Journal of Scientific Computing, Vol. 6, No. 4, 1991

Domain Decomposition for Wave Propagation Problems

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Received November 12, 1991

The problem posed by domain decomposition methods is to find the correct modeling of physical phenomena across the interfaces separating the subdomains. The technique described here for wave propagation problems is based on physical grounds since it relies on the fact that the wave equation can be decomposed into incoming and outgoing wave modes at the boundaries of the subdomains. The inward propagating waves depend on the solution exterior to the subdomains and therefore are computed from the appropriate boundary conditions, while the behavior of the outward propagating waves is determined by the solution inside the subdomains. The technique is applied to the anisotropic–viscoelastic wave equation, which practically includes all the possible rheologies of one-phase media.

KEY WORDS: Domain decomposition; boundary conditions; viscoelastic waves; characteristic modes.

1. INTRODUCTION

There are several advantages in the use of domain decomposition techniques. In the first place, they are particularly useful for solving problems in irregular domains and on parallel computers. From the point of view of physics, the correct description of waves through interfaces separating dissimilar rheologies needs an appropriate treatment of the boundary conditions. This is the case for a fluid–solid (acoustic–elastic) interface where the parallel component of the particle velocity need not be continuous (slip wall boundary condition). Domain decomposition is also the basis of hybrid methods, i.e., different resolution algorithms are used in different subdomains. Another useful application is the possibility of using different

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grid sizes in different subdomains, for instance in the presence of localized inhomogeneities or regions having dissimilar material properties.

In this article, the technique is applied to the linear anisotropic–viscoelastic wave equation, which includes all possible types of rheologies, as shown in Fig. 1. Combining these rheologies gives 21 different types of interface, which practically cover all the possibilities in one-phase media. The method was recently developed by Thompson (1990), and applied to the wave equation by Carcione (1990b), where he implemented different types of time-dependent boundary conditions including free surface, rigid, and nonreflecting conditions. The boundary treatment is based on characteristic variables representing one-way waves propagating with the phase velocity of the medium. The wave equation is decomposed into wave modes describing outgoing and incoming wave modes perpendicular to the boundary of the subdomain. The outgoing waves are determined by the solution inside the subdomain, while the incoming waves are calculated from the conditions at the interface, i.e., continuity of displacements and normal stresses in a solid–solid boundary, and continuity of normal displacements and stresses if one of the media is acoustic or viscoacoustic. The result of this approach is a wave equation for the boundaries that automatically includes the boundary conditions. As pointed out by Carcione (1990b), the present method is equivalent to the characteristic

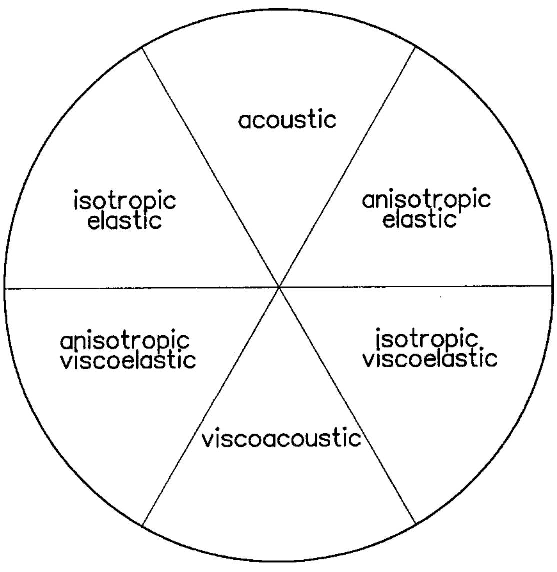

Fig. 1. Set of rheologies modeled by the linear anisotropic-viscoelastic wave equation. Combining these rheologies gives 21 different types of interface.

approach of Gottlieb *et al.* (1982), and to the upwind interface condition proposed in Canuto *et. al.* (1988).

The following section introduces the anisotropic–viscoelastic wave equation in the velocity–stress formulation. Section 3 briefly outlines the method and identifies the uncoupled outgoing and incoming waves. Next, Sec. 4 calculates the boundary equations for isotropic media as a function of the uncoupled modes. In Sec. 5, the boundary equations are explicitely calculated for interfaces separating similar and dissimilar rheologies when the boundary is horizontal. Finally, Sec. 6 extends the approach to inclined boundaries.

2. THE WAVE EQUATION

The general equation of motion of a two-dimensional linear anisotropic–viscoelastic medium involves the following equations (Carcione, 1990a):

i. The Equations of Momentum Conservation:

$$\dot{v}_x = \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z}\right) + f_x \tag{2.1a}$$

$$\dot{v}_z = \frac{1}{\rho}\left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z}\right) + f_z \tag{2.1b}$$

where $\mathbf{x} = (x, z)$ are Cartesian coordinates, $\sigma_{xx}(\mathbf{x}, t)$, $\sigma_{xz}(\mathbf{x}, t)$, and $\sigma_{zz}(\mathbf{x}, t)$ are the stress components, $v_x(\mathbf{x}, t)$ and $v_z(\mathbf{x}, t)$ are the particle velocities, $\rho(\mathbf{x})$ denotes the density, and $\mathbf{f}(\mathbf{x}, t) = (f_x, f_z)$ are the body forces. In (2.1a) and (2.16) and elsewhere, time differentiation is indicated with the dot convention.

ii. The Constitutive Equations:

$$\begin{bmatrix} \dot{\sigma}_{xx} \\ \dot{\sigma}_{zz} \\ \dot{\sigma}_{xz} \end{bmatrix} = \begin{bmatrix} \hat{c}_{11} & \hat{c}_{13} & c_{15} \\ \hat{c}_{13} & \hat{c}_{33} & c_{35} \\ c_{15} & c_{35} & \hat{c}_{55} \end{bmatrix} \begin{bmatrix} \dot{\varepsilon}_{xx} \\ \dot{\varepsilon}_{zz} \\ 2\dot{\varepsilon}_{xz} \end{bmatrix} + (D - c_{55}) \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\times \sum_{l=1}^{L_1} \begin{bmatrix} \dot{e}^{(1)}_{xxl} \\ \dot{e}^{(1)}_{zzl} \\ 2\dot{e}^{(1)}_{xzl} \end{bmatrix} + c_{55} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sum_{l=1}^{L_2} \begin{bmatrix} \dot{e}^{(2)}_{xxl} \\ \dot{e}^{(2)}_{zzl} \\ 2\dot{e}^{(2)}_{xzl} \end{bmatrix} \tag{2.2}$$

where

$$\dot{\varepsilon}_{xx}=\frac{\partial v_x}{\partial x}, \qquad \dot{\varepsilon}_{zz}=\frac{\partial v_z}{\partial z}, \qquad \dot{\varepsilon}_{xz}=\frac{1}{2}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right) \tag{2.3}$$

are the time derivative of the stress components. The quantities $e_{xxl}^{(v)}$, $e_{zzl}^{(v)}$, and $e_{xzl}^{(v)}$ are defined by

$$e_{xxl}^{(v)}=\phi_{vl}*\varepsilon_{xx}, \qquad e_{zzl}^{(v)}=\phi_{vl}*\varepsilon_{zz}, \qquad e_{xzl}^{(v)}=\phi_{vl}*\varepsilon_{xz}, \qquad l=1,..., L_v, \qquad v=1, 2 \tag{2.4}$$

where

$$\phi_{vl}(t)=\frac{1}{\tau_{\sigma l}^{(v)}}\left(1-\frac{\tau_{\varepsilon l}^{(v)}}{\tau_{\sigma l}^{(v)}}\right)e^{-t/\tau_{\sigma l}^{(v)}} \tag{2.5}$$

is the response function of the lth dissipation mechanism, and $\tau_{\sigma l}^{(v)}$ and $\tau_{\varepsilon l}^{(v)}$ are material relaxation times. The three component vectors formed with the variables defined in (2.4) are termed the memory vectors. Like the strain vector in (2.2), the memory vectors are the components of a tensor whose rank is the dimension of the space. The index $v=1$ involves variables that are related to L_1 mechanisms describing the anelastic characteristics of the quasidilatational mode, and $v=2$ corresponds to variables that are related to L_2 mechanisms of the quasishear mode.

