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A 3-D time-domain wave equation for viscoacoustic saturated porous media

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ABSTRACT. — The model for porous media introduced by Biot is representative of many materials encountered in geophysical and engineering applications. However, Biot's poroelastic equations have limited practical value since the global flow mechanism does not predict the levels of attenuation and velocity dispersion observed experimentally. In this paper Biot equations on reformulated to model the attenuation values observed in laboratory experiments. Moreover, the wave equation is recasted in the time-domain which, as is well known, allows more efficient wave field computations than in the frequency-domain. Invoking the correspondence principle, Biot formally obtained a viscoelastic equation of motion which includes all possible dissipation mechanisms. The approach involves the presence of convolutional integrals which arise from the replacement of the elastic coefficients by time operators. In this work, the time operators are expressed in terms of suitable relaxation functions in order to avoid the convolutions by the introduction of additional variables in the formulation. The present work is restricted to viscoacoustic wave propagation, *i.e.*, to dilatational waves. The model allows the study of wave propagation due to specific attenuation mechanisms (fitting any general quality factor function) and the analysis of the complete wave field in arbitrary inhomogeneous structures.

1. Introduction

Porous media arise in a variety of geophysical contexts and engineering applications. The mechanics of porous solids with fluid-filled pores has received attention recently for several reasons: the recovery of oil and gas depends upon flow in porous reservoirs; pore fluids in the ground are believed to play a role in the triggering of earthquakes; underwater acoustics involve propagation in the water-saturated bottom of the ocean, etc. The study of wave propagation through porous media can give new insight into and a better understanding of these phenomena. This fact combined with the progress in new algorithms and computer technology suggests a strong need to develop an appropriate wave equation to describe wave propagation in porous media. Most of the work on numerical modeling in viscoelastic porous media was done by using frequency-domain methods, e. g., Schmitt *et al.*, [1988], but nothing is reported on time-domain wavefield simulations. The purpose of this work is to establish a 3-D time-domain wave equation for porous media which correctly describes the level of attenuation measured at typical seismic and well logging frequencies. The base model is given by Biot's poroelastic equations. This theory assumes that the material consists of a connected pore network, with macroscopic viscous fluid flow as the mechanism responsible for the anelasticity. However,

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it is well known that the attenuation values predicted by the theory are much smaller than those measured experimentally [Mochizuki, 1982]. But as Biot [1962] correctly states, the model of a purely elastic matrix saturated with a viscous fluid is only applicable in exceptional cases. Owing to the complexity of the solid-fluid system, there are many types of dissipation mechanisms which can influence the wave propagation characteristics. For instance, non-connected pores, considered as part of the solid, may cause dissipation; the fluid itself contributes due to bulk relaxation; there are interfacial energies produced by chemical reactions; the existence of local flow relaxation mechanisms; thermoelastic dissipation produces a relaxation spectrum, etc. To consider these mechanisms, Biot [1956] formally established a viscoelastic equation of motion for porous media by substituting the elastic coefficients by time operators. The term viscoelastic refers to properties of a more general nature than those of the solid alone, which arise due to complex interactions between the solid and the fluid. As mentioned above, these properties can be of mechanical, chemical or thermoelastic nature, and each dissipation process can be represented by suitable time operators [Biot, 1962]. In this work the time operators are expressed by appropriate relaxation functions, and the convolutional kernels are avoided by the introduction of additional variables called memory variables. The result is a differential wave equation in the time-domain. In this way, efficient wave field calculations can be done with direct grid methods in arbitrarily inhomogeneous structures. Applications of this approach to single-phase media can be found in Carcione *et al.*, [1988 *a*, *b*] for isotropic solids, and Carcione [1990] for anisotropic solids.

The present model is restricted to the case of a viscoacoustic porous frame, *i. e.*, only compressional waves propagate. As in the purely elastic case, two types of waves exist, commonly referred to as fast and slow waves. The first correspond to the classic P-waves, and the slow wave is generally of diffusive character but may contribute to the attenuation of the fast wave by mode conversion at discontinuities [Geertsma & Smit, 1961]. Biot's poroelastic equations, and the equation of a viscoacoustic single-phase solid (Carcione *et al.*, 1988 *a*], are particular cases of the wave equation obtained with the present theory.

