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Rayleigh waves in isotropic viscoelastic media

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SUMMARY

In the surface of a linear viscoelastic medium, two types of Rayleigh waves may propagate. One of them, which always occurs, has wave characteristics which are close to those of the corresponding elastic solid. The second surface wave, not present in the elastic case, is possible for certain values of the material parameters, and for a given range of frequencies. Its properties are different from those of the first surface wave, particularly the energy velocity which is closer to the compressional body wave velocity. In this work, the properties of the two wave modes are analysed by using energy considerations. The energy balance for the Rayleigh waves is computed, and the quality factors and energy velocities are calculated as a function of the frequency, of depth, and per unit surface area.

The main results indicate that the anelastic properties calculated from energy considerations are close, for practical purposes, to those obtained from the Rayleigh secular equation, i.e. phase velocity and attenuation factors give a good approximation to the dispersive and dissipation characteristics of the waves. In relation to the elastic case, the energy is more evenly distributed with depth, particularly in the v.e. mode. This wave has similar anelastic properties to those of the compressional body wave.

Key words: Energy balance, Rayleigh surface waves, viscoelasticity.

INTRODUCTION

Rayleigh waves are of importance in several fields, from earthquake seismology and geophysical exploration to material science (e.g. Auld 1985; Parker & Maugin 1988). The first theoretical investigations carried out by Lord Rayleigh (1885) in isotropic elastic media showed that these waves are particularly important in seismology since their propagation is confined to the surface, and therefore, they do not scatter in depth as seismic body waves.

Hardtwig (1943) was the first to study viscoelastic Rayleigh waves, though he erroneously restricts their existence to a particular choice of the complex Lamé parameters. Scholte (1947) rectifies this mistake, and verifies that the wave always exists in a viscoelastic solid. He also predicts the existence of a second surface wave, mainly periodic with depth, whose exponential damping is due to anelasticity and not to the Rayleigh character (referred to later as v.e. mode). Caloi (1948) and Horton (1953) analysed the anelastic characteristics and displacements of the waves considering a Voigt-type dissipation mechanism with small viscous damping and a Poisson solid. Press & Healy (1957) relate the Rayleigh wave quality factor to the body wave quality factors in a low-loss solid, and tested the result successfully with laboratory experiments in Plexiglas. The first to derive the attenuation coefficient from energy considerations were King & Sheard (1969). They obtained a formula for a Voigt-type anisotropic solid in terms of a viscosity tensor. Their predictions for quartz agree fairly well with the experimental values. Borcherdt (1973) further analyses the particle motion at the free surface and concludes that the differences between elastic and viscoelastic Rayleigh waves arise from the differences of their components: the usual inhomogeneous plane waves in the elastic case, and viscoelastic inhomogeneous plane waves in the anelastic case which allow any angle between the propagation and attenuation vectors.

A complete analysis was carried out by Currie, Hayes & O'Leary (1977), Currie & O'Leary (1978) and Currie (1979). They show that for viscoelastic Rayleigh waves: (i) more than one wave is possible; (ii) the particle motion may be either direct or retrograde at the surface; (iii) the motion may change sense at many or no levels with depth; and (iv) the wave energy velocity may be greater than the body wave energy velocities. They refer to the wave that corresponds to the usual elastic surface wave as quasi-elastic (q.e.), and viscoelastic (v.e.) the wave that only exists in the viscoelastic medium. This mode is possible only for certain combinations of the complex Lamé constants, and for a given range of frequencies.

The purpose of this work is to investigate the Rayleigh wave characteristics from the standpoint of energy. Besides the phase velocity and attenuation factor which result directly from the Rayleigh secular equation, the calculation of the energy velocity and quality factor from energy considerations gives new insight into the anelastic properties of the waves. Moreover, a general viscoelastic medium is considered, in the sense that it is not restricted to a low-loss solid and to any particular type of dissipation mechanism, i.e., the complex Lamé parameters may have any frequency dependence.

The content of the next sections is, in outline, as follows. First, the equation of motion and the constitutive relation of the isotropic linear viscoelastic solid are derived in terms of the complex Lamé parameters. In the next section, the cubic secular equation determining the complex velocities is derived. There follows the determination of the displacement field and phase velocities associated with the surface wave and each mode component. The calculation of the energy velocities and quality factors requires energy considerations. The energy balance equation describes the dynamic process of wave propagation and allows the calculation of the anelastic characteristics of the Rayleigh waves with depth and as a function of the frequency. Both energy velocity and quality factor are computed per unit surface area. Finally, special types of viscoelastic media are studied, for instance the incompressible solid which describes surface waves in polymers, or the Poisson solid, frequently used in seismological applications. The example studies the wave characteristics of a solid where two modes are possible, the quasi-elastic, and one viscoelastic surface wave.

CONSTITUTIVE RELATION AND EQUATION OF MOTION

The constitutive relation of an isotropic viscoelastic solid can be expressed as (Carcione 1990)

$$\mathbf{T}(\mathbf{x}, t) = \mathbf{\Psi}(\mathbf{x}, t) * \dot{\mathbf{S}}(\mathbf{x}, t) \quad \text{or} \quad T_I = \psi_{IJ} * \dot{S}_J, \quad I, J = 1, \ldots, 6, \tag{1}$$

with

$$\mathbf{T}^{\mathrm{T}} = (T_1, T_2, T_3, T_4, T_5, T_6) = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}) \tag{2}$$

the stress vector, where σ_{ij}, i, $j = 1, \ldots, 3$ are the stress components, and

$$\mathbf{S}^{\mathrm{T}} = (S_1, S_2, S_3, S_4, S_5, S_6) = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{13}, 2\varepsilon_{12}) \tag{3}$$

the strain vector, where ε_{ij}, i, $j = 1, \ldots, 3$ are the strain components; $\mathbf{\Psi}$ is the symmetrix relaxation matrix with components

$$\psi_{IJ} = \begin{cases} \hat{\lambda} + 2\hat{\mu}\delta_{IJ} & I, J \leq 3, \\ \hat{\mu}\delta_{IJ}, & I, J > 3, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where $\hat{\lambda}(t)$ and $\hat{\mu}(t)$ are Lamé relaxation functions and δ_{IJ} is the Kronecker delta, t is the time variable, $\mathbf{x}$ is the position vector, the symbol $*$ indicates time convolution, a dot above a variable implies time differentiation, and the Einstein convention for repeated indices is used. Vectors are written as columns with the superscript T denoting transpose.

The Lamé relaxations functions can be expressed as (Carcione 1990)

$$\hat{\lambda} = (\lambda^{\mathrm{e}} + \tfrac{2}{3}\mu^{\mathrm{e}})\chi_1 - \tfrac{2}{3}\mu^{\mathrm{e}}\chi_2, \qquad \hat{\mu} = \mu^{\mathrm{e}}\chi_2, \tag{5a, b}$$

with λ^{e} and μ^{e} the elastic Lamé constants. $\chi_1(t)$ and $\chi_2(t)$ are adimensional stress relaxation kernels in dilatation and shear, respectively. This is proved in Appendix A where also some expressions for these kernels are given. Applying the convolutional theorem to equation (1), the rheological relation takes the form

$$\tilde{T}_I = \check{\dot{\psi}}_{IJ}\tilde{S}_J \equiv p_{IJ}\tilde{S}_J, \tag{6}$$

where the tilde means time Fourier transform. Equation (6) defines the frequency-domain complex stiffness matrix as

$$p_{IJ}(\omega) = \check{\dot{\psi}}_{IJ} \equiv r_{IJ}(\omega) + iq_{IJ}(\omega), \tag{7}$$

with ω the angular frequency. In matrix notation equation (7) is

$$\mathbf{P} = \mathbf{R} + i\mathbf{Q}, \tag{8}$$

where

$$\mathbf{R} = \mathcal{R}e(\mathbf{P}), \qquad \mathbf{Q} = \mathcal{I}m(\mathbf{P}) \tag{9a, b}$$

are real matrices. The operators $\mathcal{R}e$ and $\mathcal{I}m$ take real and imaginary parts, respectively.

The complex stiffness components are from (4) and (7),

$$p_{IJ} = \begin{cases} \lambda + 2\mu\delta_{IJ}, & I, J \leq 3, \\ \mu\delta_{IJ}, & I, J > 3, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

where

$$\lambda = (\lambda^{\mathrm{e}} + \tfrac{2}{3}\mu^{\mathrm{e}})M_1 - \tfrac{2}{3}\mu^{\mathrm{e}}M_2, \qquad \mu = \mu^{\mathrm{e}}M_2, \tag{11a, b}$$

are the complex Lamé constants, with $M_\nu = \check{\dot{\chi}}_\nu$, $\nu = 1, 2$ the adimensional complex given by equation (A4a).