The material properties are given by

$$\begin{aligned}
\hat{c}_{11}&=c_{11}-D+(D-c_{55})\,M_{u1}+c_{55}M_{u2},\\
\hat{c}_{13}&=c_{13}+2c_{55}-D+(D-c_{55})\,M_{u1}-c_{55}M_{u2}\\
\hat{c}_{33}&=c_{33}-D+(D-c_{55})\,M_{u1}+c_{55}M_{u2},\\
\hat{c}_{55}&=c_{55}M_{u2}
\end{aligned} \tag{2.6}$$

which are high-frequency elasticities, with c_{11}, c_{13}, c_{15}, c_{33}, c_{35}, and c_{55} the low-frequency elasticities, and $D=(c_{11}+c_{33})/2 \cdot M_{uv}$, $v=1, 2$ are relaxation functions evaluated at $t=0$. For a general standard linear solid rheology they are given by

$$M_{uv}=1-\sum_{l=1}^{L_v}\tau_{\sigma l}^{(v)}\phi_{vl}, \qquad \phi_{vl}\equiv\phi_{vl}(t=0) \tag{2.7}$$

The number of variables can be reduced to three sets by defining the so-called memory variables,

$$e_{1l} = e_{xxl}^{(1)} + e_{zzl}^{(1)}, \qquad l = 1,\dots, L_1 \tag{2.8a}$$

$$e_{2l} = e_{xxl}^{(2)} - e_{zzl}^{(2)}, \qquad e_{3l} = 2e_{xzl}^{(2)}, \qquad l = 1,\dots, L_2 \tag{2.8b, c}$$

Then, the stress-strain relations (2.2) becomes

$$\begin{bmatrix} \dot\sigma_{xx} \\ \dot\sigma_{zz} \\ \dot\sigma_{xz} \end{bmatrix} = \begin{bmatrix} \hat c_{11} & \hat c_{13} & c_{15} \\ \hat c_{13} & \hat c_{33} & c_{35} \\ c_{15} & c_{35} & \hat c_{55} \end{bmatrix} \begin{bmatrix} \dot\varepsilon_{xx} \\ \dot\varepsilon_{zz} \\ 2\dot\varepsilon_{xz} \end{bmatrix} + \begin{bmatrix} D - c_{55} & c_{55} & 0 \\ D - c_{55} & -c_{55} & 0 \\ 0 & 0 & c_{55} \end{bmatrix} \begin{bmatrix} \sum_{l=1}^{L_1} \dot e_{1l} \\ \sum_{l=1}^{L_2} \dot e_{2l} \\ \sum_{l=1}^{L_2} \dot e_{3l} \end{bmatrix} \tag{2.9}$$

In the anisotropic–elastic limit, i.e., when $\tau_{\varepsilon l}^{(\nu)} \to \tau_{\sigma l}^{(\nu)}$, and the memory variables (2.4) vanish, Eq. (2.9) becomes Hooke's law. In the isotropic–viscoelastic limit we have

$$\hat c_{11},\ \hat c_{33} \to \hat\lambda + 2\hat\mu = (\lambda + \mu)\, M_{u1} + \mu M_{u2}, \qquad \hat c_{13} \to \hat\lambda = (\lambda + \mu)\, M_{u1} - \mu M_{u2}$$

$$\hat c_{55} \to \mu M_{u2}, \qquad c_{15},\ c_{35} \to 0 \tag{2.10}$$

where λ and μ are the elastic Lame constants. In this limit, Eqs. (2.9) become the isotropic–viscoelastic constitutive relation introduced in Carcione *et al.* (1988).

ii. The Memory Variables Equations:

$$\ddot e_{1l} = \phi_{1l}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z}\right) - \frac{\dot e_{1l}}{\tau_{\sigma l}^{(1)}}, \qquad l = 1,\dots, L_1 \tag{2.11a}$$

$$\ddot e_{2l} = \phi_{2l}\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_z}{\partial z}\right) - \frac{\dot e_{2l}}{\tau_{\sigma l}^{(2)}}, \qquad l = 1,\dots, L_2 \tag{2.11b}$$

$$\ddot e_{3l} = \phi_{2l}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) - \frac{\dot e_{3l}}{\tau_{\sigma l}^{(2)}}, \qquad l = 1,\dots, L_2 \tag{2.11c}$$

The equations given in (i), (ii), and (iii) are the basis for the numerical solution algorithm. The formulation requires recasting the equation governing wave propagation as

$$-\frac{\partial \mathbf{v}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{v}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{v}}{\partial z} + \mathbf{d} = 0 \tag{2.12}$$

where

$$\mathbf{v}=\begin{bmatrix} v_x \\ v_z \\ \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{xz} \\ \langle \dot{e}_{1l} \rangle_{L_1} \\ \langle \dot{e}_{2l} \rangle_{L_2} \\ \langle \dot{e}_{3l} \rangle_{L_2} \end{bmatrix}, \quad \mathbf{d}=\begin{bmatrix} f_x \\ f_z \\ (D-c_{55}) \sum_{l=1}^{L_1} \dot{e}_{1l} + c_{55} \sum_{l=1}^{L_2} \dot{e}_{2l} \\ (D-c_{55}) \sum_{l=1}^{L_1} \dot{e}_{1l} - c_{55} \sum_{l=1}^{L_2} \dot{e}_{2l} \\ \sum_{l=1}^{L_2} \dot{e}_{3l} \\ \langle -\dot{e}_{1l}/\tau_{\sigma l}^{(1)} \rangle_{L_1} \\ \langle -\dot{e}_{2l}/\tau_{\sigma l}^{(2)} \rangle_{L_2} \\ \langle -\dot{e}_{3l}/\tau_{\sigma l}^{(2)} \rangle_{L_2} \end{bmatrix} \tag{2.13a, b}$$

and

$$\mathbf{A}=\begin{bmatrix} 0 & 0 & \rho^{-1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \rho^{-1} & 0 & \cdots & 0 \\ \hat{c}_{11} & c_{15} & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hat{c}_{13} & c_{35} & 0 & 0 & 0 & 0 & \cdots & 0 \\ c_{15} & \hat{c}_{55} & 0 & 0 & 0 & 0 & \cdots & 0 \\ \langle \phi_{1l} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \rangle_{L_1} \\ \langle \phi_{2l} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \rangle_{L_2} \\ \langle 0 & \phi_{2l} & 0 & 0 & 0 & 0 & \cdots & 0 \rangle_{L_2} \end{bmatrix},$$

$$\mathbf{B}=\begin{bmatrix} 0 & 0 & 0 & 0 & \rho^{-1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \rho^{-1} & 0 & 0 & \cdots & 0 \\ c_{15} & \hat{c}_{13} & 0 & 0 & 0 & 0 & \cdots & 0 \\ c_{35} & \hat{c}_{33} & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hat{c}_{55} & c_{35} & 0 & 0 & 0 & 0 & \cdots & 0 \\ \langle 0 & \phi_{1l} & 0 & 0 & 0 & 0 & \cdots & 0 \rangle_{L_1} \\ \langle 0 & -\phi_{2l} & 0 & 0 & 0 & 0 & \cdots & 0 \rangle_{L_2} \\ \langle \phi_{2l} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \rangle_{L_2} \end{bmatrix} \tag{2.14a, b}$$

The notation $\langle \ \rangle_{L\nu}$ denotes a vertical succession of elements from $l=1,\ldots, L_\nu$, $\nu=1, 2$. The vectors in (2.12) have dimension $m=5+L_1+2L_2$, and matrices are of size $m \times m$.

