1991) presented further micromechanical analysis of the constitutive

equations. Recently, Cheng (1997) related the Hookean constants to the engineering constants (obtained from laboratory measurements), including explicit relations for the orthorhombic and transverse isotropy material symmetries. This theory assumes that the solid constituent is isotropic and that anisotropy is due to arrangements of the grains (i.e. the frame is anisotropic). Complete experimental data for anisotropic media is scarce. Experiments on real rocks can be found in Lo *et al*. (1986) and Aoki *et al*. (1993).

The poroelastic equations of motion combine the constitutive equations with the equations of momentum conservation and the dynamic Darcy's law in the framework of Biot's theory. This theory predicts two compressional waves (the fast P-wave and the so-called Biot slow wave) and two shear waves. Wave propagation in anisotropic poroelastic rocks was investigated by Norris (1993), Ben-Menahem & Gibson (1993) and Gelinsky & Shapiro (1997), who studied plane layered systems and the effects of anisotropic permeability. Numerical simulations of wave propagation for the transversely isotropic case (in rocks and synthetic materials) are given in Carcione (1996).

The behaviour of anisotropic, viscoelastic waves departs substantially from the behaviour of isotropic, elastic waves. Anisotropy implies that, in general, the wavefield is not pure longitudinal or pure transverse, and therefore there is not a single relation between the propagation direction and the direction of particle displacement. As a consequence, wavefronts are not spherical and the direction of energy flux (ray) does not coincide with the wavenumber direction. On the other hand, in viscoelastic media, the existence of the so-called inhomogeneous waves (not the interface waves of elastic media) is necessary to satisfy the boundary conditions at interfaces. For these waves, the propagation direction does not coincide with the attenuation direction, and particle motions are in general elliptical (Buchen 1971). Carcione & Cavallini (1993) showed that in single-phase anisotropic-viscoelastic media the phase velocity is the projection of the energy velocity vector onto the propagation direction, and generalized other similar relations valid in the isotropic viscoelastic case. Here, those relations are further generalized for anisotropic poro-viscoelastic media. First, they provide a simple and useful means for evaluating the time-average kinetic, strain and dissipated energy densities from the wavenumber, attenuation and energy flow vectors. Second, they can be used to verify the kinematic and dynamic (in terms of energy) properties of complex porous materials. For instance, the above relation between phase and energy velocities has immediate implications for ultrasonic experiments. If a pulse of acoustic energy is radiated by a plane wave transducer, the Fourier components travel along the wavenumber direction, which is normal to the transducer surface, but the wave packet modulation envelope travels in the direction of the energy velocity. This means that the receiving transducer must be offset in order to intercept the acoustic pulse, and the corresponding angle is the angle between the wavenumber and energy velocity vectors. Although that relation between the velocities is well known for anisotropic lossles media (see, for example, Auld 1991), it is not immediately evident that it holds for poro-viscoelastic and anisotropic media. To our knowledge, the derived relations are not known for any other particular case, such as anisotropic poroelasticity or isotropic poro-viscoelasticity, mainly due to the fact that the published research is mainly devoted to the constitutive equations, and a complete dynamic formulation of the problem in terms of energy is not provided (see, for example, Thompson & Willis 1991).

We consider wave propagation in one of the planes of mirror symmetry of an orthorhombic material (human femoral bone). Bulk viscoelasticity is modelled by using the concept of eigenstrain (Carcione & Cavallini 1994) and the low-frequency viscodynamic operator is used to model Biot-type dissipation.

In the following, the spatial variables x, y and z are denoted by the indices $i, j = 1, 2, 3$, respectively, a partial derivative with respect to a variable x_i with ∂_i, and the upper case indices $I, J = 1, \ldots, 6$ indicate the shortened matrix notation, where pairs of subscripts (i, j) are replaced by a single number (I or J) according to the correspondence $(11) \rightarrow 1$, $(22) \rightarrow 2$, $(33) \rightarrow 3$, $(23) = (32) \rightarrow 4$, $(13) = (31) \rightarrow 5$, $(12) = (21) \rightarrow 6$. Matrix transposition is denoted by a superscript T and complex conjugation by a superscript $*$.

2. Stress–strain relations

The constitutive equations for anisotropic poroelasticity were introduced by Biot (1955, 1956). They can be expresses in terms of the microstructural properties as (Cheng 1997)

$$\boldsymbol{T} = \boldsymbol{C}^u \cdot \boldsymbol{S}, \tag{2.1}$$

where

$$\boldsymbol{T}^{\mathrm{T}} = [\tau_{11}, \tau_{22}, \tau_{33}, \tau_{23}, \tau_{13}, \tau_{12}, -p] \tag{2.2}$$

is the stress vector, with τ_{ij} the components of the total stress and p the fluid pressure,

$$\boldsymbol{S}^{\mathrm{T}} = [\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{23}, 2\epsilon_{13}, 2\epsilon_{12}, -\zeta] \tag{2.3}$$

is the strain vector, with ϵ_{ij} the strain components of the porous frame and ζ the variation of fluid content,

$$\boldsymbol{C}^u = \begin{bmatrix} c^u_{11} & c^u_{12} & c^u_{13} & c^u_{14} & c^u_{15} & c^u_{16} & M\alpha_1 \\ c^u_{12} & c^u_{22} & c^u_{23} & c^u_{24} & c^u_{25} & c^u_{26} & M\alpha_2 \\ c^u_{13} & c^u_{23} & c^u_{33} & c^u_{34} & c^u_{35} & c^u_{36} & M\alpha_3 \\ c^u_{14} & c^u_{24} & c^u_{34} & c^u_{44} & c^u_{45} & c^u_{46} & M\alpha_4 \\ c^u_{15} & c^u_{25} & c^u_{35} & c^u_{45} & c^u_{55} & c^u_{56} & M\alpha_5 \\ c^u_{16} & c^u_{26} & c^u_{36} & c^u_{46} & c^u_{56} & c^u_{66} & M\alpha_6 \\ M\alpha_1 & M\alpha_2 & M\alpha_3 & M\alpha_4 & M\alpha_5 & M\alpha_6 & M \end{bmatrix}, \tag{2.4}$$

where

$$c^u_{IJ} = c_{IJ} + M\alpha_I\alpha_J \tag{2.5}$$

are the components of the undrained stiffness tensor, with c_{IJ} the dry-rock stiffness components, M the fluid/solid coupling modulus and $\alpha_I = \alpha_{ij}$ the effective stress components. The components of the tensor given in (2.4) can be expressed in terms of the properties of the frame and of the single constituents (see the appendix). The variation of fluid content is given by

$$\zeta = -\operatorname{div}[\phi(\boldsymbol{u}_{\mathrm{f}} - \boldsymbol{u})], \tag{2.6}$$

where $\boldsymbol{u}_{\rm f}$ and $\boldsymbol{u}$ are the average fluid and solid displacements vectors, respectively.

The rate of the strain vector can be written as

$$\partial_{\rm t}\boldsymbol{S} = \nabla^{\rm T}\cdot\boldsymbol{V}, \tag{2.7}$$

where

$$\boldsymbol{V} \equiv [v_1, v_2, v_3, q_1, q_2, q_3]^{\rm T}, \tag{2.8}$$

with v and q denoting the solid and fluid (relative to the solid) particle velocities, respectively ($\boldsymbol{q} = \phi(\partial_t\boldsymbol{u}_{\rm f} - \boldsymbol{v})$ and $\boldsymbol{v} = \partial_{\rm t}\boldsymbol{u}$), and

$$\nabla = \begin{bmatrix} \partial_1 & 0 & 0 & 0 & \partial_3 & \partial_2 & 0 \\ 0 & \partial_2 & 0 & \partial_3 & 0 & \partial_1 & 0 \\ 0 & 0 & \partial_3 & \partial_2 & \partial_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial_3 \end{bmatrix}. \tag{2.9}$$

The form (2.7) relating the particle velocities to the strain components and the differential operator (2.9) are generalizations of those used by Auld (1991).