The equation of motion for a linear anelastic medium is

$$\nabla \cdot \mathbf{T} = \rho\ddot{\mathbf{u}} + \mathbf{f} \quad \text{or} \quad \nabla_{iJ}T_J = \rho\ddot{u}_i + f_i, \tag{12}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the displacement vector, $\mathbf{f}(\mathbf{x}, t)$ is the body forces vector, $\rho(\mathbf{x})$ is the density, and '$\nabla\cdot$' is a divergence operator defined by

$$\nabla\cdot \rightarrow \nabla_{iJ} = \begin{pmatrix} \partial/\partial x & 0 & 0 & 0 & \partial/\partial z & \partial/\partial y \\ 0 & \partial/\partial y & 0 & \partial/\partial z & 0 & \partial/\partial x \\ 0 & 0 & \partial/\partial z & \partial/\partial y & \partial/\partial x & 0 \end{pmatrix}. \tag{13}$$

The strain-displacement relation can be written as

$$\mathbf{S} = \nabla^{\mathrm{T}}\mathbf{u} \quad \text{or} \quad S_K = \nabla_{Kj}u_j. \tag{14}$$

Considering zero body forces and Fourier transforming equation (12) with respect to the time gives

$$(\nabla_{iJ}p_{JK}\nabla_{Kj} + \rho\omega^2\delta_{ij})\tilde{u}_j = 0, \tag{15}$$

where equation (14) has been used. Equation (15) is the frequency-domain equation of motion for a general anisotropic linear viscoelastic medium. The equation for the isotropic solid is obtained by using the complex components (10).

COMPLEX VELOCITY

Since the medium is isotropic, it can be assumed without loss in generality that the wave propagation is in the (x, z) plane with $z = 0$ the free surface. Let a plane wave solution to equation (15) be of the form

$$\mathbf{u} = \mathbf{U}e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}, \tag{16}$$

where $\mathbf{k}$ is the complex wavenumber. Substitution of (16) into the equation of motion (15) gives the solutions for the compressional and shear waves. Let $m = 1$ denote the compressional mode and $m = 2$ the shear mode. Then, the following dispersion relations are obtained:

$$|\mathbf{k}^{(m)}|^2 = \frac{\omega^2}{V_m^2}, \quad m = 1, 2, \qquad V_1^2 = \frac{\lambda + 2\mu}{\rho}, \qquad V_2^2 = \frac{\mu}{\rho}, \tag{17a, b, c}$$

and

$$\mathbf{U}^{(1)} = U_0\mathbf{k}^{(1)}, \qquad \mathbf{U}^{(2)}\cdot\mathbf{k}^{(2)} = 0, \tag{18a, b}$$

with V_1 and V_2 the complex velocities for homogeneous viscoelastic waves, and U_0 a scalar quantity.

A general solution is given by the superposition of the compressional and shear modes,

$$\mathbf{u} = \mathbf{U}^{(m)}e^{i(\omega t - \mathbf{k}^{(m)}\cdot\mathbf{x})}. \tag{19}$$

At the free surface ($z = 0$),

$$T_3 = \lambda S_1 + (\lambda + 2\mu)S_3 = 0, \qquad T_5 = \mu S_5 = 0. \tag{20a, b}$$

These boundary conditions imply the horizontal wavenumber to be the same for each mode,

$$k_x^{(1)} = k_x^{(2)} \equiv k \equiv \kappa - i\alpha. \tag{21}$$

Denoting for simplicity

$$k_z^{(m)} \equiv k_m \equiv \kappa_m - i\alpha_m, \qquad m = 1, 2, \tag{22}$$

the displacement components become

$$u_x = F(z)e^{i(\omega t - kx)}, \qquad F(z) = U_x^{(m)}e^{-ik_m z}, \tag{23a, b}$$

$$u_z = G(z)e^{i(\omega t - kx)}, \qquad G(z) = U_z^{(m)}e^{-ik_m z}. \tag{24a, b}$$

From (17a) and (18a, b),

$$k_m^2 = \frac{\omega^2}{V_m^2} - k^2 = \omega^2\left(\frac{1}{V_m^2} - \frac{1}{V^2}\right), \qquad m = 1, 2 \tag{25}$$

$$\frac{U_z^{(1)}}{U_x^{(1)}} = \frac{k_1}{k} = \left(\frac{V^2}{V_1^2} - 1\right)^{1/2}, \qquad \frac{U_z^{(2)}}{U_x^{(2)}} = -\frac{k}{k_2} = -\left(\frac{V^2}{V_2^2} - 1\right)^{-1/2}, \tag{26a, b}$$

where

$$V = \omega/k \tag{27}$$

is the Rayleigh wave complex velocity in the x direction. The boundary conditions (20a, b) and equations (26a, b) imply that

$$\frac{U_x^{(2)}}{U_x^{(1)}} = \frac{V^2}{2V_2^2} - 1 \equiv A, \tag{28}$$

and

$$A^2 + \frac{k_1 k_2}{k^2} = 0. \tag{29}$$

Squaring (29) and reordering terms gives a cubic equation for the complex velocity,

$$q^3 - 8q^2 + \left(24 - 16\frac{V_2^2}{V_1^2}\right)q - 16\left(1 - \frac{V_2^2}{V_1^2}\right) = 0, \qquad q = \frac{V^2}{V_2^2}. \tag{30}$$

This equation together with (29) may determine one or more wave systems. One mode, the q.e. surface wave, is always possible since it is the equivalent of the elastic Rayleigh wave. The other surface waves, called v.e. modes, are possible depending on the frequency and the material parameters.

DISPLACEMENT FIELD

The amplitude coefficients may be referred to $U_x^{(1)} = 1$ without loss in generality. Thus, from (26a, b) and (28)

$$U_x^{(1)} = 1, \qquad U_x^{(2)} = A, \qquad U_z^{(1)} = \frac{k_1}{k}, \qquad U_z^{(2)} = \frac{k_1}{kA}. \tag{31a, b, c, d}$$

From (31a, b, c, d), the displacements (23a) and (24a) become

$$u_x = (e^{-ik_1 z} + Ae^{-ik_2 z})e^{i(\omega t - kx)}, \tag{32a}$$

$$u_z = \frac{k_1}{k}\left(e^{-ik_1 z} + \frac{e^{-ik_2 z}}{A}\right)e^{i(\omega t - kx)}. \tag{32b}$$

The displacements (32a, b) are a combination of compressional and shear modes containing the following phase and decay factors:

$$e^{i[\omega t - (\kappa x + \kappa_m z)]}e^{-(\alpha x + \alpha_m z)}, \qquad m = 1, 2, \tag{33}$$

in virtue of equations (21) and (22). It is clear from (33) that in order to have attenuating waves, a physical solution of equation (30) must satisfy the following conditions:

$$\alpha > 0, \qquad \alpha_1 > 0, \qquad \alpha_2 > 0, \qquad \kappa > 0. \tag{34a, b, c, d}$$

The last one imposes wave propagation along the positive x direction. In terms of the complex velocities, these conditions read

$$-\omega\mathcal{I}m\left(\frac{1}{V}\right) > 0, \qquad -\omega^2\mathcal{I}m\left[\left(\frac{1}{V_m^2} - \frac{1}{V^2}\right)^{1/2}\right] > 0,$$

$$m = 1, 2, \qquad \omega\mathcal{R}e\left(\frac{1}{V}\right) > 0. \tag{35a, b, c}$$

Also, equation (29) must be verified to avoid spurious roots.

PHASE VELOCITIES

The phase velocity in the x direction is defined as the frequency divided by the real wavenumber κ,

$$c \equiv \frac{\omega}{\kappa} = \frac{\omega}{\mathcal{Re}\,(k)} = \mathcal{Re}^{-1}\left(\frac{1}{V}\right). \tag{36}$$

From (33), the phase velocities associated with each component wave mode are

$$\mathbf{c}_m = \omega \frac{\kappa\hat{\mathbf{x}} + \kappa_m\hat{\mathbf{z}}}{\kappa^2 + \kappa_m^2}, \qquad m = 1, 2, \qquad \mathbf{c}_m = \frac{\omega}{\kappa}\hat{\mathbf{x}} \ \text{(elastic case)}. \tag{37a, b}$$

In the elastic case, there is only a single and real physical solution to equation (30). Moreover, since $V < V_2 < V_1$, and they are real quantities, k_1 and k_2 are both purely imaginary, and, $\kappa_m = 0$. Hence, $c_m = c$, and (33) reduces to

$$e^{i(\omega t - \kappa x)} e^{-\alpha_m z}, \tag{38}$$

with $\kappa = k$ and $\alpha_m = ik_m$. In this case, the propagation vector is along the surface, and the attenuation vector is normal to the surface; but in a viscoelastic solid, according to equation (38), these vectors are inclined with respect to the previous directions.