The equivalent velocity–stress formulation for a viscoacoustic medium is

$$\dot{v}_x = -\frac{1}{\rho}\frac{\partial p}{\partial x} + f_x \tag{2.15a}$$

$$\dot{v}_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + f_z \tag{2.15b}$$

$$-\dot{p} = \hat{\lambda}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z}\right) + \lambda \sum_{l=1}^{L} \dot{e}_l \tag{2.15c}$$

$$\ddot{e}_l = \phi_l\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z}\right) - \frac{\dot{e}_l}{\tau_{\sigma l}}, \qquad l = 1, \ldots, L \tag{2.15d}$$

where equation (2.15d) is equivalent to (2.11a) since here only the dilatational field exists. The set of equations (2.15a)–(2.15d) can be written as a first-order matricial equation in time of the form (2.12), where

$$\mathbf{v} = \begin{bmatrix} v_x \\ v_z \\ -\dot{p} \\ \langle \dot{e}_l \rangle_L \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} f_x \\ f_z \\ \lambda \sum_{l=1}^{L} \dot{e}_l \\ \langle -\dot{e}_l/\tau_{\sigma l} \rangle_L \end{bmatrix} \tag{2.16a, b}$$

and

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & \rho^{-1} & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \hat{\lambda} & 0 & 0 & \cdots & 0 \\ \langle \phi_l & 0 & 0 & \cdots & 0 \rangle_L \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \rho^{-1} & \cdots & 0 \\ 0 & \hat{\lambda} & 0 & \cdots & 0 \\ \langle 0 & \phi_l & 0 & \cdots & 0 \rangle_L \end{bmatrix} \tag{2.17a, b}$$

Implementation of the boundary conditions along a given direction requires the characteristic equation corresponding to (2.12) in that direction.

3. THE BOUNDARY TREATMENT

The method is based on characteristics, and was applied to the wave equation by Carcione (1990b). Let the boundary be normal to the z direction; then the characteristic equation corresponding to (2.12) is

$$-\mathbf{S}^{-1}\frac{\partial \mathbf{v}}{\partial t} + \mathscr{H} + \mathbf{S}^{-1}\mathbf{C}_z = 0, \qquad \mathbf{C}_z = \mathbf{A}\frac{\partial \mathbf{v}}{\partial x} + \mathbf{d} \tag{3.1}$$

or

$$-\frac{\partial \mathbf{v}}{\partial t} + \mathbf{S}\mathscr{H} + \mathbf{C}_z = 0 \tag{3.2}$$

where

$$\mathscr{H} \equiv \mathbf{\Lambda} \mathbf{S}^{-1} \frac{\partial \mathbf{v}}{\partial z} \quad \text{and} \quad \mathbf{\Lambda} = \mathbf{S}^{-1} \mathbf{B} \mathbf{S} \tag{3.3}$$

with $\mathbf{\Lambda}$ the diagonal matrix corresponding to $\mathbf{B}$, and eigenvalues λ_i, $i = 1,..., m$; $\mathbf{S}$ is formed by the columns of the right eigenvectors of $\mathbf{B}$; $\mathscr{H}$ includes each decoupled characteristic wave mode in the z-direction. Since the system of equations is hyperbolic, the eigenvalues of $\mathbf{B}$ are real. Some of the eigenvalues give the characteristic velocities of outgoing and incoming waves at the boundary.

Equation (3.2) completely defines $\partial \mathbf{V}/\partial t$ at the boundaries in terms of the decoupled outgoing and incoming modes. The boundary conditions are implemented in the following way. Assume that $a \leqslant z \leqslant b$. For points (z, a), compute $\mathscr{H}_i(\lambda_i < 0$ outgoing waves) from Eq. (3.3), and $\mathscr{H}_i(\lambda_i > 0$ incoming waves) from the boundary conditions. Similarly, for points (z, b), compute $\mathscr{H}_i(\lambda_i > 0)$ from Eq. (3.3), and $\mathscr{H}_i(\lambda_i < 0)$ from the boundary conditions. Then, solve Eq. (2.12) for the interior region, and Eq. (3.2) at the boundaries.

For the anisotropic–viscoelastic rheology the eigenvalues are

$$\lambda_1 = c_P = (2\rho)^{-1/2} \{\hat{c}_{55} + \hat{c}_{33} + [(\hat{c}_{33} - \hat{c}_{55})^2 + 4c_{35}^2]^{1/2}\}^{-1/2}, \qquad \lambda_2 = -c_P \tag{3.4a}$$

$$\lambda_3 = c_S = (2\rho)^{-1/2} \{\hat{c}_{55} + \hat{c}_{33} - [(\hat{c}_{33} - \hat{c}_{55})^2 + 4c_{35}^2]^{1/2}\}^{-1/2}, \qquad \lambda_4 = -c_S \tag{3.4b}$$

$$\lambda_i = 0, \qquad i = 5,..., m. \tag{3.4c}$$

The zero eigenvalues arise from the fact that $\mathbf{B}$ has $m - 5$ zero columns. The first four eigenvalues satisfy

$$(\rho\lambda_i^2 - \hat{c}_{55})(\rho\lambda_i^2 - \hat{c}_{33}) - c_{35}^2 = 0, \qquad i = 1,..., 4 \tag{3.5}$$

Eigenvalues λ_1 and λ_2 are the phase velocities of quasicompressional waves moving in the positive and negative z-directions; while λ_3 and λ_4 are the corresponding velocities of the quasishear mode [see Carcione (1990a) for the expression of the phase velocities in linear anisotropic-viscoelastic media].