The first section establishes the constitutive relation of the porous medium as a convolutional relation. Then, memory variables are introduced in order to avoid the convolutional integrals. In the next section, Biot's equations and the memory variable differential equations are combined to give a single first-order matricial differential equation in time to solve for the solid and fluid dilatation fields. A numerical solution algorithm is proposed for high accuracy calculations. Next follows the calculation of the phase velocities and quality factors; and finally, examples are given of how to pose particular problems.

2. Constitutive relation

The material consists of a porous frame with a statistical distribution of interconnected pores. These define the effective porosity, while sealed pores are considered as part of the solid. It is assumed that the fluid is compressible.

The stress-strain relation for a porous viscoelastic solid was obtained by Biot (1956) from the thermodynamics of irreversible processes. The constitutive relation describing viscoacoustic propagation, *i. e.*, dilatational waves can be written (Appendix A) as

(1) $$\begin{bmatrix} p \\ p_f \end{bmatrix} = \begin{bmatrix} \psi_1 & \psi_2 \\ -\psi_2 & \psi_3 \end{bmatrix} * \begin{bmatrix} \dot{e} \\ \dot{\zeta} \end{bmatrix},$$ or, in compact notion,

(2) $$\mathbf{P} = \underset{\sim}{\Psi} * \dot{\mathbf{E}},$$

where p and p_f are the pressure fields of the matrix-fluid system and fluid, respectively; e and ζ are the dilatation fields of the solid matrix, and fluid relative to the solid, respectively. ψ_r, $r=1, \ldots, 3$ are relaxation functions, which may be represented by viscoelastic mechanical models of by more general frequency-domain rational functions (*see* Appendix B). This is a necessary condition for obtaining a differential wave equation in the time-domain, which for general initial value problems can be solved with higher efficiency than a frequency-domain equation. The symbol * indicates time convolution, and a dot above a variable denotes time differentiation.

As shown in Appendix B, the relaxation functions can be expressed as

(3) $$\psi_r(t) = \left[A_{Rr} + \sum_{l=1}^{L_r} A_{rl} e^{-t/\tau_l^{(r)}} \right] H(t), \qquad r=1, \ldots, 3$$

with A_{rl}, A_{Rr} and $\tau_l^{(r)}$, space-dependent functions, and $H(t)$ the step function.

3. Introduction of the memory variables

Eq. (2) can be alternatively written as

(4) $$\mathbf{P} = \dot{\underset{\sim}{\Psi}} * \mathbf{E}.$$

Since the relaxation matrix $\underset{\sim}{\Psi}$ contains a step function, further development implies

(5) $$\mathbf{P} = \underset{\sim}{\Psi}(0)\,\mathbf{E} + \sum_{l=1}^{L} \underset{\sim}{\Phi}_l * \mathbf{E},$$

where $L = \max(L_r)$, $r=1, \ldots, 3$, for convenience. $\underset{\sim}{\Phi}_l$, called the response matrix (from response function Adelman [1980]), is such that

(6) $\sum_{l=1}^{L} \underset{\sim}{\Phi}'_l = \dot{\underset{\sim}{\Psi}}'$, where the prime means that for a given causal function, $g(t)$ is $g = g'\,H$.

Hence,

(7 *a*, *b*) $\underset{\sim}{\Phi}_l = \begin{bmatrix} \varphi_{1l} & \varphi_{2l} \\ -\varphi_{2l} & \varphi_{3l} \end{bmatrix}$, with $\varphi'_{rl} = -\dfrac{A_{rl}}{\tau_l^{(r)}} e^{-t/\tau_l^{(r)}}$ by virtue of equation (3). Defining

(8 *a*, *b*) $\mathbf{V}_l = \begin{bmatrix} e_{1l} \\ \zeta_{3l} \end{bmatrix}$, and $\mathbf{W}_l = \begin{bmatrix} e_{2l} \\ \zeta_{2l} \end{bmatrix}$, as the memory vectors, with

(9 *a*, *b*, *c*) $e_{1l}=\varphi_{1l}*e,\ e_{2l}=-\varphi_{2l}*e,$ and $\zeta_{rl}=\varphi_{rl}*\zeta,$ $r=2,3$

the memory variables, the rheological relation of the porous medium (5) can be written as