ENERGY BALANCE EQUATION

The complex Poynting's theorem for a general medium is given by (Auld 1973; Carcione 1990)

$$\int_S \mathbf{p} \cdot \hat{e}_n dS - i\omega[(E_s)_{\text{peak}} - (E_v)_{\text{peak}}] + (P_d)_{\text{AV}} = P_s, \tag{39}$$

with $\mathbf{p}$ the complex Poynting vector defined as

$$\mathbf{p} = -\tfrac{1}{2}\mathbf{v}^* \cdot \mathbf{T}, \qquad \mathbf{v} = (v_x, v_z) = \dot{\mathbf{u}}, \tag{40}$$

where the superscript '*' denotes complex conjugate. The real part of the Poynting vector gives the average power flow density over a cycle. The surface integral in (39) is the total power flow outward in the direction of $\hat{e}_n$, through a closed surface S which includes a volume V. The quantities

$$(E_s)_{\text{peak}} = \int_V (\varepsilon_s)_{\text{peak}}\, dV, \qquad (\varepsilon_s)_{\text{peak}} = \tfrac{1}{2}\mathbf{S}:\mathbf{R}:\mathbf{S}^*, \tag{41a, b}$$

and

$$(E_v)_{\text{peak}} = \int_V (\varepsilon_v)_{\text{peak}}\, dV, \qquad (\varepsilon_v)_{\text{peak}} = \tfrac{1}{2}\rho|\mathbf{v}|^2, \tag{42a, b}$$

are the peak strain and peak kinetic total energies, with $(\varepsilon_s)_{\text{peak}}$ and $(\varepsilon_v)_{\text{peak}}$, the respective energy densities. The matrix $\mathbf{R}$ is given by equation (9a). The double dot product ':' is defined by summation over a single abbreviated subscript. For instance, $\mathbf{S}:\mathbf{R}:\mathbf{S}^* \equiv S_I r_{IJ} S_J^*$. The quantity $(P_d)_{\text{AV}}$ is the time-average power loss due to anelasticity,

$$(P_d)_{\text{AV}} = \omega \int_V (\varepsilon_d)_{\text{AV}}\, dV, \qquad (\varepsilon_d)_{\text{AV}} = \tfrac{1}{2}\mathbf{S}:\mathbf{Q}:\mathbf{S}^*, \tag{43a, b}$$

with $(\varepsilon_d)_{\text{AV}}$ the dissipated energy density, and $\mathbf{Q}$ given by equation (9b). Finally, P_s is the complex power supplied by the sources.

The energy balance equation (39) can be expressed in terms of the energy densities. In the absence of sources, application of the Gauss theorem to the first term yields

$$\nabla \cdot \mathbf{p} - i\omega[(\varepsilon_s)_{\text{peak}} - (\varepsilon_v)_{\text{peak}}] + \omega(\varepsilon_d)_{\text{AV}} = 0. \tag{44}$$

In an elastic medium is $(\varepsilon_d)_{\text{AV}} = 0$, and since in the absence of sources the net flow into, or out of S must vanish, $\nabla \cdot \mathbf{p} = 0$, giving that the peak kinetic energy equals the peak potential energy.

The average stored energy density is

$$\varepsilon_{\text{AV}} = \frac{(\varepsilon_v)_{\text{peak}} + (\varepsilon_s)_{\text{peak}}}{2} = \tfrac{1}{4}(\rho|\mathbf{v}|^2 + \mathbf{S}:\mathbf{R}:\mathbf{S}^*). \tag{45}$$

The Poynting vector and energy densities are derived in Appendix B. The calculation is done as a function of depth and per unit surface area.

ENERGY VELOCITY

The energy velocity is defined as the ratio of the average power flow density to the mean energy density (45). The average power flow density is the real part of the complex Poynting vector. Hence,

$$\mathbf{c}_e = \frac{2\mathcal{Re}\,(\mathbf{p})}{(\varepsilon_v)_{\text{peak}} + (\varepsilon_s)_{\text{peak}}}. \tag{46}$$

The energy velocity per unit area of free surface is given by (46) through integration over z of equations (B3a, b), (B4) and (B5):

$$\bar{\mathbf{c}}_e = \frac{2\displaystyle\int_0^\infty \mathcal{Re}\,(\mathbf{p})\, dz}{\displaystyle\int_0^\infty [(\varepsilon_v)_{\text{peak}} + (\varepsilon_s)_{\text{peak}}]\, dz}. \tag{47}$$

Substitution of equations (B3a, b), (B4) and (B5) into (46) gives the z-dependent energy velocity. At the free surface the formula simplifies to give

$$\mathbf{c}_e = 2|V|^2\,\mathcal{Re}\left(\frac{V_0^2}{V}\right)\left[1 + \mathcal{Re}\,(V_0^2) + \left|\frac{k_1}{k_2}\right|\right]^{-1}\hat{\mathbf{x}}. \tag{48}$$

By using (B15), the energy velocity results in

$$\mathbf{c}_e = \left[\mathcal{Re}\left(\frac{1}{V}\right) + \tfrac{1}{2}\,\mathcal{Re}^{-1}\left(\frac{V_0^2}{V}\right)\mathcal{Re}\left(\left|\frac{k_1}{k_2}\right| - \frac{k_1}{k_2}\right)\right]^{-1}\hat{\mathbf{x}}, \tag{49}$$

where the property that $\mathcal{Re}\,[b(|a|^2 + a^2)] = 2\mathcal{Re}\,(a)\mathcal{Re}\,(ab)$, with a and b complex numbers, has been used. The second term in the denominator of equation (49) makes the energy velocity at the free surface different from the phase velocity (36). This term is zero for the incompressible and Poisson solids as can be seen in the next sections.

In general, for a viscoelastic solid the energy velocity is not parallel to the surface, unlike in the elastic case where λ, μ, k and A are real quantities, and k_1 and k_2 are purely imaginary, giving $\mathcal{Re}\,(p_z) = 0$ from (B3b). In the elastic case, the energy velocity (as a function of depth and per unit area) equals the phase velocity since there is no energy flux in the vertical direction, and the medium is non-dispersive.

ABSORPTION COEFFICIENT AND QUALITY FACTOR

The absorption coefficient in the x direction is given by

$$\alpha = -\omega \mathcal{I}m \left(\frac{1}{V}\right). \tag{50}$$

Each wave mode has an attenuation vector given by

$\alpha\hat{\mathbf{x}} + \alpha_m \hat{\mathbf{z}}$, $m = 1, 2$, $\alpha_m \hat{\mathbf{z}}$, $m = 1, 2$ (elastic case).

The quality factor is defined as the ratio of the peak strain energy density (41b) to the loss in energy density due to anelasticity (43b). Then

$$Q = \frac{(\varepsilon_s)_{\text{peak}}}{(\varepsilon_d)_{\text{AV}}} = \frac{\mathbf{S}:\mathbf{R}:\mathbf{S}^*}{\mathbf{S}:\mathbf{Q}:\mathbf{S}^*}. \tag{51}$$

As with the energy velocity, a quality factor per unit surface can be calculated by using equations (B4) and (B6),

$$\bar{Q} \equiv \frac{\int_0^\infty \mathbf{S}:\mathbf{R}:\mathbf{S}^* \, dz}{\int_0^\infty \mathbf{S}:\mathbf{Q}:\mathbf{S}^* \, dz} = \frac{\mathcal{R}e\,(W)}{\mathcal{I}m\,(W)}, \tag{52}$$

with W given by equation (B19). At the surface the quality factor takes the simple form

$$Q = \frac{\mathcal{R}e\,(V_0^2)}{\mathcal{I}m\,(V_0^2)}, \tag{53}$$

with V_0 a reference velocity defined in (B14). Unlike with body waves for which (e.g., Carcione, Kosloff & Kosloff 1988)

$$Q_m = \frac{\mathcal{R}e\,(V_m^2)}{\mathcal{I}m\,(V_m^2)}, \qquad m = 1, 2, \tag{54}$$

the quality factor for Rayleigh waves is not of the form $\mathcal{R}e\,(V^2)/\mathcal{I}m\,(V^2)$. This fact precludes the use of the body wave relation between absorption coefficient and quality factor, $\alpha = \kappa[(Q^2+1)^{1/2} - Q]$, for Rayleigh waves. Replacing (B15) into equation (53) gives

$$Q = \frac{\mathcal{R}e\,[V^2(1 + k_1/k_2)]}{\mathcal{I}m\,[V^2(1 + k_1/k_2)]}. \tag{55}$$

For k_1/k_2 real, the quality factor takes the body wave form. This is the case for the incompressible and Poisson solids. However, for k_1/k_2 complex, the absorption coefficient α can be related to the quality factor by substituting Q by $(QR + I)/(R - QI)$ in the previous formula relating α and Q, with $R = \mathcal{R}e\,(1 + k_1/k_2)$ and $I = \mathcal{I}m\,(k_1/k_2)$. Strictly, not even the quality factor per unit area $\bar{Q}$ can be related to α through the body wave relation previously mentioned. However, as seen in the example, for practical purposes the relation can be used with confidence.

At the limit $z \to \infty$, it can be shown that the quality factor is very close to Q_1 if $\alpha_1 < \alpha_2$, and gives exactly Q_2 if $\alpha_1 > \alpha_2$.

SPECIAL VISCOELASTIC SOLIDS

Incompressible solid

Incompressibility implies that $\lambda \to \infty$, or, equivalently, $V_1 \to \infty$. Hence, from (30), the complex velocity satisfies the following cubic equation:

$$q^3 - 8q^2 + 24q - 16 = 0, \qquad q = \frac{V^2}{V_2^2}. \tag{56}$$

The roots are $q_1 = 3.5437 + 2.2303i$, $q_2 = 3.5437 - 2.2303i$ and $q_3 = 0.9126$. As shown by Currie *et al.* (1977), two Rayleigh waves are possible, the quasi-elastic mode represented by q_3, and the viscoelastic mode, represented by q_1, which is admissible if $\mathcal{I}m\,(V_2^2)/\mathcal{R}e\,(V_2^2) > 0.159$, in order to fulfill conditions (34a, b, c, d). In Currie *et al.* (1977) the viscoelastic root is given by q_2 since they use the opposite sign convention to compute the time-Fourier transform (see also Currie 1979). In this sense, a correction has to be made in Christensen (1982, p. 226) where the correct root should be q_1.