The quantities $\mathscr{H}_i$ relevant for the implementation of the boundary conditions are given by (Carcione, 1990b):

$$\mathscr{H}_1 = \frac{c_P}{N} \left[\frac{\partial v_z}{\partial z} + \frac{1}{Z_P} \frac{\partial \sigma_{zz}}{\partial z} + \delta \left(\frac{\partial v_x}{\partial z} + \frac{1}{Z_P} \frac{\partial \sigma_{xz}}{\partial z} \right) \right] \tag{3.6a}$$

$$\mathcal{H}_2 = -\frac{c_P}{N}\left[\frac{\partial v_z}{\partial z} - \frac{1}{Z_P}\frac{\partial \sigma_{zz}}{\partial z} + \delta\left(\frac{\partial v_x}{\partial z} - \frac{1}{Z_P}\frac{\partial \sigma_{xz}}{\partial z}\right)\right] \tag{3.6b}$$

$$\mathcal{H}_3 = \frac{c_S}{M}\left[\frac{\partial v_x}{\partial z} + \frac{1}{Z_S}\frac{\partial \sigma_{xz}}{\partial z} + \gamma\left(\frac{\partial v_z}{\partial z} + \frac{1}{Z_S}\frac{\partial \sigma_{zz}}{\partial z}\right)\right] \tag{3.6c}$$

$$\mathcal{H}_4 = -\frac{c_S}{M}\left[\frac{\partial v_x}{\partial z} - \frac{1}{Z_S}\frac{\partial \sigma_{xz}}{\partial z} + \gamma\left(\frac{\partial v_z}{\partial z} - \frac{1}{Z_S}\frac{\partial \sigma_{zz}}{\partial z}\right)\right] \tag{3.6d}$$

where

$$Z_P = \rho c_P, \qquad Z_S = \rho c_S \tag{3.7}$$

are the unrelaxed impedances of the medium, and

$$\delta = \frac{1}{c_{35}}(\rho c_P^2 - \hat{c}_{33}), \qquad \gamma = \frac{1}{c_{35}}(\rho c_S^2 - \hat{c}_{55}) \tag{3.8a, b}$$

The normalization factors are given by

$$N^2 = 2\left(1 + \frac{\rho c_P^2 - \hat{c}_{33}}{\rho c_P^2 - \hat{c}_{55}}\right), \qquad M^2 = 2\left(1 + \frac{\rho c_S^2 - \hat{c}_{55}}{\rho c_S^2 - \hat{c}_{33}}\right) \tag{3.9a, b}$$

The product $\mathbf{S}\mathcal{H}$ that is required in Eq. (3.2) is

$$\mathbf{S}\mathcal{H} = \begin{bmatrix} \frac{\delta}{N}(\mathcal{H}_1 + \mathcal{H}_2) + \frac{1}{M}(\mathcal{H}_3 + \mathcal{H}_4) \\ \frac{1}{N}(\mathcal{H}_1 + \mathcal{H}_2) + \frac{\gamma}{M}(\mathcal{H}_3 + \mathcal{H}_4) \\ \frac{\hat{c}_{13} + \delta c_{15}}{N c_P}(\mathcal{H}_1 - \mathcal{H}_2) + \frac{c_{15} + \delta \hat{c}_{13}}{M c_S}(\mathcal{H}_3 - \mathcal{H}_4) \\ \frac{Z_P}{N}(\mathcal{H}_1 - \mathcal{H}_2) + \frac{\gamma Z_S}{M}(\mathcal{H}_3 - \mathcal{H}_4) \\ \frac{\delta Z_P}{N}(\mathcal{H}_1 - \mathcal{H}_2) + \frac{Z_S}{M}(\mathcal{H}_3 - \mathcal{H}_4) \\ \left\langle \frac{\phi_{1l}}{N c_P}(\mathcal{H}_1 - \mathcal{H}_2) + \frac{\gamma \phi_{1l}}{M c_S}(\mathcal{H}_3 - \mathcal{H}_4) \right\rangle_{L_1} \\ \left\langle \frac{\phi_{2l}}{N c_P}(\mathcal{H}_2 - \mathcal{H}_1) + \frac{\gamma \phi_{2l}}{M c_S}(\mathcal{H}_4 - \mathcal{H}_3) \right\rangle_{L_2} \\ \left\langle \frac{\delta \phi_{2l}}{N c_P}(\mathcal{H}_1 - \mathcal{H}_2) + \frac{\phi_{2l}}{M c_S}(\mathcal{H}_3 - \mathcal{H}_4) \right\rangle_{L_2} \end{bmatrix} \tag{3.10}$$

For each subdomain an equation of the form (3.2) has to be solved at the boundaries. For simplicity, the formulation is applied in the next section to the isotropic problem, which can be easily solved analytically.

4. THE ISOTROPIC PROBLEM

4.1. Boundary Equations for a Viscoelastic Medium

In the isotropic limit, it can be verified that $\delta \to 0$, $\gamma \to 0$, $N \to \sqrt{2}$, and $M \to \sqrt{2}$. Then, Eqs. (3.6a)–(3.6d) become

$$\mathscr{H}_1 = \frac{c_P}{\sqrt{2}}\left(\frac{\partial v_z}{\partial z} + \frac{1}{Z_P}\frac{\partial \sigma_{zz}}{\partial z}\right) \tag{4.1a}$$

$$\mathscr{H}_2 = -\frac{c_P}{\sqrt{2}}\left(\frac{\partial v_z}{\partial z} - \frac{1}{Z_P}\frac{\partial \sigma_{zz}}{\partial z}\right) \tag{4.1b}$$

$$\mathscr{H}_3 = \frac{c_S}{\sqrt{2}}\left(\frac{\partial v_x}{\partial z} + \frac{1}{Z_S}\frac{\partial \sigma_{xz}}{\partial z}\right) \tag{4.1c}$$

$$\mathscr{H}_4 = -\frac{c_S}{\sqrt{2}}\left(\frac{\partial v_x}{\partial z} - \frac{1}{Z_S}\frac{\partial \sigma_{xz}}{\partial z}\right) \tag{4.1d}$$

where $c_P = [(\hat{\lambda} + 2\hat{\mu})/\rho]^{1/2}$ and $c_s = (\hat{\mu}/\rho)^{1/2}$ are the unrelaxed compressional and shear wave velocities. Substitution of (3.10) into Eq. (3.2) gives the wave propagation equations in terms of the decoupled outgoing and incoming modes:

$$\dot{v}_x = \frac{1}{\rho}\frac{\partial \sigma_{xx}}{\partial x} + \frac{1}{\sqrt{2}}(\mathscr{H}_3 + \mathscr{H}_4) + f_x \tag{4.2a}$$

$$\dot{v}_z = \frac{1}{\rho}\frac{\partial \sigma_{xz}}{\partial x} + \frac{1}{\sqrt{2}}(\mathscr{H}_1 + \mathscr{H}_2) + f_z \tag{4.2b}$$

$$\dot{\sigma}_{xx} = (\hat{\lambda} + 2\hat{\mu})\frac{\partial v_x}{\partial x} + \frac{1}{\sqrt{2}}\frac{\hat{\lambda}}{c_P}(\mathscr{H}_1 - \mathscr{H}_2) + (\lambda + \mu)\sum_{l=1}^{L_1} \dot{e}_{1l} + \mu \sum_{l=1}^{L_2} \dot{e}_{2l} \tag{4.2c}$$

$$\dot{\sigma}_{zz} = \hat{\lambda}\frac{\partial v_x}{\partial x} + \frac{Z_P}{\sqrt{2}}(\mathscr{H}_1 - \mathscr{H}_2) + (\lambda + \mu)\sum_{l=1}^{L_1} \dot{e}_{1l} - \mu \sum_{l=1}^{L_2} \dot{e}_{2l} \tag{4.2d}$$

$$\dot{\sigma}_{xz} = \hat{\mu}\frac{\partial v_z}{\partial x} + \frac{Z_S}{\sqrt{2}}(\mathscr{H}_3 - \mathscr{H}_4) + \mu \sum_{l=1}^{L_2} \dot{e}_{3l} \tag{4.2e}$$

$$\ddot{e}_{1l} = \phi_{1l}\left[\frac{\partial v_x}{\partial x} + \frac{1}{\sqrt{2}\, c_P}(\mathscr{H}_1 - \mathscr{H}_2)\right] - \frac{\dot{e}_{1l}}{\tau_{\sigma l}^{(1)}}, \qquad l = 1,\ldots, L_1 \tag{4.2f}$$

$$\ddot{e}_{2l}=\phi_{2l}\left[\frac{\partial v_x}{\partial x}+\frac{1}{\sqrt{2}\,c_P}(\mathscr{H}_2-\mathscr{H}_1)\right]-\frac{\dot{e}_{2l}}{\tau_{\sigma l}^{(2)}}, \qquad l=1,..., L_2 \tag{4.2g}$$

$$\ddot{e}_{3l}=\phi_{2l}\left[\frac{\partial v_z}{\partial x}+\frac{1}{\sqrt{2}\,c_S}(\mathscr{H}_3-\mathscr{H}_4)\right]-\frac{\dot{e}_{3l}}{\tau_{\sigma l}^{(2)}}, \qquad l=1,..., L_2 \tag{4.2h}$$

These equations are used at the boundaries normal to the z-direction, where the quantities $\mathscr{H}_i$, representing incoming variables, are calculated from the boundary conditions.