(10) $$\mathbf{P}=\underset{\sim}{\Psi}(0)\,\mathbf{E}+\sum_{l=1}^{L}(\mathbf{V}_l+\mathbf{J}\mathbf{W}_l),\qquad \text{where}\quad \underset{\sim}{\mathbf{J}}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

4. Wave equation

Biot (1956) established the equations for wave propagation in a porous viscoelastic solid as a consequence of the application of the correspondence principle to the elastic wave equations. In the viscoacoustic case, the equations can be written in matrix notation as

(11) $$\mathbf{grad}\ \mathbf{P}=\underset{\sim}{\Gamma}\,\ddot{\mathbf{U}}+\underset{\sim}{\mathrm{H}}\,\dot{\mathbf{U}}+\mathbf{S},$$

where $\underset{\sim}{\Gamma}$ and $\underset{\sim}{\mathrm{H}}$ are the mass and damping matrices whose elements, γ_{rs} and η_{rs}, r, $s=1$, 2 respectively, as functions of the material parameters, are given in Appendix A;

(12) $$\mathbf{U}=\begin{bmatrix}\mathbf{u}\\ -\mathbf{w}\end{bmatrix},$$

with **u** the displacement of the solid, and **w** a vector representing the flow of the fluid relative to the solid, such that

(13) $$\mathbf{E}=\mathrm{div}\ \mathbf{U};$$

$\mathbf{S}^T=[\mathbf{s},\ \mathbf{s}_f]$ is the body force vector, with **s** acting on the matrix-fluid system, and $\mathbf{s}_f$ acting on the fluid phase. Multiplying both sides of (11) by $\underset{\sim}{\Gamma}^{-1}$ and recordering terms, yields

(14) $$\ddot{\mathbf{U}}=\underset{\sim}{\Gamma}^{-1}(\mathbf{grad}\ \mathbf{P}-\underset{\sim}{\mathrm{H}}\,\dot{\mathbf{U}}-\mathbf{S}),\quad \text{where}$$

(15 *a*, *b*) $$\underset{\sim}{\Gamma}^{-1}=\frac{1}{\det\underset{\sim}{\Gamma}}\begin{bmatrix}\gamma_{22} & -\gamma_{12}\\ \gamma_{12} & \gamma_{11}\end{bmatrix},\qquad \det\underset{\sim}{\Gamma}=\gamma_{11}\gamma_{22}+\gamma_{12}^2,$$

since $\gamma_{21}=-\gamma_{12}$. Taking divergence in equation (14) gives

(16 *a*, *b*) $$\ddot{\mathbf{E}}=\underset{\sim}{\Delta}\,\mathbf{P}-\mathrm{div}\,\underset{\sim}{\Gamma}^{-1}(\underset{\sim}{\mathrm{H}}\,\dot{\mathbf{U}}+\mathbf{S}),\qquad \text{and}\qquad \underset{\sim}{\Delta}=\mathrm{div}\,\underset{\sim}{\Gamma}^{-1}\,\mathbf{grad},$$

where (13) has been used. The operator matrix $\underset{\sim}{\Delta}$ acting on a vector $\mathbf{A}^T=[a_1,\ a_2]$ is simply

(17) $$\underset{\sim}{\Delta}\,\mathbf{A}=\partial_i\underset{\sim}{\Gamma}^{-1}\begin{bmatrix}\partial_i a_1\\ \partial_i a_2\end{bmatrix}\qquad i=1,\ldots,3,$$

where ∂_i means the spatial derivative with respect to the cartesian coordinate x_i, and implicit summation over repeated indices is assumed. Note that for a constant mass matrix $\underset{\sim}{\Gamma}$, $\underset{\sim}{\Delta}$ becomes

(18) $$\underset{\sim}{\Delta}=\underset{\sim}{\Gamma}^{-1}\Delta, \quad \text{where } \Delta \text{ is the Laplacian operator.}$$

Replacing the constitutive relation (10) into (16 *a*) gives

(19) $$\ddot{\mathbf{E}}=\underset{\sim}{\Delta}\Big[\underset{\sim}{\Psi}(0)\,\mathbf{E}+\sum_{l=1}^{L}(\mathbf{V}_l+\underset{\sim}{\mathbf{J}}\mathbf{W}_l)\Big]-\operatorname{div}\underset{\sim}{\Gamma}^{-1}(\underset{\sim}{\mathrm{H}}\,\dot{\mathbf{U}}+\mathbf{S}),$$