The reference velocity V_0 (equation B14) becomes in this case $V_0 = 2V_2$, and $k_1 = -ik$ according to (34d). For the quasi-elastic mode, $k_1/k_2 = (1 - q_3)^{-1/2}$ is real, and from (49), the energy velocity equals the phase velocity at the free surface. For the viscoelastic mode the energy velocity becomes

$$\mathbf{c}_e = \left[\mathcal{R}e\left(\frac{1}{V}\right) + \tfrac{1}{2}\mathcal{R}e^{-1}\left(\frac{V_2^2}{V}\right)\mathcal{R}e\left(\left|\frac{k}{k_2}\right| + i\frac{k}{k_2}\right)\right]^{-1}\hat{\mathbf{x}}. \tag{57}$$

The quality factor at the free surface (equation 53) becomes the quality factor of shear waves:

$$Q = \frac{\mathcal{R}e\,(V_2^2)}{\mathcal{I}m\,(V_2^2)}. \tag{58}$$

To obtain the z-dependent Poynting vector and energy densities, it is necessary to analyse in detail the form of the constitutive equation. This is done in Appendix B. The energy velocities can be calculated by using equations (B5), (B7a, b) and (B8), and the quality factor by using (B8) and (B9). The quality factors calculated per unit volume (equation 52), and per unit area become simply Q_2.

Poisson solid

A Poisson solid has $\lambda = \mu$, so that $V_1 = \sqrt{3}\,V_2$, and equation (30) becomes

$$3q^3 - 24q^2 + 56q - 32 = 0, \qquad q = \frac{V^2}{V_2^2}. \tag{59}$$

This equation has three real roots: $q_1 = 4$, $q_2 = 2 + 2/\sqrt{3}$, and $q_3 = 2 - 3/\sqrt{3}$. The last root corresponds to the q.e. mode. The other two roots do not satisfy equations (29), and therefore, there are no v.e. modes in a Poisson solid. The Rayleigh wave satisfies $k_1/k_2 = [(1 - q_3/3)/(1 - q_3)]^{1/2}$, a real number. Therefore, as with the incompressible solid, the surface energy velocity of the q.e. mode equals the phase velocity. Similarly, the quality factors per unit volume and unit area are those of shear body waves, as in equation (58).

Hardtwig solid

Hardtwig (1943) studied the properties of a viscoelastic Rayleigh wave for which $\mathcal{R}e\,(\lambda)/\mathcal{R}e\,(\mu) = \mathcal{I}m\,(\lambda)/\mathcal{I}m\,(\mu)$. From equations (17b, c) and (55), this condition implies that

$Q_1 = Q_2$, i.e. for this type of solid the compressional wave and the shear wave have similar anelastic characteristics. Therefore, equation (51) together with (B4) and (B6) shows that the quality factor of the Rayleigh wave is that of the body waves. Moreover, the coefficients of equation (30) are real, ensuring at least one real root corresponding to the q.e. mode. As before, this implies that the surface energy velocity coincides with the phase velocity. A Poisson medium is a particular type of Hardtwig solid.

EXAMPLE

The viscoelastic medium is defined by: (i) the low-frequency limit Lamé constants, $\lambda^e = 8$ GPa, $\mu^e = 4.5$ GPa, and density $\rho = 2000\ \mathrm{kg\,m^{-3}}$, which give compressional and shear velocities of 2000 and $1500\ \mathrm{m\,s^{-1}}$, respectively; and (ii)

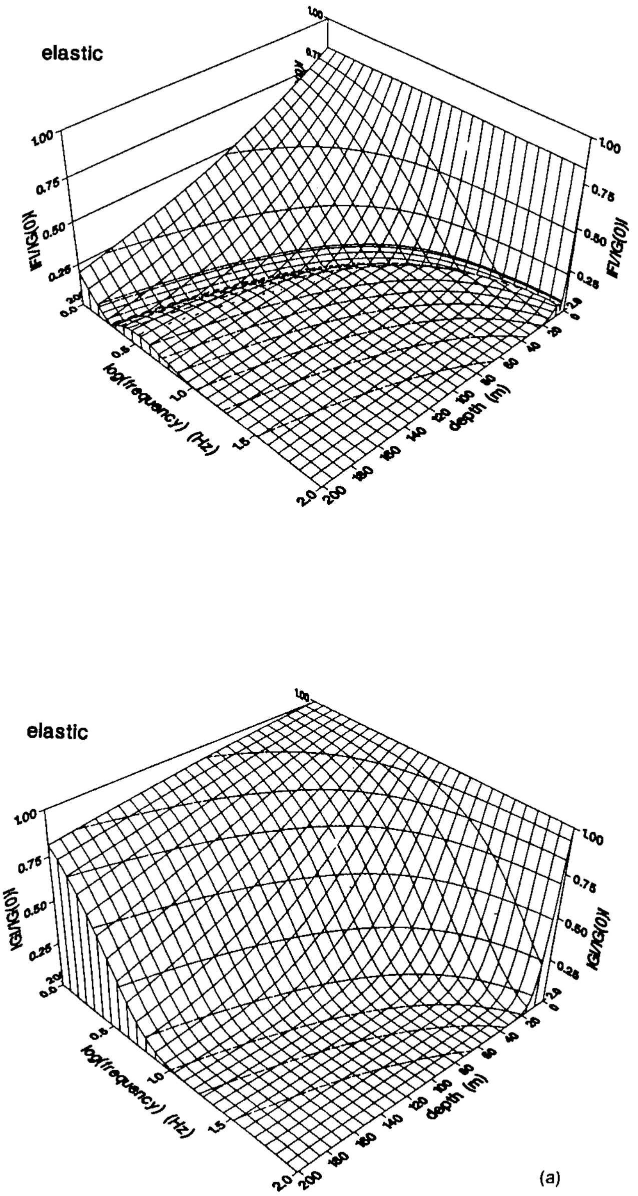

Figure 1. Normalized displacements components for (a) elastic case, (b) q.e. mode, and (c) v.e. mode. The normalization factor is $|G(z = 0\ \mathrm{m}, f = 20\ \mathrm{Hz})|$.

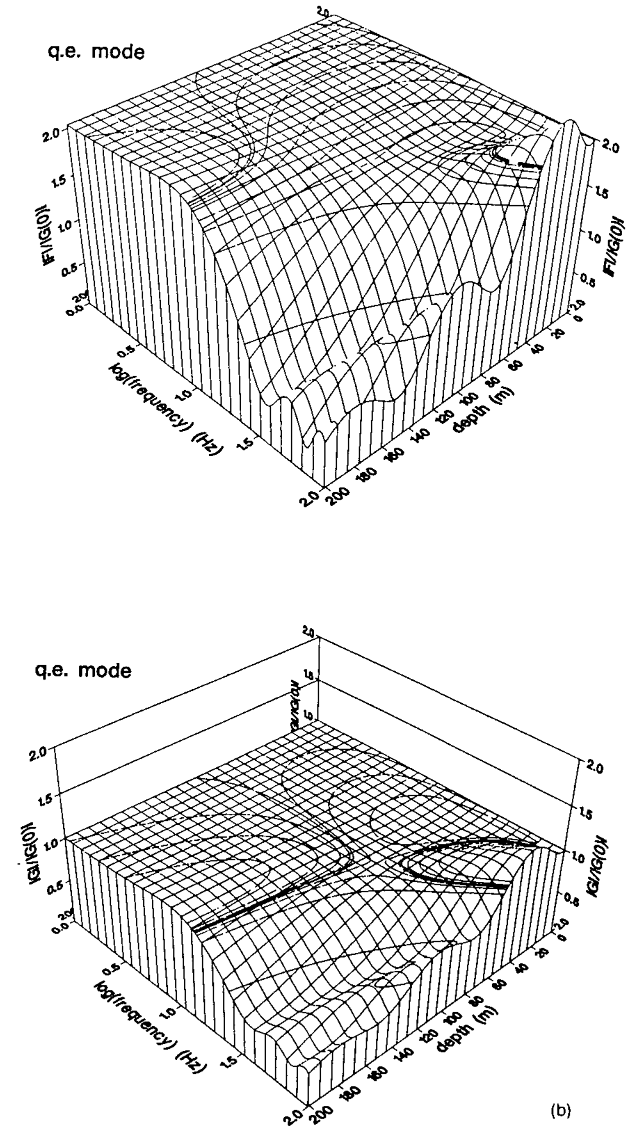

Figure 1. *(continued)*

complex moduli of the type given by equation (A4a) with $L = 1$, such that the dilatational modulus M_1 is defined by $\theta^{(1)} = 0.008$ s and $\tau^{(1)} = 0.007$ s, and the shear modulus M_2 is defined by $\theta^{(2)} = 0.008$ s and $\tau^{(2)} = 0.0065$ s. These are standard linear solid elements with one dissipation mechanism each. They give minima in the body wave quality factors at $f_0 \simeq 20$ Hz, with $Q_1(f_0) \simeq 15$ and $Q_2(f_0) \simeq 9.6$. Two roots satisfy equations (29) and (30); $q_1 = 0.7109 - i0.0046$ is the q.e. mode, and $q_2 = 1.7640 - i0.0156$ is the v.e. mode. It can be seen that these waves satisfy the energy balance equation (44). The v.e. mode disappears if the shear velocity is taken less than $1407\ \mathrm{m\,s^{-1}}$.