4.2. Boundary Equations for a Viscoacoustic Medium

For a viscoacoustic medium the quantities $\mathscr{H}_i$ are

$$\mathscr{H}_1=\frac{c_P}{\sqrt{2}}\left(\frac{\partial v_z}{\partial z}-\frac{1}{Z_P}\frac{\partial p}{\partial z}\right) \tag{4.3a}$$

$$\mathscr{H}_2=-\frac{c_P}{\sqrt{2}}\left(\frac{\partial v_z}{\partial z}+\frac{1}{Z_P}\frac{\partial p}{\partial z}\right) \tag{4.3b}$$

The term $\mathbf{S}\mathscr{H}$ in Eq. (3.2) becomes

$$\mathbf{S}\mathscr{H}=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ \mathscr{H}_1+\mathscr{H}_2 \\ Z_P(\mathscr{H}_1-\mathscr{H}_2) \\ \dfrac{\phi_l}{c_P}(\mathscr{H}_1-\mathscr{H}_2) \end{bmatrix} \tag{4.4}$$

Substitution of (4.4) into (3.2) yields the equations for the boundaries,

$$\dot{v}_x=-\frac{1}{\rho}\frac{\partial p}{\partial x}+f_x \tag{4.5a}$$

$$\dot{v}_z=\frac{1}{\sqrt{2}}(\mathscr{H}_1+\mathscr{H}_2)+f_z \tag{4.5b}$$

$$-\dot{p}=\hat{\lambda}\frac{\partial v_x}{\partial x}+\frac{Z_P}{\sqrt{2}}(\mathscr{H}_1-\mathscr{H}_2)+\lambda\sum_{l=1}^{L}\dot{e}_l \tag{4.5c}$$

$$\ddot{e}_l=\phi_l\left[\frac{\partial v_x}{\partial x}+\frac{1}{\sqrt{2}\,c_P}(\mathscr{H}_1-\mathscr{H}_2)\right]-\frac{\dot{e}_l}{\tau_{\sigma l}}, \qquad l=1,..., L \tag{4.5d}$$

5. DOMAIN DECOMPOSITION FOR A HORIZONTAL BOUNDARY

5.1. Viscoelastic–Viscoelastic boundary

Let the upper and lower media be indicated by medium A and medium B, respectively, with z increasing toward the upper medium. Then, the incoming characteristics in medium A are represented by $\mathscr{H}_1(A)$ and $\mathscr{H}_3(A)$, while in medium B they are $\mathscr{H}_2(B)$ and $\mathscr{H}_4(B)$. These quantities are computed from the boundary conditions at the interface as indicated in Fig. 2. Actually, each $\mathscr{H}_i$ represents a one-way wave motion which is assumed to be continuous across the interface. In a solid–solid boundary, continuity of v_x, v_z, σ_{zz}, and σ_{xz} is required. From Eqs. (4.2a)–(4.2g) these conditions imply that

$$\frac{1}{\rho(A)}\frac{\partial\sigma_{xx}}{\partial x}(A)+\frac{1}{\sqrt{2}}[\mathscr{H}_3(A)+\mathscr{H}_4(A)]+f_x(A) = \frac{1}{\rho(B)}\frac{\partial\sigma_{xx}}{\partial x}(B)+\frac{1}{\sqrt{2}}[\mathscr{H}_3(B)+\mathscr{H}_4(B)]+f_x(B) \tag{5.1a}$$

$$\frac{1}{\rho(A)}\frac{\partial\sigma_{xz}}{\partial x}(A)+\frac{1}{\sqrt{2}}[\mathscr{H}_1(A)+\mathscr{H}_2(A)]+f_z(A) = \frac{1}{\rho(B)}\frac{\partial\sigma_{xz}}{\partial x}(B)+\frac{1}{\sqrt{2}}[\mathscr{H}_1(B)+\mathscr{H}_2(B)]+f_z(B) \tag{5.1b}$$

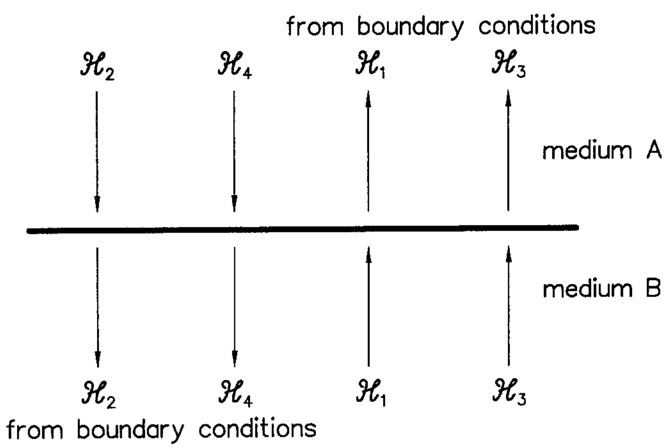

Fig. 2. The incoming characteristics variables $\mathscr{H}_1$ and $\mathscr{H}_3$ in medium A, and $\mathscr{H}_2$ and $\mathscr{H}_4$ in medium B, are calculated from the boundary conditions. The diagram represents continuity of one-way motion across the interface.

$$\hat{\lambda}(A)\frac{\partial v_x}{\partial x}(A)+\frac{Z_P(A)}{\sqrt{2}}[\mathscr{H}_1(A)-\mathscr{H}_2(A)]$$

$$+[\lambda(A)+\mu(A)]\sum_{l=1}^{L_1(A)}\dot{e}_{1l}(A)-\mu(A)\sum_{l=1}^{L_2(A)}\dot{e}_{2l}(A)$$

$$=\hat{\lambda}(B)\frac{\partial v_x}{\partial x}(B)+\frac{Z_P(B)}{\sqrt{2}}[\mathscr{H}_1(B)-\mathscr{H}_2(B)]$$

$$+[\lambda(B)+\mu(B)]\sum_{l=1}^{L_1(B)}\dot{e}_{1l}(B)-\mu(B)\sum_{l=1}^{L_2(B)}\dot{e}_{2l}(B) \tag{5.1c}$$

$$\hat{\mu}(A)\frac{\partial v_z}{\partial x}(A)+\frac{Z_S(A)}{\sqrt{2}}[\mathscr{H}_3(A)-\mathscr{H}_4(A)]+\mu(A)\sum_{l=1}^{L_2(A)}\dot{e}_{3l}(A)$$

$$=\hat{\mu}(B)\frac{\partial v_z}{\partial x}(B)+\frac{Z_S(B)}{\sqrt{2}}[\mathscr{H}_3(B)-\mathscr{H}_4(B)]+\mu(B)\sum_{l=1}^{L_2(B)}\dot{e}_{3l}(B) \tag{5.1d}$$