On the other hand, since φ'_{rl} in (7 *b*) satisfies the following differential equation: $\dot{\varphi}'_{rl}=-\varphi'_{rl}/\tau_l^{(r)}$, differentiating the memory vectors (8 *a*, *b*) with respect to time, and using (9 *a*, *b*, *c*) yields

(20 *a*) $$\dot{\mathbf{V}}_l=\begin{bmatrix}\varphi_{1l}(0) & 0\\ 0 & \varphi_{3l}(0)\end{bmatrix}\mathbf{E}-\begin{bmatrix}1/\tau_l^{(1)} & 0\\ 0 & 1/\tau_l^{(3)}\end{bmatrix}\mathbf{V}_l, \qquad l=1,\ldots,\mathrm{L},$$

(20 *b*) $$\dot{\mathbf{W}}_l=\begin{bmatrix}-\varphi_{2l}(0) & 0\\ 0 & \varphi_{2l}(0)\end{bmatrix}\mathbf{E}-\frac{1}{\tau_l^{(2)}}\underset{\sim}{\mathbf{I}}\mathbf{W}_l, \qquad l=1,\ldots,\mathrm{L},$$

where $\underset{\sim}{\mathbf{I}}$ is the 2×2 identity matrix.

Eq (14), (19) and (20 *a*, *b*) fully describe the response of the 3-D viscoacoustic porous medium and are the basis for solving wave propagation problems in the time-domain. They can be reformulated as a first order differential equation in time:

(21) $$\dot{\mathbf{D}}=\underset{\sim}{\mathbf{M}}\mathbf{D}+\mathbf{F},$$

where $\mathbf{D}^{\mathrm{T}}=[\mathbf{E},\ \dot{\mathbf{E}},\ \dot{\mathbf{U}},\ \mathbf{V}_1,\ldots\ \mathbf{V}_{\mathrm{L}},\ \mathbf{W}_1,\ldots\ \mathbf{W}_{\mathrm{L}}]$ is the vector of the unknown variables, $\mathbf{F}^{\mathrm{T}}=[\mathbf{0},\ -\operatorname{div}\underset{\sim}{\Gamma}^{-1}\mathbf{S},\ -\underset{\sim}{\Gamma}^{-1}\mathbf{S},\ \mathbf{0},\ldots\ \mathbf{0},\ \mathbf{0},\ldots\ \mathbf{0}]$, is the body forces vector, and $\underset{\sim}{\mathbf{M}}$ is a matrix operator which contains the spatial derivatives and the material parameters. The superscript T denotes transposition. For instance, for $\mathrm{L}=1$, $\underset{\sim}{\mathbf{M}}$ is formed by 2×2 operator matrices in the following way:

(22) $$\underset{\sim}{\mathbf{M}}=\begin{bmatrix}\underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{I}} & \underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{0}}\\ \underset{\sim}{\mathbf{M}}_{21} & \underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{M}}_{23} & \underset{\sim}{\mathbf{M}}_{24} & \underset{\sim}{\mathbf{M}}_{25}\\ \underset{\sim}{\mathbf{M}}_{31} & \underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{M}}_{33} & \underset{\sim}{\mathbf{M}}_{34} & \underset{\sim}{\mathbf{M}}_{35}\\ \underset{\sim}{\mathbf{M}}_{41} & \underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{M}}_{44} & \underset{\sim}{\mathbf{0}}\\ \underset{\sim}{\mathbf{M}}_{51} & \underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{M}}_{55}\end{bmatrix}$$

where $\underset{\sim}{\mathbf{0}}$ is the zero matrix, and

(23 *a*, *b*, *c*, *d*) $$\underset{\sim}{\mathbf{M}}_{21}=\underset{\sim}{\Delta}\underset{\sim}{\Psi}(0), \quad \underset{\sim}{\mathbf{M}}_{23}=-\operatorname{div}\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\mathrm{H}}, \quad \underset{\sim}{\mathbf{M}}_{24}=\underset{\sim}{\Delta}, \quad \underset{\sim}{\mathbf{M}}_{25}=\underset{\sim}{\Delta}\underset{\sim}{\mathbf{J}},$$