Biot (1956) developed a generalization of the constitutive equations to the viscoelastic case by invoking the correspondence principle and using relaxation functions based on mechanical models of viscoelastic behaviour. Viscoelasticity is due to a variety of dissipation mechanisms. One of these mechanisms is the squirt-flow (Biot 1962; Dvorkin *et al*. 1994), by which a force applied to the area of contact between two grains produces a displacement of the surrounding fluid in and out of this area. Since the fluid is viscous, the motion is not instantaneous and energy dissipation occurs. Other important attenuation mechanisms are discussed by Biot (1962). Using the correspondence principle (Ben-Menahem & Singh 1981), we generalize to relaxation functions the elements of matrix $\boldsymbol{C}^u$ and (2.1) becomes

$$\boldsymbol{T} = \boldsymbol{\Psi} * \partial_{\rm t}\boldsymbol{S}, \tag{2.10}$$

where $\boldsymbol{\Psi}$ is the relaxation matrix and the asterisk denotes time convolution and matrix product. The matrix $\boldsymbol{C}^u$ is obtained from $\boldsymbol{\Psi}$ when $t \to 0$ if we consider that the elastic Biot's poroelastic theory corresponds to the unrelaxed state.

3. Biot–Newton's equation

The dynamic equations describing wave propagation in heterogeneous porous media were obtained by Biot (1962). They are

$$\partial_1\tau_{11} + \partial_2\tau_{12} + \partial_3\tau_{13} = \rho\partial_{\rm t}v_1 + \rho_{\rm f}\partial_{\rm t}q_1 + f_1, \tag{3.1}$$

$$\partial_1\tau_{12} + \partial_2\tau_{22} + \partial_3\tau_{23} = \rho\partial_{\rm t}v_2 + \rho_{\rm f}\partial_{\rm t}q_2 + f_2, \tag{3.2}$$

$$\partial_1\tau_{13} + \partial_2\tau_{23} + \partial_3\tau_{33} = \rho\partial_{\rm t}v_3 + \rho_{\rm f}\partial_{\rm t}q_3 + f_3, \tag{3.3}$$

where f denotes body force and $\rho = (1-\phi)\rho_{\rm s} + \phi\rho_{\rm f}$ is the composite density, with $\rho_{\rm s}$ and $\rho_{\rm f}$ the solid and fluid densities. On the other hand, dynamic Darcy's law, generalized to the anisotropic case, can be expressed as

$$-\partial_1 p = \rho_{\rm f}\partial_{\rm t}v_1 + \psi_1 * \partial_{\rm t}q_1, \tag{3.4}$$

$$-\partial_2 p = \rho_{\rm f}\partial_{\rm t}v_2 + \psi_2 * \partial_{\rm t}q_2, \tag{3.5}$$

$$-\partial_3 p = \rho_{\rm f}\partial_{\rm t}v_3 + \psi_3 * \partial_{\rm t}q_3, \tag{3.6}$$

where the asterisk denotes time convolution and ψ_l, $l = 1, \ldots, 3$, are time-dependent functions related to Biot's viscodynamic effects (Biot 1962). In matrix form, equations (3.1)–(3.6) can be written as

$$\nabla \cdot \boldsymbol{T} = \boldsymbol{R} \cdot \partial_{\mathrm{t}} \boldsymbol{V} + \boldsymbol{F}, \tag{3.7}$$

where

$$\boldsymbol{F} \equiv [f_1, f_2, f_3, 0, 0, 0, 0]^{\mathrm{T}} \tag{3.8}$$

and

$$\boldsymbol{R} = \begin{bmatrix} \rho & 0 & 0 & \rho_{\mathrm{f}} & 0 & 0 \\ 0 & \rho & 0 & 0 & \rho_{\mathrm{f}} & 0 \\ 0 & 0 & \rho & 0 & 0 & \rho_{\mathrm{f}} \\ \rho_{\mathrm{f}} & 0 & 0 & \psi_1 * & 0 & 0 \\ 0 & \rho_{\mathrm{f}} & 0 & 0 & \psi_2 * & 0 \\ 0 & 0 & \rho_{\mathrm{f}} & 0 & 0 & \psi_3 * \end{bmatrix} \tag{3.9}$$

is the density matrix. We refer to (3.7) as Biot–Newton's equation.

4. Time-harmonic fields

Let us consider a time-harmonic field $\exp(\mathrm{i}\omega t)$, where ω is the angular frequency and $\mathrm{i} = \sqrt{-1}$. The stress–strain relation (2.10) becomes

$$\boldsymbol{T} = \boldsymbol{C} \cdot \boldsymbol{S}, \quad \boldsymbol{C} = \mathcal{F}[\partial_{\mathrm{t}} \boldsymbol{\Psi}], \tag{4.1}$$

where $\boldsymbol{C}$ is the complex and frequency-dependent stiffness matrix, and the operator $\mathcal{F}$ denotes time Fourier transform. Equation (2.7) becomes

$$\mathrm{i}\omega \boldsymbol{S} = \nabla^{\mathrm{T}} \cdot \boldsymbol{V}. \tag{4.2}$$

Substituting (4.2) into (4.1) gives

$$\mathrm{i}\omega \boldsymbol{T} = \boldsymbol{C} \cdot (\nabla^{\mathrm{T}} \cdot \boldsymbol{V}). \tag{4.3}$$

On the other hand, Biot–Newton's equation (3.7) becomes

$$\nabla \cdot \boldsymbol{T} = \mathrm{i}\omega \boldsymbol{R} \cdot \boldsymbol{V} + \boldsymbol{F}, \tag{4.4}$$

where

$$\boldsymbol{R} = \begin{bmatrix} \rho & 0 & 0 & \rho_{\mathrm{f}} & 0 & 0 \\ 0 & \rho & 0 & 0 & \rho_{\mathrm{f}} & 0 \\ 0 & 0 & \rho & 0 & 0 & \rho_{\mathrm{f}} \\ \rho_{\mathrm{f}} & 0 & 0 & Y_1/(\mathrm{i}\omega) & 0 & 0 \\ 0 & \rho_{\mathrm{f}} & 0 & 0 & Y_2/(\mathrm{i}\omega) & 0 \\ 0 & 0 & \rho_{\mathrm{f}} & 0 & 0 & Y_3/(\mathrm{i}\omega) \end{bmatrix} \tag{4.5}$$

and

$$Y_l(\omega) = \mathcal{F}[\partial_{\mathrm{t}} \psi_l] \tag{4.6}$$

are Biot's viscodynamic operators for the x-, y- and z-directions. In the low-frequency range (Biot 1962; Auriault *et al.* 1985), i.e. for frequencies lower than $\omega_c = \min(\omega_l)$, where $\omega_l = \eta/(m_l\kappa_l)$,

$$\psi_l(t) = m_l\delta(t) + \frac{\eta}{\kappa_l}H(t), \tag{4.7}$$

where $m_l = T_l\rho_f/\phi$, with T_l the tortuosity, η is the dynamic viscosity, κ_l, $l = 1, 2, 3$, are the principal components of the global permeability tensor, $\delta(t)$ is Dirac's function and $H(t)$ is the Heaviside function. From (4.6),

$$Y_l(\omega) = \mathrm{i}\omega m_l + \frac{\eta}{\kappa_l}. \tag{4.8}$$

In terms of mechanical models, equation (4.8) represents a Kelvin–Voigt element (Ben-Menahem & Singh 1981). In the high-frequency range ($\omega \geqslant \omega_c$), the viscodynamic operator is strongly influenced by the pore geometry, and a precise evaluation of its frequency dependence requires an experimental determination (Auriault *et al.* 1985).