Equations (5.1a)–(5.1d) are solved for the four unknowns $\mathscr{H}_1(A)$, $\mathscr{H}_3(A)$, $\mathscr{H}_2(B)$, and $\mathscr{H}_4(B)$, while the quantities representing outgoing waves $\mathscr{H}_2(A)$, $\mathscr{H}_4(A)$, $\mathscr{H}_1(B)$, and $\mathscr{H}_3(B)$ are given by Eqs. (4.1a)–(4.1d). Substituting the results into (4.2a)–(4.2g) yields the equations for the interface:

$$\dot{v}_x^{(\text{new})}=\frac{1}{Z_S(A)+Z_S(B)}\times[Z_S(B)\,\dot{v}_x^{(\text{old})}(B)+Z_S(A)\,\dot{v}_x^{(\text{old})}(A)-\dot{\sigma}_{xz}^{(\text{old})}(A)+\dot{\sigma}_{xz}^{(\text{old})}(B)] \tag{5.2a}$$

$$\dot{v}_z^{(\text{new})}=\frac{1}{Z_P(A)+Z_P(B)}\times[Z_P(B)\,\dot{v}_z^{(\text{old})}(B)+Z_P(A)\,\dot{v}_z^{(\text{old})}(A)-\dot{\sigma}_{zz}^{(\text{old})}(A)+\dot{\sigma}_{zz}^{(\text{old})}(B)] \tag{5.2b}$$

$$\dot{\sigma}_{zz}^{(\text{new})}=\frac{Z_P(A)\,Z_P(B)}{Z_P(A)+Z_P(B)}\times\left[\dot{v}_z^{(\text{old})}(B)-\dot{v}_z^{(\text{old})}(A)+\frac{\dot{\sigma}_{zz}^{(\text{old})}(A)}{Z_P(A)}+\frac{\dot{\sigma}_{zz}^{(\text{old})}(B)}{Z_P(B)}\right] \tag{5.2c}$$

$$\dot{\sigma}_{xz}^{(\text{new})}=\frac{Z_S(A)\,Z_S(B)}{Z_S(A)+Z_S(B)}\times\left[\dot{v}_x^{(\text{old})}(B)-\dot{v}_x^{(\text{old})}(A)+\frac{\dot{\sigma}_{xz}^{(\text{old})}(A)}{Z_S(A)}+\frac{\dot{\sigma}_{xz}^{(\text{old})}(B)}{Z_S(B)}\right] \tag{5.2d}$$

$$\dot{\sigma}_{xx}^{(\text{new})}(\bullet)=\dot{\sigma}_{xx}^{(\text{old})}(\bullet)+\frac{\hat{\lambda}(\bullet)}{\hat{\lambda}(\bullet)+2\hat{\mu}(\bullet)}\,[\dot{\sigma}_{zz}^{(\text{new})}-\dot{\sigma}_{zz}^{(\text{old})}(\bullet)],$$
$$(\bullet)=A \text{ or } B \tag{5.2e}$$

$$\ddot{e}_{1l}^{(\text{new})}(\bullet)=\ddot{e}_{1l}^{(\text{old})}(\bullet)+\frac{\phi_{1l}(\bullet)}{\hat{\lambda}(\bullet)+2\hat{\mu}(\bullet)}$$
$$\times\,[\dot{\sigma}_{zz}^{(\text{new})}-\dot{\sigma}_{zz}^{(\text{old})}(\bullet)],\qquad l=1,\dots,L_1(\bullet),\ (\bullet)=A \text{ or } B \tag{5.2f}$$

$$\ddot{e}_{2l}^{(\text{new})}(\bullet)=\ddot{e}_{2l}^{(\text{old})}(\bullet)-\frac{\phi_{2l}(\bullet)}{\hat{\lambda}(\bullet)+2\hat{\mu}(\bullet)}$$
$$\times\,[\dot{\sigma}_{zz}^{(\text{new})}-\dot{\sigma}_{zz}^{(\text{old})}(\bullet)],\qquad l=1,\dots,L_2(\bullet),\ (\bullet)=A \text{ or } B \tag{5.2g}$$

$$\ddot{e}_{3l}^{(\text{new})}(\bullet)=\ddot{e}_{3l}^{(\text{old})}(\bullet)+\frac{\phi_{2l}(\bullet)}{\hat{\mu}(\bullet)}\,[\dot{\sigma}_{xz}^{(\text{new})}-\dot{\sigma}_{xz}^{(\text{old})}(\bullet)],$$
$$l=1,\dots,L_2(\bullet),\ \ (\bullet)=A \text{ or } B \tag{5.2h}$$

where $\dot{\mathbf{v}}^{(\text{new})}$ is the left-hand side of Eqs. (4.2a)–(4.2g), and $\dot{\mathbf{v}}^{(\text{old})}$ is the right-hand side of the equations within the computational volume, i.e., from (2.12),

$$\dot{\mathbf{v}}^{(\text{old})}=\mathbf{A}\,\frac{\partial\mathbf{v}}{\partial x}+\mathbf{B}\,\frac{\partial\mathbf{v}}{\partial z}+\mathbf{d} \tag{5.3}$$

5.2. Viscoacoustic–Viscoelastic Boundary

Let medium A be viscoacoustic. Then, the characteristics corresponding to $\mathscr{H}_1(A)$, $\mathscr{H}_2(B)$, and $\mathscr{H}_4(B)$ have to be computed from the boundary conditions. These involve continuity of v_z and normal stresses, i.e., $-p=\sigma_{zz}$. Moreover, zero shear stress imposes $\sigma_{xz}=0$, from which the quantity $\mathscr{H}_4(B)$ can be computed. Continuity implies that

$$\frac{1}{\sqrt{2}}\,[\mathscr{H}_1(A)+\mathscr{H}_2(A)]+f_z(A)$$
$$=\frac{1}{\rho(B)}\frac{\partial\sigma_{xz}}{\partial x}(B)+\frac{1}{\sqrt{2}}\,[\mathscr{H}_1(B)+\mathscr{H}_2(B)]+f_z(B) \tag{5.4a}$$

$$\hat{\lambda}(A)\frac{\partial v_x}{\partial x}(A)+\frac{1}{\sqrt{2}}Z_P(A)[\mathscr{H}_1(A)-\mathscr{H}_2(A)]+\lambda(A)\sum_{l=1}^{L(A)}\dot{e}_l(A)$$
$$=\hat{\lambda}(B)\frac{\partial v_x}{\partial x}(B)+\frac{1}{\sqrt{2}}Z_P(B)[\mathscr{H}_1(B)-\mathscr{H}_2(B)]$$
$$+\,[\lambda(B)+\mu(B)]\sum_{l=1}^{L_1(B)}\dot{e}_{1l}(B)-\mu(B)\sum_{l=1}^{L_2(B)}\dot{e}_{2l}(B) \tag{5.4b}$$