(23 *e*, *f*, *g*, *h*) $$\underset{\sim}{\mathbf{M}}_{31}=\underset{\sim}{\Gamma}^{-1}\,\mathbf{grad}\,\underset{\sim}{\Psi}(0), \quad \underset{\sim}{\mathbf{M}}_{33}=-\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\mathrm{H}},$$
$$\underset{\sim}{\mathbf{M}}_{34}=\underset{\sim}{\Gamma}^{-1}\,\mathbf{grad}, \quad \underset{\sim}{\mathbf{M}}_{35}=\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\mathbf{J}}\,\mathbf{grad},$$

(23 *i*, *j*) $$\underset{\sim}{\mathbf{M}}_{41}=\begin{bmatrix}\varphi_1(0) & 0\\ 0 & \varphi_3(0)\end{bmatrix},\qquad \underset{\sim}{\mathbf{M}}_{44}=-\begin{bmatrix}1/\tau^{(1)} & 0\\ 0 & 1/\tau^{(3)}\end{bmatrix},$$

(23 *k*, *l*) $$\underset{\sim}{\mathbf{M}}_{51}=\begin{bmatrix}-\varphi_2(0) & 0\\ 0 & \varphi_2(0)\end{bmatrix},\quad \text{and}\quad \underset{\sim}{\mathbf{M}}_{55}=-\frac{1}{\tau^{(2)}}\underset{\sim}{\mathbf{I}},$$

Formally, the vector $\dot{\mathbf{U}}$ has to be included in the unknown vector **D**, although due to the form of the damping matrix $\underset{\sim}{\mathbf{H}}$ (Eq. (A 6 *b*)), only $-\dot{\mathbf{w}}$ is involved, as implied by Eq. (18). Actually, this variable appears by virtue of the Biot dissipation mechanism. This is included in the theory to account for the diffusive character of the slow wave since, as is well known, the Biot mechanism cannot explain the attenuation characteristics of the fast wave, at least at seismic and sonic frequencies [Mochizuki, 1982]. On the other hand, this mechanism can be simulated by choosing appropriate relaxation functions. It can be seen that when the mass and damping matrices are constants, or when the damping matrix is zero (zero viscosity), $\dot{\mathbf{U}}$ is not involved in the calculations.

As an example, let only the relaxation function ψ_2 be time-dependent (*i. e.* assume $\tau^{(1)}$, $\tau^{(3)}\to\infty$), with $L_2=1.$, and consider a zero viscosity fluid. Then, from Eq. (1) and (3), the relaxation matrix at $t=0$ is

(24) $$\underset{\sim}{\Psi}(0)=\begin{bmatrix}A_{R1} & (A_{R2}+A_2)\\ -(A_{R2}+A_2) & A_{R3}\end{bmatrix};$$

the response matrix (7 *a*) reduces to

(25) $$\underset{\sim}{\Phi}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\varphi_2.$$

Then, the memory vectors contains only two variables, since

(26 *a*, *b*) $$\mathbf{V}_1=\begin{bmatrix}0\\0\end{bmatrix},\quad \text{and}\quad \mathbf{W}_1=\begin{bmatrix}e_2\\ \zeta_2\end{bmatrix}.$$

The unknown variable vector is $\mathbf{D}^T=[e, \zeta, \dot{e}, \dot{\zeta}, e_2, \zeta_2]$, and

(27 *a*, *b*, *c*) $$\underset{\sim}{\mathbf{M}}_{41}=\underset{\sim}{\mathbf{M}}_{44}=\underset{\sim}{\mathbf{0}},\qquad \underset{\sim}{\mathbf{M}}_{51}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\frac{A_2}{\tau},\qquad \underset{\sim}{\mathbf{M}}_{55}=-\frac{1}{\tau}\underset{\sim}{\mathbf{I}},$$

with τ the relaxation time corresponding to ψ_2. As mentioned above, since the damping matrix $\underset{\sim}{\mathbf{H}}$ is zero, the vector $\dot{\mathbf{U}}$ disappears from the wave equation (*see* Eq. (19)), and (14) is no longer necessary. The purely acoustic wave equation is obtained by taking $A_{rl}=0$ in Eq. (3) or $\theta_l^{(r)}=\tau_l^{(r)}$ in Eq. (B 2 *a*). Hence, the response matrix $\underset{\sim}{\Phi}_l$ and the memory vectors vanish, and

(28 *a*, *b*) $$\mathbf{D}^T=[\mathbf{E}, \dot{\mathbf{E}}, \dot{\mathbf{U}}],\qquad \underset{\sim}{\mathbf{M}}=\begin{bmatrix}\underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{I}} & \underset{\sim}{\mathbf{0}}\\ \underset{\sim}{\mathbf{M}}_{21} & \underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{M}}_{23}\\ \underset{\sim}{\mathbf{M}}_{31} & \underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{M}}_{33}\end{bmatrix},$$

which implies Biot's poroelastic equations [Biot, 1962].