In the following, all the 3-D graphics display the physical variables as a function of depth and frequency, with the surface represented by orthogonal grid lines and isolines. The normalized displacements are plotted in Fig. 1: (a) elastic case, (b) q.e. mode and (c) v.e. mode. The graphs represent equations (23b) and (24b) normalized by $|G(z = 0\ \mathrm{m}, f = 20\ \mathrm{Hz})|$. The behaviour of the q.e. mode

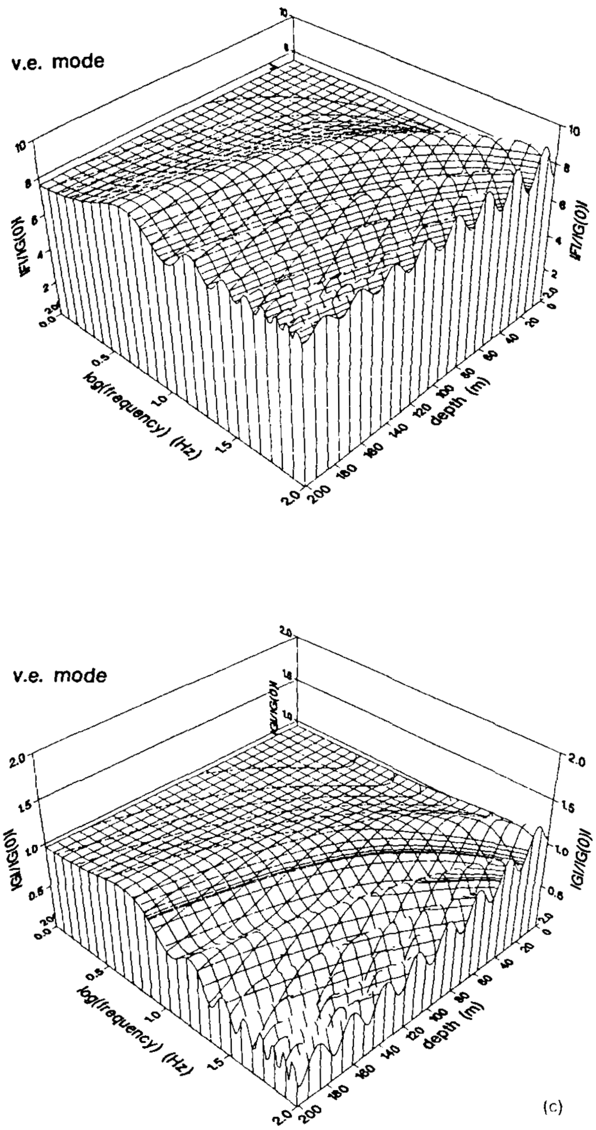

Figure 1. *(continued)*

compared to the elastic wave is quite different. The elastic horizontal displacement vanishes at a depth of 0.192Λ, where $\Lambda = 2\pi/\mathcal{Re}(k)$ is the wavelength (e.g. Pilant 1979), indicating that the motion changes sense at that level. This is not the case for the q.e. wave. Moreover, this mode shows an oscillating behaviour and a less pronounced decay with depth. These effects are stronger in the v.e. mode.

Figure 2 displays (a) peak potential energy density, (b) peak kinetic energy density, and (c) average energy loss, for the two wave modes. A normalization factor $\rho\omega^2|U_x^{(1)}|^2/8$ is applied to study particularly the dependence with depth. As can be seen, for the q.e. mode, the energy is confined to the surface and, relatively in the low frequencies compared to the v.e. mode. For this wave, the energy is not confined to the free surface showing an oscillating behaviour with depth and at high frequencies. The energy loss peaks at $f_0 \simeq 20$ Hz [$\log(f_0) \simeq 1.3$] in both cases, with a maximum at certain depth for the v.e. mode. However, the real measure of energy dissipation is given by the quality factors, whose inverse is the energy loss per unit of potential energy. They are represented in Fig. 3. The minimum is in both cases at $f_0 \simeq 20$ Hz, and both modes have quite similar values at the surface, with the q.e. wave presenting more attenuation with depth.

The energy flux in an elastic medium is in the horizontal direction since $\mathcal{Re}(p_z) = 0$. In the viscoelastic case, this is true only at the surface, to satisfy the boundary conditions, but not in depth. The energy velocity components are plotted in Fig. 4: (a) q.e. mode and (b) v.e. mode. They show dispersion with remarkably higher velocity values for the v.e. mode. The horizontal velocities do not show appreciable variations with depth, particularly the q.e. mode. On the contrary, the vertical component shows a maximum at maximum attenuation for the q.e. mode, and a

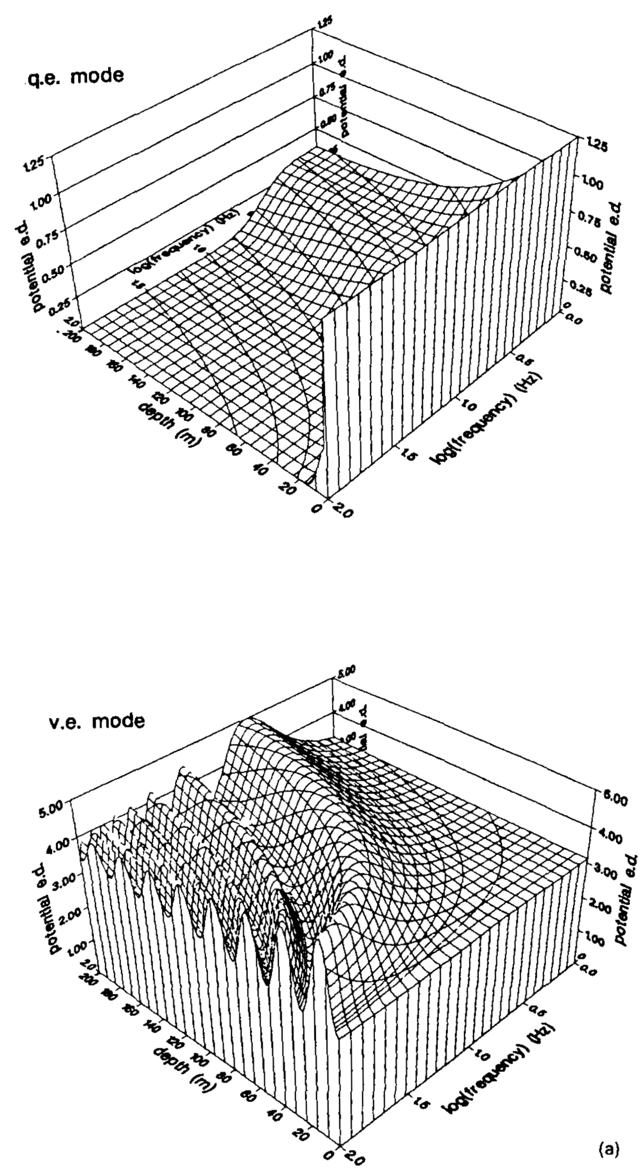

Figure 2. Normalized energy densities for the q.e. and v.e. modes, where (a) is peak potential, (b) peak kinetic, and (c) average loss. The normalization factor is $\rho\omega^2|U_x^{(1)}|^2/8$.