As before, solving these equations and replacing the results into (4.2a)–(4.2g) gives the equations for both sides of the interface:

$$\dot{v}_x^{(\text{new})}(A) = \dot{v}_x^{(\text{old})}(A) \tag{5.5a}$$

$$\dot{v}_z^{(\text{new})} = \frac{1}{Z_P(A) + Z_P(B)} \times [Z_P(B)\, \dot{v}_z^{(\text{old})}(B) + Z_P(A)\, \dot{v}_z^{(\text{old})}(A) + \dot{p}^{(\text{old})}(A) + \dot{\sigma}_{zz}^{(\text{old})}(B)] \tag{5.5b}$$

$$-\dot{p}^{(\text{new})} = \dot{\sigma}_{zz}^{(\text{new})} = \frac{Z_P(A)\, Z_P(B)}{Z_P(A) + Z_P(B)} \times \left[\dot{v}_z^{(\text{old})}(B) - \dot{v}_z^{(\text{old})}(A) - \frac{\dot{p}^{(\text{old})}(A)}{Z_P(A)} + \frac{\dot{\sigma}_{zz}^{(\text{old})}(B)}{Z_P(B)} \right] \tag{5.5c}$$

$$\ddot{e}_l^{(\text{new})}(A) = \ddot{e}_l^{(\text{old})}(A) + \frac{\phi_l(A)}{\hat{\lambda}(A)} [\dot{p}^{(\text{old})}(A) - \dot{p}^{(\text{new})}], \qquad l = 1, \ldots, L(A) \tag{5.5d}$$

$$\dot{v}_x^{(\text{new})}(B) = \dot{v}_x^{(\text{old})}(B) + \frac{\dot{\sigma}_{xz}^{(\text{old})}(B)}{Z_S(B)} \tag{5.5e}$$

$$\dot{\sigma}_{xz}^{(\text{new})}(B) = 0 \tag{5.5f}$$

$$\dot{\sigma}_{xx}^{(\text{new})}(B) = \dot{\sigma}_{xx}^{(\text{old})}(B) + \frac{\hat{\lambda}(B)}{\hat{\lambda}(B) + 2\hat{\mu}(B)} [\dot{\sigma}_{zz}^{(\text{new})} - \dot{\sigma}_{zz}^{(\text{old})}(B)] \tag{5.5g}$$

$$\ddot{e}_{1l}^{(\text{new})}(B) = \ddot{e}_{1l}^{(\text{old})}(B) + \frac{\phi_{1l}(B)}{\hat{\lambda}(B) + 2\hat{\mu}(B)} [\dot{\sigma}_{zz}^{(\text{new})} - \dot{\sigma}_{zz}^{(\text{old})}(B)], \qquad l = 1, \ldots, L_1(B) \tag{5.5h}$$

$$\ddot{e}_{2l}^{(\text{new})}(B) = \ddot{e}_{2l}^{(\text{old})}(B) - \frac{\phi_{2l}(B)}{\hat{\lambda}(B) + 2\hat{\mu}(B)} [\dot{\sigma}_{zz}^{(\text{new})} - \dot{\sigma}_{zz}^{(\text{old})}(B)], \qquad l = 1, \ldots, L_2(B) \tag{5.5i}$$

$$\ddot{e}_{3l}^{(\text{new})}(B) = \ddot{e}_{3l}^{(\text{old})}(B) - \frac{\phi_{2l}(B)}{\hat{\mu}(B)} \dot{\sigma}_{xz}^{(\text{old})}(B), \qquad l = 1, \ldots, L_2(B) \tag{5.5j}$$

When medium A is vacuum $[Z_P(A) = 0]$, or in many applications air $[Z_P(A) \approx 0]$, the interface obeys free surface boundary conditions, which imply that normal stresses vanish, i.e., $\sigma_{xz} = 0$, and $\sigma_{zz} = 0$, In this case, (5.5e)–(5.5*j*) reduce to the boundary equations for free surface implemented in Carcione (1991).

5.3. Viscoacoustic–Viscoacoustic Boundary

Continuity of normal displacements and pressures leads to the following equations at the interface:

$$\dot{v}_x^{(\text{new})}(\bullet) = \dot{v}_x^{(\text{old})}(\bullet), \qquad (\bullet) = A \text{ or } B \tag{5.6a}$$

$$\dot{v}_z^{(\text{new})} = \frac{1}{Z_P(A) + Z_P(B)} \times [Z_P(B)\, \dot{v}_z^{(\text{old})}(B) + Z_P(A)\, \dot{v}_z^{(\text{old})}(A) + \dot{p}^{(\text{old})}(A) - \dot{p}^{(\text{old})}(B)] \tag{5.6b}$$

$$-\dot{p}^{(\text{new})} = \frac{Z_P(A)\, Z_P(B)}{Z_P(A) + Z_P(B)} \times \left[\dot{v}_z^{(\text{old})}(B) - \dot{v}_z^{(\text{old})}(A) - \frac{\dot{p}^{(\text{old})}(A)}{Z_P(A)} - \frac{\dot{p}^{(\text{old})}(B)}{Z_P(B)}\right] \tag{5.6c}$$

$$\ddot{e}_l^{(\text{new})}(\bullet) = \ddot{e}_l^{(\text{old})}(\bullet) + \frac{\phi_l(\bullet)}{\hat{\lambda}(\bullet)} [\dot{p}^{(\text{old})}(\bullet) - \dot{p}^{(\text{new})}], \quad l = 1, \ldots, L(\bullet), \quad (\bullet) = A \text{ or } B \tag{5.6d}$$

It can be shown that in the elastic case, i.e., when $\tau_{\varepsilon l}^{(\nu)} \to \tau_{\sigma l}^{(\nu)}$, and the memory variables vanish, the boundary equations are equivalent to the interface equation found by Tessmer *et al.* (1990) and Kessler and Kosloff (1991).

The equations obtained for a horizontal interface can be easily extended for inclined boundaries. This is done in the next section.

6. DOMAIN DECOMPOSITION FOR INCLINED BOUNDARIES

Consider that the boundary is not perpendicular to any of the Cartesian coordinate axes, i.e., that say the z' direction normal to the boundary makes an angle θ with the z axis where the problem is solved (see Fig. 3). For convenience, these coordinate systems are denoted by S' and S, respectively. The constitutive relation (2.2) in S can be written in compact form as

$$\mathbf{T} = \mathbf{\Psi S} + \mathbf{D}_\nu \sum_{l=1}^{L_\nu} \mathbf{E}_l^{(\nu)} \tag{6.1}$$

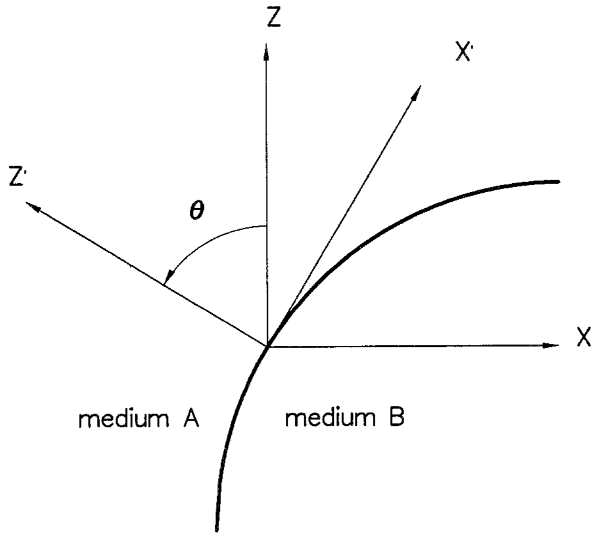

Fig. 3. Disposition of coordinate systems for an inclined interface. The z' axis is perpendicular to the interface.

where **T** and **S** are the stress and strain vectors, $\mathbf{\Psi}$ is the unrelaxed matrix multiplying the strain vector, $\mathbf{E}_\nu$, $\nu = 1, 2$ are the memory vectors for dilation and shear, respectively, and $\mathbf{D}_\nu$, $\nu = 1, 2$ are the matrices which multiply the memory vectors. Implicit summation over the index ν is assumed.