The derivation of the energy balance equation is straightforward when using complex notation. The same procedure given in Carcione & Cavallini (1993) for single-phase media is used here. The dot product of the complex conjugate of (4.2) with $-\boldsymbol{T}^{\mathrm{T}}$ gives

$$-\boldsymbol{T}^{\mathrm{T}}\cdot\nabla^{\mathrm{T}}\cdot\boldsymbol{V}^* = \mathrm{i}\omega\boldsymbol{T}^{\mathrm{T}}\cdot\boldsymbol{S}^*. \tag{4.9}$$

On the other hand, the dot product of $-\boldsymbol{V}^{*\mathrm{T}}$ with (4.4) is

$$-\boldsymbol{V}^{*\mathrm{T}}\cdot\nabla\cdot\boldsymbol{T} = -\mathrm{i}\omega\boldsymbol{V}^{*\mathrm{T}}\cdot\boldsymbol{R}\cdot\boldsymbol{V} - \boldsymbol{V}^{*\mathrm{T}}\cdot\boldsymbol{F}. \tag{4.10}$$

Adding (4.9) and (4.10) gives

$$-\boldsymbol{T}^{\mathrm{T}}\cdot\nabla^{\mathrm{T}}\cdot\boldsymbol{V}^* - \boldsymbol{V}^{*\mathrm{T}}\cdot\nabla\cdot\boldsymbol{T} = \mathrm{i}\omega\boldsymbol{T}^{\mathrm{T}}\cdot\boldsymbol{S}^* - \mathrm{i}\omega\boldsymbol{V}^{*\mathrm{T}}\cdot\boldsymbol{R}\cdot\boldsymbol{V} - \boldsymbol{V}^{*\mathrm{T}}\cdot\boldsymbol{F}. \tag{4.11}$$

The left-hand side is simply

$$-\boldsymbol{T}^{\mathrm{T}}\cdot\nabla^{\mathrm{T}}\cdot\boldsymbol{V}^* - \boldsymbol{V}^{*\mathrm{T}}\cdot\nabla\cdot\boldsymbol{T} = 2\,\mathrm{div}(\boldsymbol{P}), \tag{4.12}$$

where

$$\boldsymbol{P} = -\frac{1}{2}\begin{bmatrix}\tau_{11} & \tau_{12} & \tau_{13} & -p & 0 & 0\\ \tau_{12} & \tau_{22} & \tau_{23} & 0 & -p & 0\\ \tau_{13} & \tau_{23} & \tau_{33} & 0 & 0 & -p\end{bmatrix}\cdot\boldsymbol{V}^* \tag{4.13}$$

is the complex Umov–Poynting vector and 'div' is the divergence operator. Using equation (4.12) and the stress–strain relation (4.1), equation (4.11) gives

$$2\,\mathrm{div}(\boldsymbol{P}) = \mathrm{i}\omega\boldsymbol{S}^{\mathrm{T}}\cdot\boldsymbol{C}\cdot\boldsymbol{S}^* - \mathrm{i}\omega\boldsymbol{V}^{*\mathrm{T}}\cdot\boldsymbol{R}\cdot\boldsymbol{V} - \boldsymbol{V}^{*\mathrm{T}}\cdot\boldsymbol{F}, \tag{4.14}$$

where we used the fact that $\boldsymbol{C}$ is a symmetric matrix. Equation (4.14) can be rewritten as

$$\begin{aligned}\mathrm{div}(\boldsymbol{P}) = {}& 2\mathrm{i}\omega[\tfrac{1}{4}\,\mathrm{Re}(\boldsymbol{S}^{\mathrm{T}}\cdot\boldsymbol{C}\cdot\boldsymbol{S}^*) - \tfrac{1}{4}\,\mathrm{Re}(\boldsymbol{V}^{*\mathrm{T}}\cdot\boldsymbol{R}\cdot\boldsymbol{V})]\\ & + 2\omega[-\tfrac{1}{4}\,\mathrm{Im}(\boldsymbol{S}^{\mathrm{T}}\cdot\boldsymbol{C}\cdot\boldsymbol{S}^*) + \tfrac{1}{4}\,\mathrm{Im}(\boldsymbol{V}^{*\mathrm{T}}\cdot\boldsymbol{R}\cdot\boldsymbol{V})] - \tfrac{1}{2}\boldsymbol{V}^{*\mathrm{T}}\cdot\boldsymbol{F},\end{aligned} \tag{4.15}$$

where 'Re' and 'Im' take real and imaginary parts, respectively. The significance of this equation becomes clear when we recognize that each of its terms has a precise physical meaning on a time-average basis. When using complex notation for plane waves, the field variables are obtained as the real part of the corresponding complex fields. For generic field variables $\boldsymbol{A}$ and $\boldsymbol{B}$ and a symmetric matrix $\boldsymbol{D}$, the time-average over a cycle of period $2\pi/\omega$ has the following properties

$$\langle \mathrm{Re}(\boldsymbol{A}^{\mathrm{T}}) \cdot \mathrm{Re}(\boldsymbol{B}) \rangle = \tfrac{1}{2}\,\mathrm{Re}(\boldsymbol{A}^{\mathrm{T}} \cdot \boldsymbol{B}^{*}) \tag{4.16}$$

(Booker 1992), and

$$\langle \mathrm{Re}(\boldsymbol{A}^{\mathrm{T}}) \cdot \mathrm{Re}(\boldsymbol{D}) \cdot \mathrm{Re}(\boldsymbol{A}) \rangle = \tfrac{1}{2}\,\mathrm{Re}(\boldsymbol{A}^{\mathrm{T}} \cdot \boldsymbol{D} \cdot \boldsymbol{A}^{*}), \tag{4.17}$$

$$\langle \mathrm{Re}(\boldsymbol{A}^{\mathrm{T}}) \cdot \mathrm{Im}(\boldsymbol{D}) \cdot \mathrm{Re}(\boldsymbol{A}) \rangle = \tfrac{1}{2}\,\mathrm{Im}(\boldsymbol{A}^{\mathrm{T}} \cdot \boldsymbol{D} \cdot \boldsymbol{A}^{*}) \tag{4.18}$$

(Carcione & Cavallini 1993). Using these relations, we identify

$$\tfrac{1}{4}\,\mathrm{Re}(\boldsymbol{S}^{\mathrm{T}} \cdot \boldsymbol{C} \cdot \boldsymbol{S}^{*}) = \tfrac{1}{2}\langle \mathrm{Re}(\boldsymbol{S}^{\mathrm{T}}) \cdot \mathrm{Re}(\boldsymbol{C}) \cdot \mathrm{Re}(\boldsymbol{S}) \rangle \equiv \langle S \rangle \tag{4.19}$$

as the strain energy density,

$$\tfrac{1}{4}\,\mathrm{Re}(\boldsymbol{V}^{*\mathrm{T}} \cdot \boldsymbol{R} \cdot \boldsymbol{V}) = \tfrac{1}{2}\langle \mathrm{Re}(\boldsymbol{V}^{*\mathrm{T}}) \cdot \mathrm{Re}(\boldsymbol{R}) \cdot \mathrm{Re}(\boldsymbol{V}) \rangle \equiv \langle K \rangle \tag{4.20}$$

as the kinetic energy density,

$$\begin{aligned} -\tfrac{1}{2}\omega\,\mathrm{Im}(\boldsymbol{S}^{\mathrm{T}} \cdot \boldsymbol{C} \cdot \boldsymbol{S}^{*}) + \tfrac{1}{2}\omega\,\mathrm{Im}(\boldsymbol{V}^{*\mathrm{T}} \cdot \boldsymbol{R} \cdot \boldsymbol{V}) \\ = -\omega\langle \mathrm{Re}(\boldsymbol{S}^{\mathrm{T}}) \cdot \mathrm{Im}(\boldsymbol{C}) \cdot \mathrm{Re}(\boldsymbol{S}) \rangle + \omega\langle \mathrm{Re}(\boldsymbol{V}^{\mathrm{T}}) \cdot \mathrm{Im}(\boldsymbol{R}) \cdot \mathrm{Re}(\boldsymbol{V}) \rangle \\ \equiv -\langle \dot{D}_{\mathrm{S}} \rangle - \langle \dot{D}_{\mathrm{K}} \rangle \end{aligned} \tag{4.21}$$