The equation for a viscoacoustic single-phase solid [Carcione *et al.*, 1988 *a*] is obtained with $A_{Rr} = A_{rl} = 0$, $r = 2,3$, fluid density $\rho_f = 0$, and fluid viscosity $\eta = 0$. Then, the response function and the memory vectors reduce to

$$(29\,a,\,b,\,c)\qquad \underset{\sim}{\Phi}_l = \begin{bmatrix} \varphi_{1l} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{V}_l = \begin{bmatrix} e_{1l} \\ 0 \end{bmatrix}, \qquad \mathbf{W}_l = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{and, for } L = 1,$$

$$(30\,a,\,b)\qquad \mathbf{D}^{\mathrm{T}} = [e, \dot{e}, e_1], \qquad \underset{\sim}{\mathbf{M}} = \begin{bmatrix} 0 & 1 & 0 \\ M_{21} & 0 & M_{23} \\ M_{31} & 0 & M_{33} \end{bmatrix}, \quad \text{where}$$

$$(31\,a,\,b,\,c)\qquad M_{21} = \partial_i \rho_s^{-1} \partial_i \psi(0), \qquad M_{31} = \varphi(0), \qquad M_{33} = -1/\tau_\sigma, \qquad M_{23} = \partial_i \rho_s^{-1} \partial_i,$$

with ρ_s the density of the solid.

The differential equation (21) represents the wave equation of the viscoacoustic porous medium which correctly describes the anelastic effects in wave propagation within the framework of linear response theory. The solution of (21) subject to the initial condition

$$(32)\qquad \mathbf{D}(t=0) = \mathbf{D}_0 \quad \text{is formally given by}$$

$$(33)\qquad \mathbf{D}(t) = e^{t\underset{\sim}{\mathbf{M}}} \mathbf{D}_0 + \int_0^t e^{\theta \underset{\sim}{\mathbf{M}}} \mathbf{F}(t-\theta)\, d\theta.$$

In Eq. (33), $e^{t\underset{\sim}{\mathbf{M}}}$ is called the evolution operator of the system. Most frequently, an explicit or implicit finite-difference scheme is used to march the solution in time. This technique is based on a Taylor expansion of the evolution operator. An alternative and more effective approach specially designed to solve wave propagation problems in linear viscoelastic media was developed by Tal-Ezer *et al.*, [1990]. The approach is based on a polynomial interpolation of the exponential function in the complex domain of the eigenvalues of the operator $\underset{\sim}{\mathbf{M}}$, over a set of points which is known to have some maximal properties. In this way, the interpolating polynomial is almost best. The eigenvalues should have negative real parts; this is verified for viscoacoustic porous media in Appendix C, where the eigenvalues of $\underset{\sim}{\mathbf{M}}$ are calculated for the one-dimensional wave propagation problem. To balance time integration and spatial accuracies, the spatial derivatives can be computed by means of the Fourier pseudospectral method, although finite-differences or finite-elements can also by used.

5. Complex velocities, phase velocities and quality factors

To analyse if the wave equation (26) describes correctly the anelastic characteristics of the dilatational waves, it is necessary to calculate the phase velocities and quality factors. Applying the convolutional theorem to equation (4), the rheological relation in the frequency-domain takes the form

$$(34\,a,\,b)\qquad \tilde{\mathbf{P}} = \underset{\sim}{\mathbf{B}} \tilde{\mathbf{E}}, \qquad \underset{\sim}{\mathbf{B}} = \tilde{\underset{\sim}{\Psi}},$$

where the tilde over the variables means time Fourier transformation. The complex bulk matrix is given by