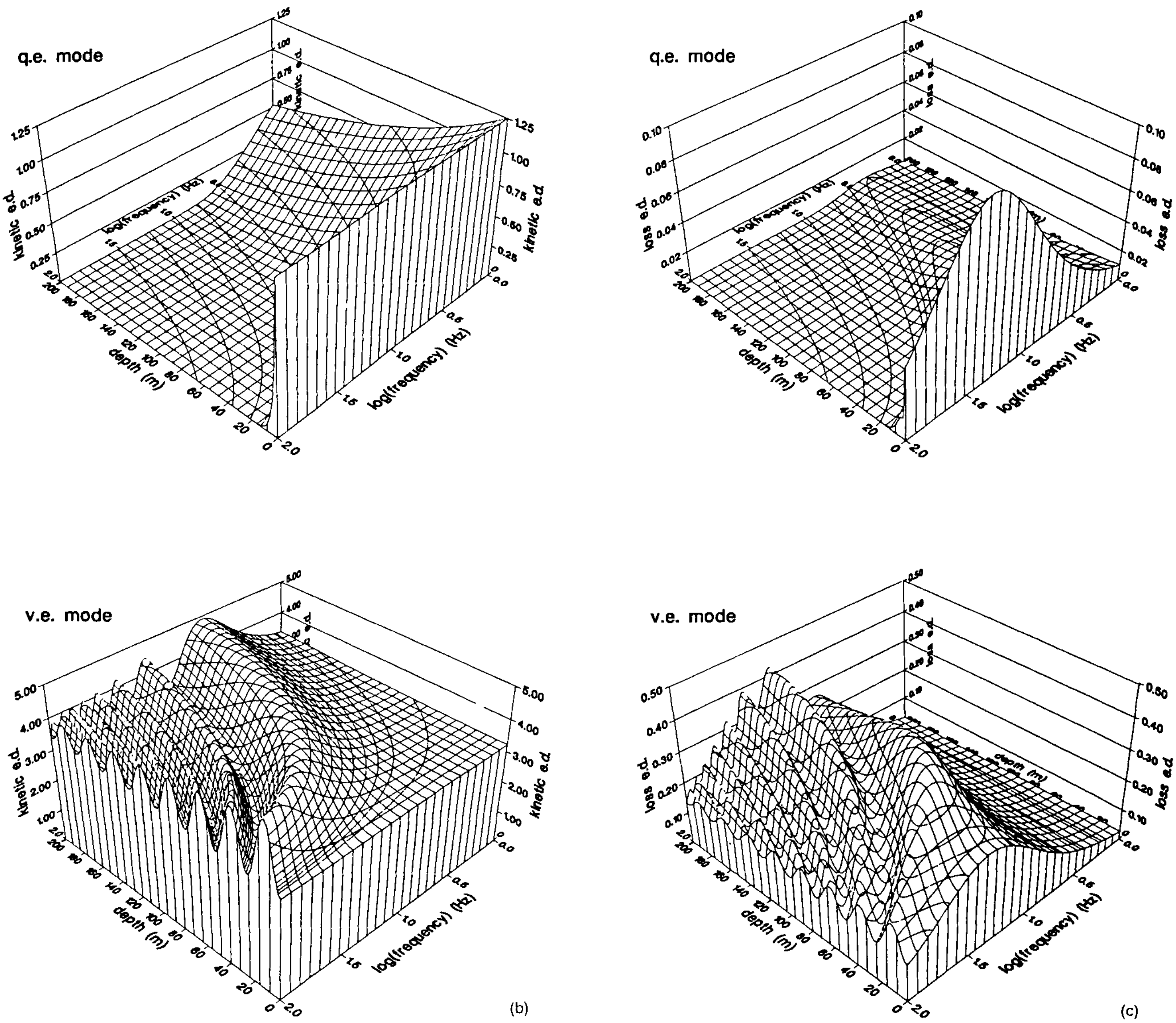

Figure 2. *(continued)*

component towards the surface at high frequencies for the v.e. mode.

Figure 5 displays the phase velocities c and the horizontal components of the energy velocities per unit area $(\bar{c}_e)_x$. Both velocities are very close to each other; for practical purposes they are the same. The v.e. mode presents the most interesting characteristics; phase and horizontal energy velocities closely follow the dispersion curve of the compressional body wave, $c_P = \omega / \mathcal{R}e\,(k_1^2 + k^2)^{1/2}$, with $(\bar{c}_e)_x < c_P < c$. Moreover, the phase velocities associated with each wave component (equation 37a), whose moduli are $c_m = \omega / [\mathcal{R}e^2\,(k_m) + \mathcal{R}e^2\,(k)]$, follow the respective dispersion curves of the body waves. On the other hand, the q.e. mode shows a completely different behaviour. All the velocities involved have values that are less than the body wave velocities with $c < (\bar{c}_e)_x$. This mode has the expected characteristics that one should expect from an anelastic Rayleigh wave. Fig. 6 represents the quality factors per unit area compared to the body wave quality factors. In principle, one may consider that the Rayleigh wave quality factor takes the body wave form $Q_b = \mathcal{R}e\,(V^2) / \mathcal{I}m\,(V^2)$. For the q.e. mode, $\bar{Q} \simeq Q_b$, and the low- and high-frequency limits, as well as for low-loss solids, $\bar{Q} \rightarrow Q_b$; while for the v.e. mode, $\bar{Q} \simeq Q_1 \approx Q_b$, regardless of the degree of anelasticity. Since Q_b is easier to compute than $\bar{Q}$, for practical purposes, it can be used as the Rayleigh wave quality factor.

CONCLUSIONS

Looking at anelastic Rayleigh wave propagation from the standpoint of energy gives new insights into the physical processes involved. The model analysed in the example considers two dissipation mechanisms, one for the compressional wave and one for the shear wave. As is well known, the anelastic characteristics of a general medium can be described by a set of such relaxation mechanisms. The media has fairly low wave velocities and relatively high

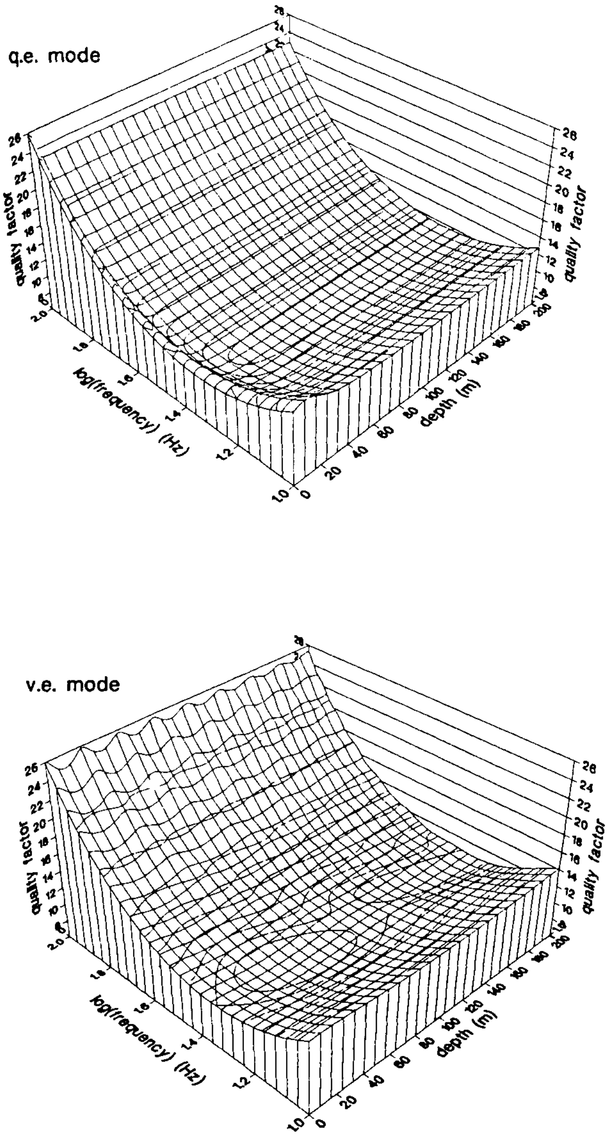

Figure 3. Quality factors for the q.e. and v.e. modes.

dissipation, and may represent, for instance, an unconsolidated weathering zone containing the water table and very soft sediments.

The analysis shows that, in contrast to elastic materials, the energy flow is not along the surface, and that the energy velocity is not equal to the phase velocity. However, the horizontal energy velocity per unit area is very close to the phase velocity. Strictly, neither the quality factor at the surface nor the quality factor per unit area can be related to the attenuation factor with the usual body wave relation. However, for practical purposes, the relation can be used with confidence, i.e. the quality factor of the Rayleigh waves can be calculated as $\mathcal{R}e\,(V^2)/\mathcal{I}m\,(V^2)$, with V the complex velocity. At increasing depths the quality factors approach the P or S quality factors depending on the value of the body wave attenuations.

Furthermore, the decay with depth of the displacements is less pronounced and oscillating compared to the elastic case, with no changes in the sense of the motion; the energy of the v.e. mode is distributed in depth and high frequencies, while for the q.e. mode the opposite effect takes place. Also, the anelastic properties of the v.e. mode follows closely the characteristics of the compressional body wave. For incompressible, Poisson and Hardtwig solids, the phase and energy velocities coincide at the surface. Moreover, the quality factor at the surface and per unit area equals the quality factor of shear body waves.

Further research requires the verification of the existence of the v.e. mode through numerical forward modelling. Modelling would also be useful to analyse the physical properties of anelastic Rayleigh waves, particularly their relative amplitudes in seismological applications. Extension to the anisotropic case will also be investigated in future work.

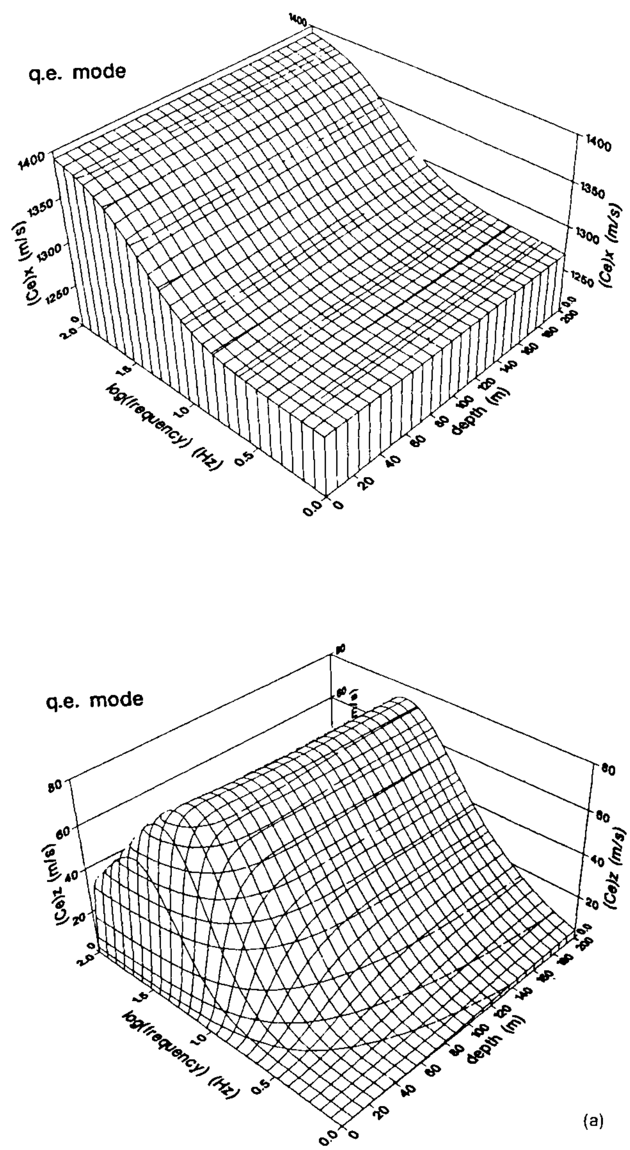

Figure 4. Energy velocity components, where (a) is q.e. mode, and (b) v.e. mode.