as minus the mean (time-average) rate of dissipated strain energy density ($-\langle \dot{D}_{\mathrm{S}} \rangle$, the first term) minus the dissipated kinetic energy density ($-\langle \dot{D}_{\mathrm{K}} \rangle$, the second term), and

$$-\tfrac{1}{2}\boldsymbol{V}^{*\mathrm{T}} \cdot \boldsymbol{F} \equiv P_{\mathrm{s}} \tag{4.22}$$

as the complex power per unit volume supplied by the body forces. We may define the corresponding time-average energy densities $\langle D_{\mathrm{S}} \rangle$ and $\langle D_{\mathrm{K}} \rangle$ by the relations

$$\langle \dot{D}_{\mathrm{S}} \rangle = \omega \langle D_{\mathrm{S}} \rangle \quad \text{and} \quad \langle \dot{D}_{\mathrm{K}} \rangle = \omega \langle D_{\mathrm{K}} \rangle. \tag{4.23}$$

Substituting the preceding expressions into (4.15) gives the energy balance equation

$$\mathrm{div}(\boldsymbol{P}) - 2\mathrm{i}\omega(\langle S \rangle - \langle K \rangle) + \omega \langle D \rangle = P_{\mathrm{s}}, \tag{4.24}$$

where

$$\langle D \rangle = \langle D_{\mathrm{S}} \rangle + \langle D_{\mathrm{K}} \rangle \tag{4.25}$$

is the total time-average dissipated energy density.

The total stored energy density is

$$\langle E \rangle = \langle S \rangle + \langle K \rangle. \tag{4.26}$$

If there is no dissipation ($\langle D \rangle = 0$) and since, in the absence of sources ($P_{\mathrm{s}} = 0$), the net energy flow into, or out of, a given closed surface must vanish, $\mathrm{div}(\boldsymbol{P}) = 0$.

Thus the average kinetic energy equals the average strain energy. As a consequence, the stored energy is twice the strain energy.

The energy balance equation as given by (4.24) is analogous to a similar relation for complex power in sinusoidal steady-state circuit theory,

$$\tfrac{1}{2}VI^* - 2\mathrm{i}\omega(\langle E_\mathrm{C}\rangle - \langle E_\mathrm{L}\rangle) + \langle P_\mathrm{R}\rangle = 0 \tag{4.27}$$

(Fano *et al*. 1960), where the first term is the complex power, V is the voltage phasor, I is the current phasor, $\langle E_\mathrm{C}\rangle$ and $\langle E_\mathrm{L}\rangle$ are the time-average stored energies in the capacitors and inductors, respectively, and $\langle P_\mathrm{R}\rangle$ is the average power dissipated in the resistors, which is equivalent to the dissipated strain energy density.

5. Inhomogeneous plane waves

A general plane wave solution for the particle velocity vector is

$$\boldsymbol{V} = \boldsymbol{V}_0 \exp[\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})], \tag{5.1}$$

where $\boldsymbol{V}_0$ represents a constant complex vector and $\boldsymbol{k}$ is the complex wavevector. The wavevector is, in general, complex and can be written as

$$\boldsymbol{k} \equiv \boldsymbol{\kappa} - \mathrm{i}\boldsymbol{\alpha} = (k_1, k_2, k_3), \tag{5.2}$$

where $\boldsymbol{\kappa}$ and $\boldsymbol{\alpha}$ are the real wavevector and attenuation vector, respectively. They indicate the directions and magnitude of the wavevector and attenuation vector. In general, these directions differ and the plane wave is termed inhomogeneous. For inhomogeneous viscoelastic plane waves, the operator (2.9) takes the form

$$\nabla \rightarrow -\mathrm{i}\boldsymbol{K} \tag{5.3}$$

in frequency domain, where

$$\boldsymbol{K} = \begin{bmatrix} k_1 & 0 & 0 & 0 & k_3 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 & 0 & k_1 & 0 \\ 0 & 0 & k_3 & k_2 & k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_3 \end{bmatrix}. \tag{5.4}$$

When the operator is applied to a conjugated field, ∇ should be replaced by $\mathrm{i}\boldsymbol{K}^*$.

Substituting the differential operator into (4.9) and (4.10) and assuming zero body forces yields

$$-\boldsymbol{T}^\mathrm{T}\cdot\boldsymbol{K}^{*\mathrm{T}}\cdot\boldsymbol{V}^* = \omega\boldsymbol{T}^\mathrm{T}\cdot\boldsymbol{S}^* \tag{5.5}$$

and

$$-\boldsymbol{V}^{*\mathrm{T}}\cdot\boldsymbol{K}\cdot\boldsymbol{T} = \omega\boldsymbol{V}^{*\mathrm{T}}\cdot\boldsymbol{R}\cdot\boldsymbol{V}, \tag{5.6}$$

respectively. The left-hand sides of (5.5) and (5.6) contain the complex Umov–Poynting vector (4.13). In fact, by virtue of (5.2), equations (5.5) and (5.6) become

$$2\boldsymbol{k}^{*\mathrm{T}}\cdot\boldsymbol{P} = \omega\boldsymbol{T}^\mathrm{T}\cdot\boldsymbol{S}^* \tag{5.7}$$

and

$$2\boldsymbol{k}^{\mathrm{T}} \cdot \boldsymbol{P} = \omega \boldsymbol{V}^{*\mathrm{T}} \cdot \boldsymbol{R} \cdot \boldsymbol{V}, \tag{5.8}$$

respectively. Adding (5.7) and (5.8), and using (5.2) ($\boldsymbol{k}^* + \boldsymbol{k} = 2\boldsymbol{\kappa}$), yields

$$4\boldsymbol{\kappa}^{\mathrm{T}} \cdot \boldsymbol{P} = \omega(\boldsymbol{T}^{\mathrm{T}} \cdot \boldsymbol{S}^* + \boldsymbol{V}^{*\mathrm{T}} \cdot \boldsymbol{R} \cdot \boldsymbol{V}). \tag{5.9}$$

Using (4.16), the time-average of the real Umov–Poynting vector (4.13),

$$-\operatorname{Re}\begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} & -p & 0 & 0 \\ \tau_{12} & \tau_{22} & \tau_{23} & 0 & -p & 0 \\ \tau_{13} & \tau_{23} & \tau_{33} & 0 & 0 & -p \end{bmatrix} \cdot \operatorname{Re}(\boldsymbol{V}), \tag{5.10}$$

is

$$\langle \boldsymbol{P} \rangle = \operatorname{Re}(\boldsymbol{P}), \tag{5.11}$$

which gives the average power flow.