(35) $\underset{\sim}{\mathbf{B}}=\begin{bmatrix} M_1 & M_2 \\ -M_2 & M_3 \end{bmatrix}$, where the complex moduli, given in Appendix B, are

$$(36\,a,\,b)\qquad M_r=\frac{A_{Rr}}{L_r}\sum_{l=1}^{L_r}\frac{1+i\omega\theta_l^{(r)}}{1+i\omega\tau_l^{(r)}},\qquad \theta_l^{(r)}=\left(1+L_r\frac{A_{rl}}{A_{Rr}}\right)\tau_l^{(r)},$$

with ω the angular frequency, and $\theta_l^{(r)}$ and $\tau_l^{(r)}$ relaxation times. Identification of the purely elastic case, either with the low-frequency or the high-frequency limits, defines the acoustic coefficients A_{er}, $r=1, \ldots, 3$ as the relaxed or the unrelaxed moduli (equations (B 5) and (B 6), respectively). They can be written as functions of the properties of the constituent material through the elastic coefficients A, R and Q of Biot theory. This is done in Appendix D.

Taking time Fourier transform of Eq. (16 *a*), and using (18), yields

$$(37)\qquad \underset{\sim}{\Gamma}^{-1}\Delta\tilde{\mathbf{P}}=(-\omega^2+i\omega\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\mathrm{H}})\tilde{\mathbf{E}},$$

where a zero source vector and constant material properties were assumed. Replacing (34 *a*) into (37) gives

$$(38)\qquad (\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\mathbf{B}}\Delta+\omega^2\underset{\sim}{\mathbf{I}}-i\omega\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\mathrm{H}})\tilde{\mathbf{E}}=\underset{\sim}{\mathbf{0}}.$$

Assume a plane wave solution to equation (38) of the form

$$(39\,a,\,b)\qquad \tilde{\mathbf{E}}=\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}},\qquad \mathbf{E}_0=\begin{bmatrix} e_0 \\ \zeta_0 \end{bmatrix},$$

where $\mathbf{k}$ is the complex wavenumber and $\mathbf{x}$ is the position vector. Replacing this solution into (38) and taking zero determinant gives

$$(40)\qquad \det\left[\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\mathbf{B}}-V^2\left(\underset{\sim}{\mathbf{I}}+\frac{1}{i\omega}\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\mathrm{H}}\right)\right]=0,$$

where (41) $V=\omega/k$, is the complex velocity. The solution of (40) give the complex velocities for the fast and slow compressional waves in a viscoacoustic porous solid. Denote them be V_α, $\alpha=1,2$, respectively. Then, for homogeneous viscoelastic waves, the phase velocities are given by the frequency divided by the real part of the complex wavenumber,

(42) $c_\alpha=\mathrm{Re}^{-1}[V_\alpha^{-1}]$, $\alpha=1,2$, and the quality factors by

$$(43)\qquad Q_\alpha=\frac{\mathrm{Re}[V_\alpha^2]}{\mathrm{Im}[V_\alpha^2]},\qquad \alpha=1,2,$$

where Re and Im take the real and imaginary parts, respectively. Eq. (43) is an extension to porous media of the quality factors for homogeneous plane waves in a viscoelastic solid [Carcione *et al.*, 1988 *b*]. A detailed frequency-domain formulation for viscoelastic porous media can be found, for instance, in Rasolofosaon [1991].

6. Solving a particular problem

To solve a given wave propagation problem, it is necessary to establish the relaxation matrix $\underset{\sim}{\Psi}$. Each relaxation function ψ_r, $r=1, \ldots, 3$ is completely determined, for instance by the relaxed moduli A_{R_r}, and the relaxation times $\theta_l^{(r)}$, $\tau_l^{(r)}$, $l=1, \ldots, L_r$. In principle, it should be possible to measure experimentally the relaxation functions and the complex moduli. Having these quantities, the relaxed moduli and the relaxation times can be obtained by curve fitting to the theoretical relaxation functions (3) or complex moduli (36 *a*). Usually, the experimental quality factors and velocity dispersion curves of the fast wave are given [Bourbie *et al.*, 1987]. In this case, the relaxation functions are obtained by fitting the theoretical quality factors (43) and phase velocities (42) to the experimental ones.