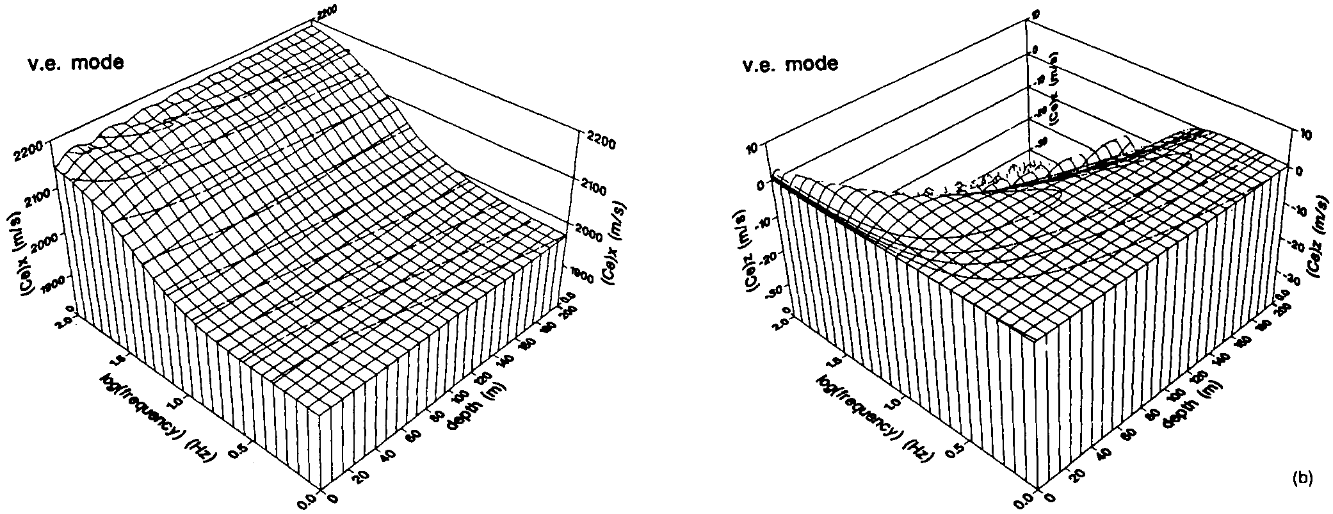

Figure 4. *(continued)*

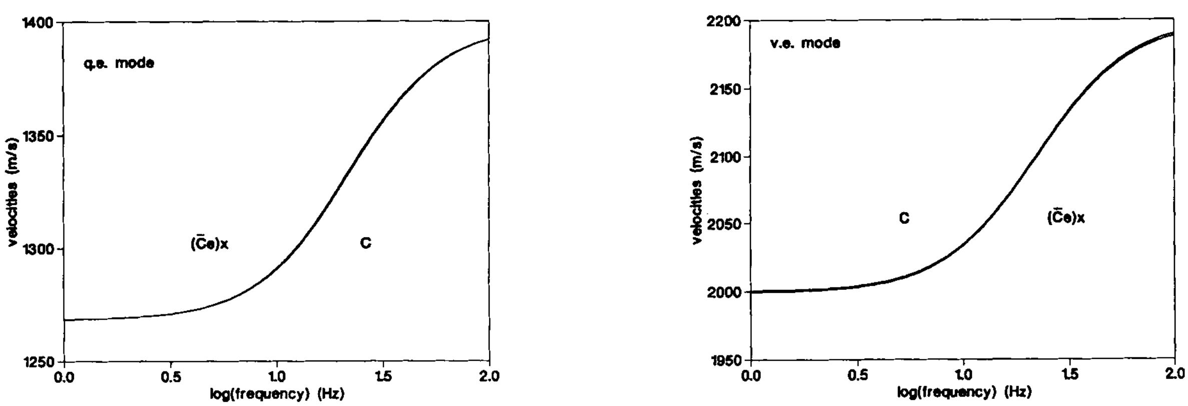

Figure 5. Horizontal energy velocities per unit area, and phase velocities for the q.e. and v.e. modes.

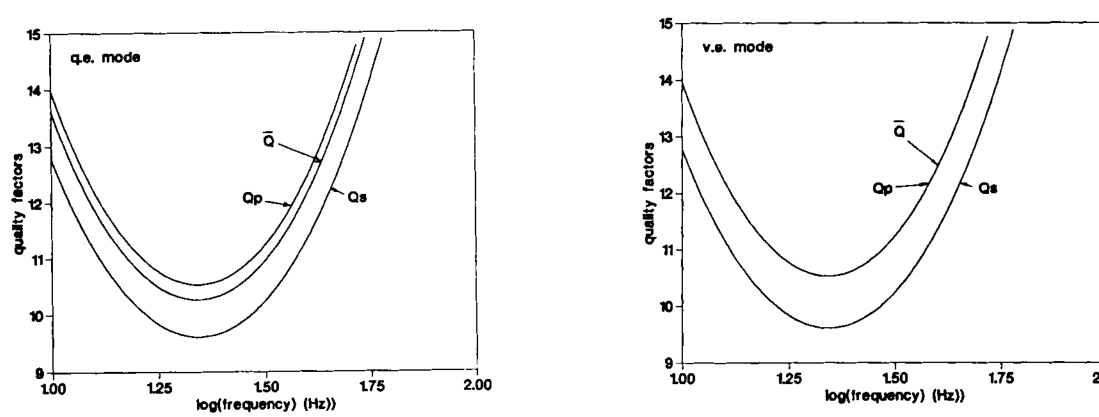

Figure 6. Quality factors per unit area compared to the body wave quality factors for the q.e. and v.e. modes.

ACKNOWLEDGMENTS

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APPENDIX A

Relaxation functions and complex moduli

Defining the mean stress as $\Theta_\sigma = \sum_{l=1}^{3} T_l/3$, and the mean strain as $\Theta_\varepsilon = \sum_{l=1}^{3} S_l/3$, and using equations (1) to (5a, b), the following relations hold:

$$\Theta_\sigma = 3(\lambda^e + \tfrac{2}{3}\mu^e)\dot{\Theta}_\varepsilon * \chi_1, \tag{A1}$$

and the deviatoric stress components,

$$T_l - \Theta_\sigma = 2\mu^e(\dot{S}_l - \dot{\Theta}_\varepsilon) * \chi_2, \qquad l \le 3, \tag{A2a}$$

$$T_l = \mu^e \chi_2 * \dot{S}_l, \qquad l > 3, \tag{A2b}$$

i.e., χ_1 and χ_2 are relaxation functions in dilatation and shear, respectively.

A relaxation function appropriate for numerical wave field calculations in the time-domain can be expressed as (Carcione 1990)

$$\chi(t) = \left(A_R + \sum_{l=1}^{L} A_l e^{-t/\tau_l}\right) H(t), \tag{A3}$$

with A_l, A_R, and τ_l, space-dependent functions, and $H(t)$ the step function. Fourier transforming the time derivative of the relaxation function gives the complex modulus, which can be written as

$$M(\omega) = \tilde{\dot{\chi}}(t) = \frac{A_R}{L}\sum_{l=1}^{L}\frac{1 + i\omega\theta_l}{1 + i\omega\tau_l}, \qquad \theta_l = \left(1 + L\frac{A_l}{A_R}\right)\tau_l,$$

$$A_l \quad = \frac{A_R}{L}\left(\frac{\theta_l}{\tau_l} - 1\right), \tag{A4a, b, c}$$

with ω the angular frequency. θ_l and τ_l are relaxation times, and the tilde indicates time Fourier transformation. For the relaxation matrix (4), $A_R = 1$, in order to obtain the elastic Lamé constants at the low-frequency limit in equations (5a, b).

Equation (A4a) is the expression of a general rational function in the variable $i\omega$. As special cases, the general standard linear solid, and the generalized Maxwell body can be represented by this complex modulus.

A parallel connection of L single standard linear elements, each with constants M_R/L, $\tau_{\varepsilon l}$ and $\tau_{\sigma l}$, $l = 1, \ldots, L$, has a complex modulus of the form (A4a), with $A_R = M_R$, $\tau_l = \tau_{\sigma l}$, and $\theta_l = \tau_{\varepsilon l}$. Similarly, a parallel connection of L Maxwell elements, each with constants k_1 and τ_l, $l = 1, \ldots, L$, plus a spring of constant M_R, gives $A_R = M_R$, $\tau_l \equiv \tau_l$, and $\theta_l = (1 + Lk_l/M_R)\tau_l$. Expressions for the complex moduli and relaxation functions of single standard linear and Maxwell elements can be found, for instance in Ben-Menahem & Singh (1981). For $\omega \to 0$ in (A4a), or $t \to \infty$ in (A3), $M(0) \to \psi(\infty) \to A_R$, the relaxed modulus associated with the long-term behaviour of the system. For $\omega \to \infty$, or $t \to 0$,

$$M(\infty) \to \psi(0) \to A_u \equiv A_R + \sum_{l=1}^{L} A_l = \frac{A_R}{L}\sum_{l=1}^{L}\frac{\theta_l}{\tau_l},$$

the unrelaxed modulus, which characterizes the instantaneous response.