As in the previous section, the time-average of the strain energy density

$$\langle S \rangle = \tfrac{1}{2} \operatorname{Re}(\boldsymbol{T}^{\mathrm{T}}) \cdot \operatorname{Re}(\boldsymbol{S}) \tag{5.12}$$

is

$$\langle S \rangle = \tfrac{1}{4} \operatorname{Re}(\boldsymbol{T}^{\mathrm{T}} \cdot \boldsymbol{S}^*) = \tfrac{1}{4} \operatorname{Re}(\boldsymbol{S}^{*\mathrm{T}} \cdot \boldsymbol{C} \cdot \boldsymbol{S}). \tag{5.13}$$

Similarly, the time-average kinetic energy density is

$$\langle K \rangle = \tfrac{1}{4} \operatorname{Re}(\boldsymbol{V}^{*\mathrm{T}} \cdot \boldsymbol{R} \cdot \boldsymbol{V}) \tag{5.14}$$

and the time-average strain and kinetic dissipated energy densities are

$$\langle D_{\mathrm{S}} \rangle = \tfrac{1}{2} \operatorname{Im}(\boldsymbol{S}^{*\mathrm{T}} \cdot \boldsymbol{C} \cdot \boldsymbol{S}) \tag{5.15}$$

and

$$\langle D_{\mathrm{K}} \rangle = -\tfrac{1}{2} \operatorname{Im}(\boldsymbol{V}^{*\mathrm{T}} \cdot \boldsymbol{R} \cdot \boldsymbol{V}), \tag{5.16}$$

respectively. The last two quantities represents the energy loss per unit volume due to viscoelastic and viscodynamic effects, respectively. The minus sign in (5.16) is due to the fact that $\operatorname{Im}(Y_I)/(\mathrm{i}\omega) < 0$ (see (4.5)). It can be shown that the dissipated energies should be defined with the opposite sign if an $\exp(-\mathrm{i}\omega t)$ kernel is used. This is the case for the dissipated kinetic energy in Carcione (1996).

Substituting (5.11), (5.13) and (5.14) into the real part of (5.9) yields

$$\boldsymbol{\kappa}^{\mathrm{T}} \cdot \langle \boldsymbol{P} \rangle = \omega(\langle S \rangle + \langle K \rangle) = \omega \langle E \rangle, \tag{5.17}$$

where $\langle E \rangle$ is the stored energy density (4.26). On the other hand, the imaginary part of (5.9) gives

$$2\boldsymbol{\kappa}^{\mathrm{T}} \cdot \operatorname{Im}(\boldsymbol{P}) = \omega(\langle D_{\mathrm{S}} \rangle - \langle D_{\mathrm{K}} \rangle). \tag{5.18}$$

The wave surface is the locus of the end of the energy velocity vector multiplied by one unit of propagation time, with the energy velocity defined as the ratio of the

average power-flow density $\langle \boldsymbol{P} \rangle$ to the total energy density $\langle E \rangle$. Since this is equal to the sum of the average kinetic and strain energy densities $\langle K \rangle$ and $\langle S \rangle$, the energy velocity is

$$\boldsymbol{V}_{\mathrm{e}} = \frac{\langle \boldsymbol{P} \rangle}{\langle K + S \rangle}. \tag{5.19}$$

Dissipation is quantified by the quality factor, which can be defined as

$$Q = \frac{2\langle S \rangle}{\langle D \rangle}. \tag{5.20}$$

Using the definition of the energy velocity and (5.17) gives

$$\hat{\boldsymbol{\kappa}}^{\mathrm{T}} \cdot \boldsymbol{V}_{\mathrm{e}} = V_{\mathrm{p}}, \tag{5.21}$$

where $V_{\mathrm{p}} = \omega/\kappa$ is the phase velocity and $\hat{\boldsymbol{\kappa}}$ is a unit vector along the wavenumber or propagation direction. Relation (5.21), as in a single-phase medium (Carcione & Cavallini 1993), means that the phase velocity is simply the projection of the energy velocity onto the propagation direction.

On the other hand, subtracting (5.7) from (5.8) and using (5.2) yields the energy balance equation,

$$-2\boldsymbol{\alpha}^{\mathrm{T}} \cdot \boldsymbol{P} = 2\mathrm{i}\omega(\langle S \rangle - \langle K \rangle) - \omega \langle D \rangle. \tag{5.22}$$

Taking the real part of (5.22) yields

$$2\boldsymbol{\alpha}^{\mathrm{T}} \cdot \langle \boldsymbol{P} \rangle = \omega \langle D \rangle. \tag{5.23}$$

This equation is the generalization of a similar relation for viscoelastic single-phase media, stating that the time-average dissipated energy can be obtained as the projection of the average power-flow density onto the attenuation direction.

As stated in § 1, the derived relations are not known for the different simplified cases of dynamic poro-viscoelasticity, as, for instance, isotropic poro-viscoelasticity or anisotropic poroelasticity. Similar relations in the isotropic viscoelastic case were first obtained by Buchen (1971). In this respect, equations (5.17), (5.21) and (5.23) are equivalent in form to (38), (40) and (34) of Buchen (1971). A substantial difference is that the dissipated kinetic energy density is zero in single-phase viscoelastic media, because the attenuation effects arise from the constitutive equations.

6. Homogeneous plane waves

For homogeneous waves, the propagation and attenuation directions coincide and the wavevector can be written as

$$\boldsymbol{k} = (\kappa - \mathrm{i}\alpha)\hat{\boldsymbol{\kappa}} \equiv k\hat{\boldsymbol{\kappa}}, \tag{6.1}$$

where

$$\hat{\boldsymbol{\kappa}} = (l_1, l_2, l_3) \tag{6.2}$$

defines the propagation direction through the directions cosines l_1, l_2 and l_3. For homogeneous waves,

$$\boldsymbol{K} \to k\boldsymbol{L} = k \begin{bmatrix} l_1 & 0 & 0 & 0 & l_3 & l_2 & 0 \\ 0 & l_2 & 0 & l_3 & 0 & l_1 & 0 \\ 0 & 0 & l_3 & l_2 & l_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & l_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & l_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & l_3 \end{bmatrix}, \tag{6.3}$$

where k is the complex wavenumber. Using (5.3), equations (4.3) and (4.4) give

$$(\boldsymbol{R}^{-1} \cdot \boldsymbol{\Gamma} - V^2 \boldsymbol{I}_6) \cdot \boldsymbol{V} = 0, \tag{6.4}$$

where

$$\boldsymbol{\Gamma} = \boldsymbol{L} \cdot \boldsymbol{C} \cdot \boldsymbol{L}^{\mathrm{T}} \tag{6.5}$$

is the Christoffel matrix,

$$V = \omega/k \tag{6.6}$$

is the complex velocity, and $\boldsymbol{I}_6$ denotes the six-dimensional unit matrix.

Making zero the determinant, equation (6.4) gives the following dispersion relation:

$$\det(\boldsymbol{R}^{-1} \cdot \boldsymbol{\Gamma} - V^2 \boldsymbol{I}_6) = 0. \tag{6.7}$$

The eigensystem formed by (6.4) and (6.7) gives six eigenvalues and the corresponding eigenvectors. Four of them correspond to the wave modes, and the others equal zero. These modes correspond to the fast and slow quasi-compressional waves, and the two quasi-shear waves.

The slowness and attenuation vectors for homogeneous waves can be expressed in terms of the complex velocity as

$$\boldsymbol{s} = \mathrm{Re}\left(\frac{1}{V}\right)\hat{\boldsymbol{\kappa}} \tag{6.8}$$

and

$$\boldsymbol{\alpha} = -\omega\, \mathrm{Im}\left(\frac{1}{V}\right)\hat{\boldsymbol{\kappa}}, \tag{6.9}$$

respectively ($\mathrm{Re}(1/V)$ is the reciprocal of the phase velocity).

The average strain energy density (5.13) can be written in terms of the density matrix $\boldsymbol{R}$ using (4.2), (5.3) and (6.3)–(6.6),

$$\langle S \rangle = \tfrac{1}{4}|V|^{-2}\, \mathrm{Re}(V^2 \boldsymbol{V}^{\mathrm{T}} \cdot \boldsymbol{R} \cdot \boldsymbol{V}^*), \tag{6.10}$$

where we used the fact that $\boldsymbol{R}$ and $\boldsymbol{\Gamma}$ are symmetric matrices.