Firstly, the boundary equations in system S' should be computed. They have already been obtained in previous sections [e.g., Eqs. (5.2a)–(5.2h)], provided that the matrices $\mathbf{\Psi}$ and $\mathbf{D}_\nu$ containing the material properties are known in S'. This implies a coordinate transformation from S to S', which is achieved with the so-called Bond matrices (Auld, 1973). The boundary equations are finally obtained by coordinate transformations from S' to S.

A rotation of the coordinate system transforms the particle velocities as

$$\begin{bmatrix} v'_x \\ v'_z \end{bmatrix} = \mathbf{R}(\theta) \begin{bmatrix} v_x \\ v_z \end{bmatrix}, \qquad \begin{bmatrix} v_x \\ v_z \end{bmatrix} = \mathbf{R}(-\theta) \begin{bmatrix} v'_x \\ v'_z \end{bmatrix} \tag{6.2a, b}$$

where

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta \sin\theta \\ -\sin\theta \cos\theta \end{bmatrix} \tag{6.3}$$

The stress, strain, and memory vectors transform as

$$\mathbf{T}' = \mathbf{M}(\theta)\,\mathbf{T}, \qquad \mathbf{S}' = \mathbf{M}^T(-\theta)\,\mathbf{S}, \qquad \mathbf{E}_l'^{(\nu)} = \mathbf{M}^T(-\theta)\,\mathbf{E}_l^{(\nu)} \qquad (6.4\text{a, b, c})$$

where

$$\mathbf{M} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin 2\theta \\ \sin^2\theta & \cos^2\theta & -\sin 2\theta \\ \dfrac{-\sin 2\theta}{2} & \dfrac{\sin 2\theta}{2} & \cos 2\theta \end{bmatrix},$$

$$\mathbf{M}^{-1}(\theta) = \mathbf{M}(-\theta), \qquad [\mathbf{M}^T(-\theta)]^{-1} = \mathbf{M}^T(\theta) \qquad (6.5\text{a, b, c})$$

are Bond transformation matrices. Making use of (6.4c), the memory variables (2.8a)–(2.8c) transform as

$$e'_{1l} = e_{1l}, \qquad l = 1,\ldots, L_1 \qquad (6.6\text{a})$$

$$\begin{bmatrix} e'_{2l} \\ e'_{3l} \end{bmatrix} = \mathbf{R}(2\theta) \begin{bmatrix} e_{2l} \\ e_{3l} \end{bmatrix}, \qquad \begin{bmatrix} e_{2l} \\ e_{3l} \end{bmatrix} = \mathbf{R}(-2\theta) \begin{bmatrix} e'_{2l} \\ e'_{3l} \end{bmatrix}, \qquad l = 1,\ldots, L_2 \qquad (6.6\text{b, c})$$

Application of Bond transformations to Eq. (6.1) implies that

$$\mathbf{\Psi}' = \mathbf{M}(\theta)\,\mathbf{\Psi}\mathbf{M}^T(\theta), \qquad \mathbf{D}'_\nu = \mathbf{M}(\theta)\,\mathbf{D}_\nu\mathbf{M}^T(\theta), \qquad \nu = 1, 2 \qquad (6.7\text{a, b})$$

which are used to determine the boundary equations in system S'. It can be shown that $\mathbf{D}'_\nu = \mathbf{D}_\nu$, and if the rheology is isotropic, $\mathbf{\Psi}' = \mathbf{\Psi}$.

Formally, the procedure for matching medium A and medium B is the following: Assume that the boundary equations in system S' can be written as [e.g., (5.2a)–(5.2h)]

$$\dot{\mathbf{v}}'^{(\text{new})}(A) = \mathbf{P}(A) \begin{bmatrix} \dot{\mathbf{v}}'^{(\text{old})}(A) \\ \dot{\mathbf{v}}'^{(\text{old})}(B) \end{bmatrix}, \qquad \dot{\mathbf{v}}'^{(\text{new})}(B) = \mathbf{P}(B) \begin{bmatrix} \dot{\mathbf{v}}'^{(\text{old})}(A) \\ \dot{\mathbf{v}}'^{(\text{old})}(B) \end{bmatrix} \qquad (6.8\text{a, b})$$

where $\mathbf{P}(A)$ and $\mathbf{P}(B)$ are matrices of size $2m \times m$ containing material properties. Moreover, applying Bond transformations to the components of $\mathbf{v}$ results in

$$\mathbf{v}' = \mathbf{N}\mathbf{v}, \qquad \mathbf{v} = \mathbf{N}^{-1}\mathbf{v}' \qquad (6.9\text{a, b})$$

with $\mathbf{N}$ a transformation matrix of size $m \times m$ which depends on the components of $\mathbf{R}$ and $\mathbf{M}$. Then, transforming (6.8a) and (6.8b) by using (6.9a) and (6.9b) yields the boundary equations in system S,

$$\dot{\mathbf{v}}^{(\mathrm{new})}(A) = \mathbf{N}^{-1}\left\{\mathbf{P}(A)\begin{bmatrix}\mathbf{N}\dot{\mathbf{v}}^{(\mathrm{old})}(A)\\ \mathbf{N}\dot{\mathbf{v}}^{(\mathrm{old})}(B)\end{bmatrix}\right\},$$
$$\dot{\mathbf{v}}^{(\mathrm{new})}(B) = \mathbf{N}^{-1}\left\{\mathbf{P}(B)\begin{bmatrix}\mathbf{N}\dot{\mathbf{v}}^{(\mathrm{old})}(A)\\ \mathbf{N}\dot{\mathbf{v}}^{(\mathrm{old})}(B)\end{bmatrix}\right\} \qquad (6.10\mathrm{a, b})$$

In general, equations (6.10a) (6.10b) must be solved at the boundaries of medium A and B, respectively. Most explicit time integration schemes compute the operation **Cv**, where

$$\mathbf{C} \equiv \mathbf{A}\frac{\partial}{\partial x} + \mathbf{B}\frac{\partial}{\partial z} \qquad (6.11)$$

is the differential operator of the right-hand side of (2.12). In these cases, the approach first calculates

$$\mathbf{v}^{(\mathrm{old})} = \mathbf{C}\mathbf{v} \qquad (6.12)$$

which is enough, inside the computational domain excluding the boundaries. At the boundaries, additional calculations are required such that

$$\mathbf{v}^{(\mathrm{old})}(\bullet) \to \text{rotation } \mathbf{N} \to \mathbf{v}'^{(\mathrm{old})}(\bullet) \to \text{matching } \mathbf{P}(\bullet) \to \mathbf{v}'^{(\mathrm{new})}(\bullet)$$
$$\to \text{rotation } \mathbf{N}^{-1} \to \mathbf{v}^{(\mathrm{new})}(\bullet) \qquad (6.13)$$

for every operation with **C**.

7. CONCLUSIONS

The problem of matching waves at interfaces separating subdomains has been solved by decoupling the wave equation and imposing appropriate boundary conditions to the incoming waves in each subdomain. Emphasis was given to the treatment of different rheologies, although the method is equally valid for use in hybrid modeling, and situations where the presence of heterogeneities requires the use of dissimilar grid sizes.

ACKNOWLEDGMENTS

This work was supported in part by the Commission of the European Communities under the GEOSCIENCE project.

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Printed in Belgium