A useful complex modulus for obtaining constant quality factors over a wide range of frequencies is given by a continuous distribution of relaxation mechanisms based on the standard linear solid. The complex modulus obtained by Liu, Anderson & Kanamori (1976), can be written as

$$M(\omega) = M \ln^{-1}\left[e\left(\frac{1 + i\omega\tau_2}{1 + i\omega\tau_1}\right)^{2/\pi\bar{Q}}\right], \tag{A5}$$

where τ_1 and τ_2 are time constants, and $\bar{Q}$ defines the value of the quality factor which remains nearly constant over the selected frequency range. In (A5), e denotes the napierian number. For the relaxation matrix (4) one should take $M = 1$.

APPENDIX B

Poynting vector and energy densities

Substitution of equations (31a, b, c, d) into (23b) and (24b) gives the z-dependent coefficients of the displacements (23a) and (24a),

$$F = e^{-ik_1z} + Ae^{-ik_2z}, \qquad G = \frac{k_1}{k}\left(e^{-ik_1z} + \frac{e^{-ik_2z}}{A}\right) \qquad \text{(B1a, b)}$$

$$F' = -i(k_1e^{-ik_1z} + k_2Ae^{-ik_2z}), \qquad \text{(B1c)}$$

$$G' = \frac{-ik_1}{k}\left(k_1e^{-ik_1z} + \frac{k_2}{A}e^{-ik_2z}\right), \qquad \text{(B1d)}$$

where the prime denotes derivative with respect to z.

The Poynting vector components are obtained from substitution of the stresses into equation (40). It gives

$$p_x = -\tfrac{1}{2}\{v_x^*[(\lambda + 2\mu)S_1 + \lambda S_3] + v_z^*\mu S_5\}, \qquad \text{(B2a)}$$

$$p_z = -\tfrac{1}{2}\{v_x^*\mu S_5 + v_z^*[\lambda S_1 + (\lambda + 2\mu)S_3]\}. \qquad \text{(B2b)}$$

From (14), and in terms of F and G, they result in

$$p_x = \tfrac{1}{2}i\omega e^{-2\alpha x}\{\lambda F^*G' + \mu G^*F' - ik[(\lambda + 2\mu)|F|^2 + \mu|G|^2]\}, \qquad \text{(B3a)}$$

$$p_z = \tfrac{1}{2}i\omega e^{-2\alpha x}\{-ik[\lambda G^*F + \mu F^*G] + (\lambda + 2\mu)G^*G' + \mu F^*F'\}. \qquad \text{(B3b)}$$

The peak potential energy density is obtained from (41b) by replacing the strain components. It yields

$$(\varepsilon_s)_{\text{peak}} = \tfrac{1}{2}e^{-2\alpha x}\mathcal{R}e\,[\lambda|G' - ikF|^2 + \mu(2\,|kF|^2 + 2|G'|^2 + |F' - ikG|^2)]. \qquad \text{(B4)}$$

The peak kinetic energy density can be obtained by substitution of the displacements (23a) and (24a) into (42b),

$$(\varepsilon_v)_{\text{peak}} = \tfrac{1}{2}\rho\omega^2e^{-2\alpha x}(|F|^2 + |G|^2). \qquad \text{(B5)}$$

The average loss energy density (43b) can be calculated in the same way as the potential energy density,

$$(\varepsilon_d)_{\text{AV}} = \tfrac{1}{2}e^{-2\alpha x}\mathcal{I}m\,[\lambda|G' - ikF|^2 + \mu(2|kF|^2 + 2|G'|^2 + |F' - ikG|^2)]. \qquad \text{(B6)}$$

In the incompressible case, $\lambda \to \infty$ and $(S_1 + S_3) \to 0$, such that $\lambda(S_1 + S_3) \to -p$, with p a reactive pressure which depends on the particular boundary conditions of the problem. Then, from (6) the stresses are $T_1 = -p + 2\mu S_1$, $T_3 = -p + 2\mu S_3$, and $T_5 = \mu S_5$. The free surface condition $T_3 = 0$ gives $p = 2\mu S_3$, which can also be extended for all z values. Then, $T_1 = 4\mu S_1$ and $T_3 = 0$, which imply that the Poynting vector components (B2a, b) take the form

$$p_x = \tfrac{1}{2}i\omega e^{-2\alpha x}\mu[G^*F' - ik(4|F|^2 + |G|^2)], \qquad \text{(B7a)}$$

$$p_z = \tfrac{1}{2}i\omega e^{-2\alpha x}\mu F^*(F' - ikG). \qquad \text{(B7b)}$$

The potential energy density is given by (41b) with $\lambda|S_1 + S_3|^2 \to 0$,

$$(\varepsilon_s)_{\text{peak}} = \tfrac{1}{2}e^{-2\alpha x}\mathcal{R}e\,(\mu)(4|kF|^2 + |F' - ikG|^2). \qquad \text{(B8)}$$

Similarly, equation (B6) becomes

$$(\varepsilon_d)_{\text{peak}} = \tfrac{1}{2}e^{-2\alpha x}\mathcal{I}m\,(\mu)(4|kF|^2 + |F' - ikG|^2). \qquad \text{(B9)}$$

At the free surface, equations (B3a, b), (B4), (B5) and (B6) simplify to give

$$p_x = \tfrac{1}{8}\rho\omega^2e^{-2\alpha x}\left|\frac{V^2}{V_2^2}\right|^2\frac{V_0^2}{V}, \qquad p_z = 0, \qquad \text{(B10a, b)}$$

$$(\varepsilon_s)_{\text{peak}} = \tfrac{1}{8}\rho\omega^2e^{-2\alpha x}\left|\frac{V}{V_2^2}\right|^2\mathcal{R}e\,(V_0^2), \qquad \text{(B11)}$$

$$(\varepsilon_v)_{\text{peak}} = \tfrac{1}{8}\rho\omega^2e^{-2\alpha x}\left|\frac{V^2}{V_2^2}\right|^2\left(1 + \left|\frac{k_1}{k_2}\right|\right), \qquad \text{(B12)}$$

$$(\varepsilon_d)_{\text{AV}} = \tfrac{1}{8}\rho\omega^2e^{-2\alpha x}\left|\frac{V}{V_2^2}\right|^2\mathcal{I}m\,(V_0^2), \qquad \text{(B13)}$$

respectively, where

$$V_0^2 = \frac{1}{\rho}\left(\frac{T_1}{S_1}\right)_{z=0} = 4V_2^2\left(1 - \frac{V_2^2}{V_1^2}\right). \qquad \text{(B14)}$$

It is important to point out that the relation

$$V_0^2 = V^2\left(1 + \frac{k_1}{k_2}\right) \qquad \text{(B15)}$$

is equivalent to equation (30). If the medium is elastic, substitution of (B15) into (B12) verifies that the peak kinetic energy equals the peak potential energy.

From equation (44), the energy balance equation requires the calculation of the divergence of the Poynting vector. At the surface one gets

$$\frac{\partial p_x}{\partial x} = -\tfrac{1}{4}\rho\omega^2\alpha e^{-2\alpha x}\left|\frac{V^2}{V_2^2}\right|^2\frac{V_0^2}{V}, \qquad \text{(B16a)}$$

$$\frac{\partial p_z}{\partial z} = -\tfrac{1}{2}i\omega\rho e^{-2\alpha x}\left(\frac{V^*}{V_2^*}\right)^2\frac{V_2^2}{k_2^*}(k_2 - k_1)(|k_1k_2| - k_1k_2^*). \qquad \text{(B16b)}$$

The Poynting vector and energy densities can also be expressed per unit area by integration over the coordinate z. All the formulae have expressions of the form

$$\int_0^\infty (a_1e^{-ik_1z} + a_2e^{-ik_2z})^*(b_1e^{-ik_1z} + b_2e^{-ik_2z})\,dz = a_1^*\left(\frac{b_1}{2\alpha_1} + i\gamma b_2\right) + a_2^*\left(\frac{b_2}{2\alpha_2} - i\gamma^*b_1\right), \qquad \text{(B17)}$$

where $\gamma = (k_1^* - k_2)^{-1}$, and a_1, a_2, b_1 and b_2 are complex numbers. After some algebra, the peak potential energy per unit area is

$$(\bar{\varepsilon}_s)_{\text{peak}} = \tfrac{1}{2}e^{-2\alpha x}\mathcal{R}e\,(W), \qquad \text{(B18)}$$

with

$$W = \frac{\lambda}{2\alpha_1}\left|\frac{\omega^2}{kV_1^2}\right|^2 + \mu\left\{\frac{|k|^2}{\alpha_1}\left(1 + \left|\frac{k_1}{k}\right|^2\right)^2 + \frac{2}{\alpha_2}(|Ak|^2 + |k_1|^2) - 4\mathcal{I}m\left[\gamma Ak\left(k^* - \frac{k_1^{*2}}{k^*}\right) - 2\gamma|k_1|^2\right]\right\}, \qquad \text{(B19)}$$

and the average loss energy per unit area is

$$(\bar{\varepsilon}_d)_{\text{AV}} = \tfrac{1}{2}e^{-2\alpha x}\mathcal{I}m\,(W). \qquad \text{(B20)}$$