Equation (6.10) is formally similar to the strain energy density in single-phase anisotropic viscoelastic media, where $\langle S \rangle = \frac{1}{4}\rho_{\mathrm{s}}|V|^{-2}V^2|\boldsymbol{V}|^2$ (see Carcione & Cavallini 1993). In the single-phase medium, every particle velocity component is equally weighted by the density. Note that, when the medium is lossless, V is real and the average strain energy density equals the average kinetic energy density (5.14).

From (5.13) and (5.14), and using the property $\boldsymbol{V}^{\mathrm{T}}\cdot\boldsymbol{R}\cdot\boldsymbol{V}^{*}=\boldsymbol{V}^{*\mathrm{T}}\cdot\boldsymbol{R}\cdot\boldsymbol{V}$ (since $\boldsymbol{R}$ is symmetric), the stored energy density (4.26) becomes

$$\langle E\rangle=\tfrac{1}{4}\,\mathrm{Re}\left[\left(1+\frac{V^2}{|V|^2}\right)\boldsymbol{V}^{\mathrm{T}}\cdot\boldsymbol{R}\cdot\boldsymbol{V}^{*}\right]. \tag{6.11}$$

When the medium is lossless, V and $\boldsymbol{R}$ are real and $\langle E\rangle$ is equal to twice the average kinetic energy (5.14).

For calculation purposes, the Umov–Poynting vector (4.13) can be expressed in terms of the eigenvector $\boldsymbol{V}$ and complex velocity V. The average power flow (5.11) can be written as

$$\langle\boldsymbol{P}\rangle=-\tfrac{1}{2}\,\mathrm{Re}[\hat{\boldsymbol{e}}_i(\boldsymbol{U}^i\cdot\boldsymbol{T}^{\mathrm{T}})\cdot\boldsymbol{V}^{*}], \tag{6.12}$$

where $\hat{\boldsymbol{e}}_i$ is the unit Cartesian vector and the Einstein convention for repeated indices is used; $\boldsymbol{U}^i$ are 6×7 matrices with most of their elements equal to zero, except U^1_{11}, U^1_{26}, U^1_{35}, U^1_{47}, U^2_{16}, U^2_{22}, U^2_{34}, U^2_{57}, U^3_{15}, U^3_{24}, U^3_{33} and U^3_{67}, which are equal to one. Substitution of the constitutive equation (4.1) into (6.12) and use of (4.2), (5.3) and (6.3)–(6.6) yields the desired expression,

$$\langle\boldsymbol{P}\rangle=\tfrac{1}{2}\,\mathrm{Re}[V^{-1}\boldsymbol{V}^{\mathrm{T}}\cdot\boldsymbol{L}\cdot\boldsymbol{C}\cdot(\hat{\boldsymbol{e}}_i\boldsymbol{U}^{i\mathrm{T}})\cdot\boldsymbol{V}^{*}]. \tag{6.13}$$

To obtain the quality factor (5.20), we follow the same steps that led to (6.10) and note that the dissipated energy (4.25) can be written as

$$\langle D\rangle=\tfrac{1}{2}\,\mathrm{Im}\left[\left(-1+\frac{V^2}{|V|^2}\right)\boldsymbol{V}^{\mathrm{T}}\cdot\boldsymbol{R}\cdot\boldsymbol{V}^{*}\right]. \tag{6.14}$$

Using (6.10) and a few calculations gives

$$Q=\frac{2\langle S\rangle}{\langle D\rangle}=\frac{\mathrm{Re}(V^2\boldsymbol{V}^{\mathrm{T}}\cdot\boldsymbol{R}\cdot\boldsymbol{V}^{*})}{2\,\mathrm{Im}(V)\,\mathrm{Re}(V\boldsymbol{V}^{\mathrm{T}}\cdot\boldsymbol{R}\cdot\boldsymbol{V}^{*})}. \tag{6.15}$$

If there are no losses due to viscosity effects ($\boldsymbol{R}$ is real and $\langle D_{\mathrm{K}}\rangle$=0), $\boldsymbol{V}^{\mathrm{T}}\cdot\boldsymbol{R}\cdot\boldsymbol{V}^{*}$ is real and

$$Q=\frac{\mathrm{Re}(V^2)}{\mathrm{Im}(V^2)}, \tag{6.16}$$

as in the single-phase case (Carcione & Cavallini 1993).

7. Example

The following example is intended to illustrate a practical application of the theory, starting from the definition of the constitutive equations to the calculation of observable quantities, such as the energy velocity and the attenuation factor. The example involves the calculation of the wavevector, the attenuation and energy flow vector and the different energy densities. The fundamental relations are used to verify the calculations.

The problem with anisotropic lossy media is the determination of the time (or frequency) dependence of the relaxation tensor (21 components in triclinic media). Most

applications use the Kelvin–Voigt constitutive law, based on 21 independent viscosity functions (Auld 1991), corresponding to imaginary constants in the frequency domain. Sometimes it has been possible to estimate all these constants satisfactorily (Hosten *et al*. 1987). Here, we use a model based on few parameters, which are not the imaginary elasticities in themselves, but real quality factors (often more readily available in the seismic practice).

Let us consider propagation of homogeneous plane waves in human femoral bone (orthorhombic symmetry), investigated by Carcione *et al*. (1998) using a single-phase theory for anisotropic and viscoelastic media. To introduce viscoelastic attenuation, they use a constitutive equation based on the fact that each eigenvector (called eigenstrain) of the stiffness matrix defines a fundamental deformation state of the medium. The six eigenvalues (called eigenstiffnesses) represent the intrinsic elastic parameters. In the elastic case, the strain energy is uniquely parametrized by the six eigenstiffnesses. These ideas date back to the middle of the 19th century when Lord Kelvin introduced the concept of 'principal strain' (eigenstrain in modern terminology) to describe the deformation state of a medium (Thomson 1856). From this fact and the correspondence principle, Carcione & Cavallini (1994) inferred that in a real medium the rheological properties depend essentially on six relaxation functions, which are the generalization of the eigenstiffnesses to the viscoelastic case. The existence of six or less complex moduli depends on the symmetry class of the medium.

We assume that the bone is saturated with water of bulk modulus $K_{\mathrm{f}} = 2.5$ GPa, density $\rho_{\mathrm{f}} = 1000\ \mathrm{kg\,m^{-3}}$ and viscosity $\eta = 1$ cP. Furthermore, the grain bulk modulus is $K_{\mathrm{s}} = 28$ GPa, the grain density is $\rho_{\mathrm{s}} = 1815\ \mathrm{kg\,m^{-3}}$, the porosity is $\phi = 0.4$, the tortuosites are $T_1 = 2$, $T_2 = 3$ and $T_3 = 3.6$, and the matrix permeabilities are $\kappa_1 = 1.2 \times 10^{-12}$, $\kappa_2 = 0.8 \times 10^{-12}$ and $\kappa_3 = 0.7 \times 10^{-12}\ \mathrm{m^2}$. The drained stiffness matrix (c_{IJ}, see the appendix) in Voigt notation is

$$\begin{pmatrix} 18 & 9.98 & 10.1 & 0 & 0 & 0 \\ 9.98 & 20.2 & 10.7 & 0 & 0 & 0 \\ 10.1 & 10.7 & 27.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6.23 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.61 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.01 \end{pmatrix},$$

in GPa (Cowin & Mehrabadi 1987). They are used to calculate the elements of matrix $\boldsymbol{C}^u$, which correspond to the high-frequency (unrelaxed) limit. It gives

$$\boldsymbol{C}^u = \begin{pmatrix} 19.8 & 11.7 & 11.5 & 0 & 0 & 0 & 3.35 \\ 11.7 & 21.8 & 12.03 & 0 & 0 & 0 & 3.14 \\ 11.5 & 12.03 & 28.7 & 0 & 0 & 0 & 2.59 \\ 0 & 0 & 0 & 6.23 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.61 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.01 & 0 \\ 3.35 & 3.14 & 2.59 & 0 & 0 & 0 & 6.12 \end{pmatrix},$$

in GPa. In order to apply Kelvin's formulation, Hooke's law has to be written in tensorial form. This implies multiplying the (44), (55) and (66) elements of matrix $\boldsymbol{C}^u$ by a factor 2 (Carcione & Cavallini 1994). Let us call this new matrix (tensor) $\bar{\boldsymbol{C}}^u$.

This matrix can be diagonalized to obtain

$$\bar{\boldsymbol{C}}^u = \boldsymbol{Q} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{Q}^{\mathrm{T}}, \tag{7.1}$$

where $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)^{\mathrm{T}}$ is the eigenvalue matrix and $\boldsymbol{Q}$ is the matrix formed with the eigenvectors of $\bar{\boldsymbol{C}}^u$, or more precisely, with the columns of the right (orthonomal) eigenvectors (note that the symmetry of $\bar{\boldsymbol{C}}^u$ implies $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^{\mathrm{T}}$). Hence, in virtue of the correspondence principle and its application to (7.1), we introduce the viscoelastic stiffness tensor

$$\bar{\boldsymbol{C}} = \boldsymbol{Q} \cdot \boldsymbol{\Lambda}^{(\mathrm{v})} \cdot \boldsymbol{Q}^{\mathrm{T}}, \tag{7.2}$$

where $\boldsymbol{\Lambda}^{(\mathrm{v})}$ is a diagonal matrix with entries

$$\lambda_I^{(\mathrm{v})}(\omega) = \lambda_I M_I(\omega), \quad I = 1, \ldots, 7. \tag{7.3}$$

The quantities M_I are complex and frequency-dependent dimensionless moduli. We describe each of them by a Zener model, whose relaxation frequency is equal to ω (Carcione *et al*. 1998). In this case, we have

$$M_I = \frac{\sqrt{Q_I^2 + 1} - 1 + \mathrm{i}Q_I}{\sqrt{Q_I^2 + 1} + 1 + \mathrm{i}Q_I}, \tag{7.4}$$

where Q_I is the quality factor associated with each modulus (we note here that if a $\exp(-\mathrm{i}\omega t)$ kernel is used, $\mathrm{i}Q_I$ should be replaced by $-\mathrm{i}Q_I$ and the dissipated strain energy should be defined with the opposite sign). To recover the Voigt notation, we should divide the (44), (55) and (66) elements of matrix $\bar{\boldsymbol{C}}$ by a factor 2. This gives the complex matrix $\boldsymbol{C}$.

In orthorhombic porous media, there are seven distinct eigenvalues, and therefore seven complex moduli. We assume that the dimensionless quality factors are defined as $Q_I = (\lambda_I/\lambda_6)Q_6$, $I = 1, \ldots, 7$, with $Q_6 = 30$. This choice implies that the higher the stiffness, the higher the quality factor (i.e. the harder the medium, the lower the attenuation). Then matrix $\boldsymbol{C}$ is given by

$$\begin{pmatrix}
19.4 + \mathrm{i}0.36 & 11.7 + \mathrm{i}0.004 & 11.5 + \mathrm{i}0.002 & & & & & \\
11.7 + \mathrm{i}0.004 & 21.4 + \mathrm{i}0.36 & 12.03 + \mathrm{i}0.002 & & & & & \\
11.5 + \mathrm{i}0.002 & 12.03 + \mathrm{i}0.002 & 28.3 + \mathrm{i}0.37 & & & & & \\
0 & 0 & 0 & & & & & \\
0 & 0 & 0 & & & & & \\
0 & 0 & 0 & & & & & \\
3.35 + \mathrm{i}0.003 & 3.14 + \mathrm{i}0.002 & 2.59 + \mathrm{i}0.0003 & & & & &
\end{pmatrix}$$

$$\begin{pmatrix}
0 & 0 & 0 & 3.35 + \mathrm{i}0.003 \\
0 & 0 & 0 & 3.14 + \mathrm{i}0.002 \\
0 & 0 & 0 & 2.59 + \mathrm{i}0.0003 \\
0 & 0 & 0 & \\
5.42 + \mathrm{i}0.18 & 0 & 0 & \\
0 & 3.82 + \mathrm{i}0.18 & 0 & \\
0 & 0 & 0 & 5.75 + \mathrm{i}0.35
\end{pmatrix}$$

in GPa.

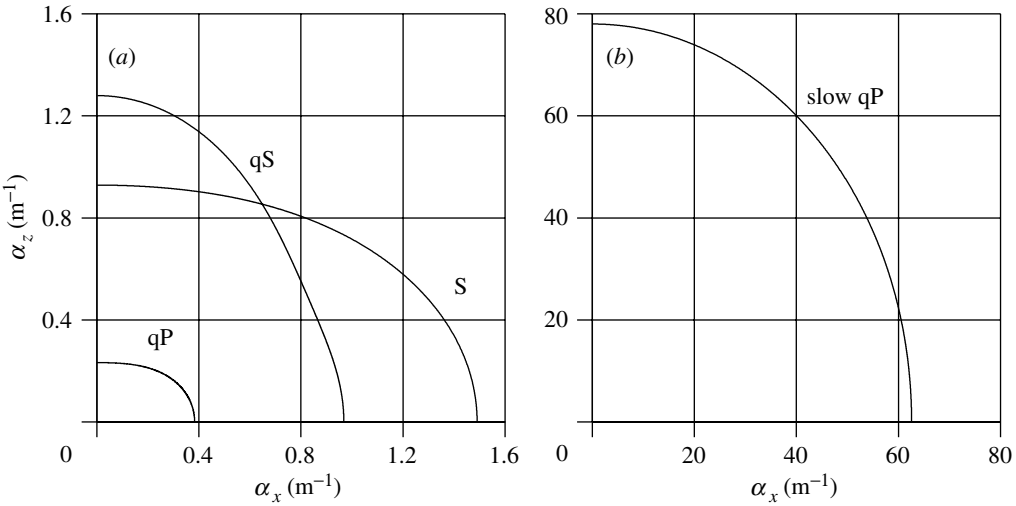

Figure 1. Polar representation of the attenuation factors in one of the planes of mirror symmetry of human femoral bone saturated with water, where (*a*) illustrates the fast quasi-compressional wave qP, the quasi-shear wave qS and the pure anti-plane shear wave S, and (*b*) the slow quasi-compressional wave. The frequency is 10 kHz.

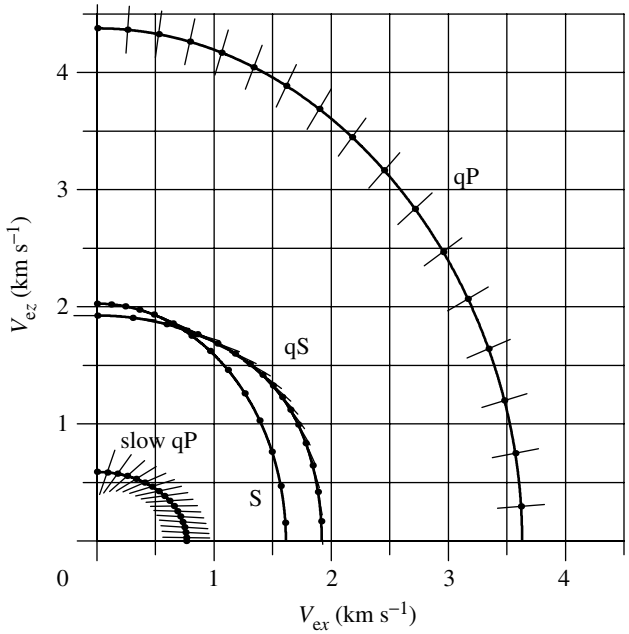

Figure 2. Polar representation of the energy velocities in one of the planes of mirror symmetries of human femoral bone saturated with water, where qP is the fast quasi-compressional wave, qS is the quasi-shear wave, S is the pure anti-plane shear wave and slow qP is the slow quasi-compressional wave. The tick-marks indicate the polarization directions $(v_1, 0, v_3)$ for the qP, slow qP and qS waves, while the polarization of the S wave is $(0, 1, 0)$. The curves correspond to a frequency of 10 kHz.

Polar representations of the attenuation factors (6.9) and energy velocities (5.19) are shown in figures 1 and 2, respectively, for the (x, z) principal plane of the medium ($l_2 = 0$). Only one-quarter of the curves are displayed because of symmetry considerations. The Cartesian planes of an orthorhombic medium are planes of symmetry, and therefore one of the shear waves, denoted by S, is a pure anti-plane mode. The tick-marks in figure 2 indicate the polarization directions $(v_1, 0, v_3)$, with the points uniformly sampled as a function of the phase angle. The curves are plotted for a frequency of $f = \omega/(2\pi) = 10$ kHz, smaller than the characteristic frequency f_c (equal to $\eta\phi/(T_3\rho_f\kappa_3) = 15$ kHz), which determines the upper limit of the low-frequency theory.

The strong dissipation of the slow qP wave is due to the Biot mechanism (i.e. the viscodynamic effect). On the other hand, $\langle D_S\rangle$ and $\langle D_K\rangle$ are comparable for the qP, qS and S waves. This is due to the choice of values for Q_I and for the pulse frequency, which is close to the centre frequencies of Biot's relaxation peaks. At low (seismic) frequencies, $\langle D_K\rangle$ is much less than $\langle D_S\rangle$. Anisotropic permeability affects the attenuation of the slow qP wave. According to Biot's theory, the lower the permeability, the higher the attenuation. In fact, the vertical attenuation factor is higher than the horizontal attenuation factor. Anisotropic tortuosity mainly affects the velocity of the slow qP wave. This is (approximately) inversely proportional to the square root of the tortuosity. Hence the vertical velocity is smaller than the horizontal velocity.

The three faster waves propagating in the (x, z)-plane of a single-phase orthorhombic medium have the following complex velocities along the coordinate axes

$$\left.\begin{aligned} V_{qS}(0) = V_{qS}(90) &= \sqrt{c_{55}/\rho}, \\ V_{qP}(0) = \sqrt{c_{33}/\rho}, \qquad V_{qP}(90) &= \sqrt{c_{11}/\rho}, \\ V_S(0) = \sqrt{c_{44}/\rho}, \qquad V_S(90) &= \sqrt{c_{66}/\rho}, \end{aligned}\right\} \tag{7.5}$$

where 0 corresponds to the z-axis and 90 to the x-axis, and c_{IJ} are, in these equations, complex stiffnesses (they should be the components of $\boldsymbol{C}$ in the porous case, do not confuse them with the drained elastic moduli defined in the appendix). The velocities (7.5) do not correspond exactly to the velocities in the porous case, since here the density is a matrix not a scalar quantity. For instance, the densities corresponding to the S and qS waves along the z-axis are $\rho - \rho_f^2/\boldsymbol{R}_{55}$ and $\rho - \rho_f^2/\boldsymbol{R}_{44}$. However, the velocities (7.5) can be used to qualitatively verify the values of the energy velocity curves along the coordinate axes. On the basis of these equations, figure 2 is in agreement with the values indicated above for matrix $\boldsymbol{C}$.

8. Conclusions

The present analysis provides fundamental relations and expressions of measurable quantities for wave motion in anisotropic porous media, including losses due to relaxation mechanisms and viscodynamic effects. The relations are

$$\begin{gathered} \boldsymbol{\kappa}^{\mathrm{T}} \cdot \langle \boldsymbol{P}\rangle = \omega(\langle S\rangle + \langle K\rangle) = \omega\langle E\rangle \quad \text{or} \quad \hat{\boldsymbol{\kappa}}^{\mathrm{T}} \cdot \boldsymbol{V}_e = V_p, \\ 2\boldsymbol{\kappa}^{\mathrm{T}} \cdot \mathrm{Im}(\boldsymbol{P}) = \omega(\langle D_S\rangle - \langle D_K\rangle), \\ 2\boldsymbol{\alpha}^{\mathrm{T}} \cdot \langle \boldsymbol{P}\rangle = \omega(\langle D_S\rangle + \langle D_K\rangle) = \omega\langle D\rangle, \\ \boldsymbol{\alpha}^{\mathrm{T}} \cdot \mathrm{Im}(\boldsymbol{P}) = \omega(\langle K\rangle - \langle S\rangle), \end{gathered}$$

where complex ω is the angular frequency, $\boldsymbol{\kappa}$ is the wavevector, $\boldsymbol{\alpha}$ is the attenuation vector, $\boldsymbol{P}$ is the Umov–Poynting vector, S and K are the strain and kinetic energy densities, D_{S} and D_{K} are the strain and kinetic dissipated energy densities, $\boldsymbol{V}_{\mathrm{e}}$ is the energy velocity vector and V_{p} is the phase velocity ($\langle\cdot\rangle$ denotes a time-average quantity and $\hat{\cdot}$ a unit vector).

The measurable quantities are generalizations of the phase velocity, attenuation factor, energy velocity and quality factor. For homogeneous plane waves, they can be explicitly written in terms of the complex velocity, eigenvectors of the Christoffel matrix, stiffness and density matrices, and direction cosines defining the propagation direction. The expressions are easily programmed and can be used to analyse the acoustic properties of complex porous media in order to design laboratory experiments and time-domain numerical simulations of wave propagation.

Appendix A. Stiffness components of anisotropic poroelasticity

Cheng (1997) gives the stiffness matrix $\boldsymbol{C}^u$ in terms of properties of the grain, porefluid and skeleton,

$$\begin{aligned} c_{IJ} &= \text{stiffness components of the drained skeleton},\\ K_{\mathrm{s}} &= \text{bulk modulus of the grain},\\ K_{\mathrm{f}} &= \text{bulk modulus of the porefluid},\\ \phi &= \text{porosity}. \end{aligned}$$

Then, under the assumptions of micro-homogeneity and micro-isotropy, we have

$$c^u_{IJ} = c_{IJ} + M\alpha_I\alpha_J, \tag{A 1}$$

$$M = \frac{K_{\mathrm{s}}}{(1-\bar{K}/K_{\mathrm{s}}) - \phi(1-K_{\mathrm{s}}/K_{\mathrm{f}})}, \tag{A 2}$$

$$\bar{K} = \tfrac{1}{9}[c_{11}+c_{22}+c_{33}+2(c_{12}+c_{13}+c_{23})] \tag{A 3}$$

and

$$\left.\begin{aligned} \alpha_1 &= 1-(c_{11}+c_{12}+c_{13})/(3K_{\mathrm{s}}),\\ \alpha_2 &= 1-(c_{12}+c_{22}+c_{23})/(3K_{\mathrm{s}}),\\ \alpha_3 &= 1-(c_{13}+c_{23}+c_{33})/(3K_{\mathrm{s}}),\\ \alpha_4 &= -(c_{14}+c_{24}+c_{34})/(3K_{\mathrm{s}}),\\ \alpha_5 &= -(c_{15}+c_{25}+c_{35})/(3K_{\mathrm{s}}),\\ \alpha_6 &= -(c_{16}+c_{26}+c_{36})/(3K_{\mathrm{s}}). \end{aligned}\right\} \tag{A 4}$$

The effective stress components α_I are a property of the solid skeleton only. The fact that these components constitute a tensor implies that pore pressure induces not only dilatational deformations, but also shear deformations. The latter is not the case in an isotropic medium (Cheng 1997).

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