For instance, assume that the dots in Figure 1 *a* correspond to hypothetical experimental values of the quality factor for a fast wave observed in brine saturated sandstone, and that the material properties of the sandstone are those given in Table I [Winkler, 1985]. The continuous lines in Figure 1 *a* represent the theoretical quality factors of the two compressional waves obtained by curve fitting, and Figure 1 *b* displays the theoretical phase velocity dispersion. Only the relaxation function ψ_1 is considered time-dependent. Table II gives the corresponding relaxation times. Of course, in this example, only the quality factor of the fast wave is realistic. An appropriate evaluation

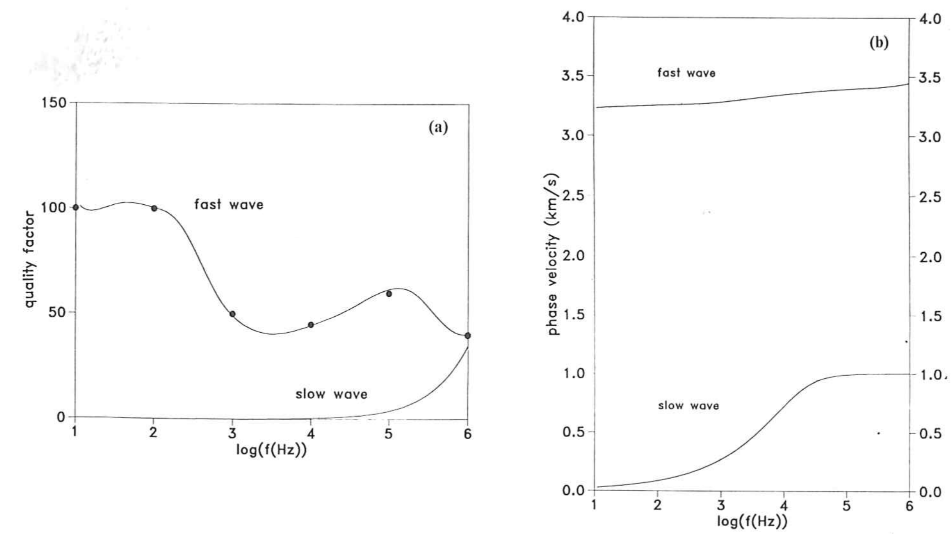

Fig. 1. — (*a*) quality factors, and (*b*) phase velocities of a brine saturated sandstone. The dots give the experimental values of the quality factor for the fast wave. The continuous lines represent the theoretical values obtained by curve fitting. Table 2 gives the relaxation times which define the relaxation functions, only ψ_1 in this particular problem.

of the relaxation functions requires knowledge of the anelastic characteristics of both the compressional modes.

TABLE 1. — Material properties of the porous medium.

Solid	bulk modulus K_s GPa	40.
	density ρ_s, kg/m^3	2500.
Matrix	bulk modulus K_m, GPa	20.
	density ρ_m, kg/m^3	2000.
	porosity β	0.2
	permeability K, m^2	0.6×10^{-12}
	tortuosity α	2.
Fluid	bulk modulus K_f, GPa	2.5
	density ρ_f, kg/m^3	1040.
	viscosity η, cP	1.

TABLE 2. — Relaxations times.

l	$\tau_l^{(1)}$ (s)	$\theta_l^{(1)}$ (s)
1	1.600×10^{-2}	1.785×10^{-2}
2	5.028×10^{-3}	5.220×10^{-3}
3	1.600×10^{-3}	1.710×10^{-3}
4	1.600×10^{-4}	1.860×10^{-4}
5	5.028×10^{-5}	6.100×10^{-5}
6	1.600×10^{-5}	1.810×10^{-5}
7	1.600×10^{-6}	1.740×10^{-6}
8	1.600×10^{-7}	2.250×10^{-7}

On the other hand, the relaxation functions can be derived by application of the correspondence principle to the acoustic coefficients. This approach allows the study of the wave field due to specific dissipation mechanisms of the constituents, and of the solid-fluid system. For example, internal dissipation in the solid or fluid, or the local flow model introduced by Biot [1962] in which the pore fluid is squeezed out of, and sucked back into, narrows cracks between the surfaces of the grains in contact. Biot demonstrates how this mechanism can be represented by a spring dashpot model, and established the form of the frequency-domain viscoelastic coefficients. These can be expanded as partial fractions, and the corresponding relaxation functions can be obtained by inverse time Fourier transform.

The following example considers a porous medium saturated with brine in which the skeleton is dissipative with a constant quality factor over part of the sonic frequency band (1 to 10 kHz). By using the correspondence principle, the solid and matrix bulk moduli, K_s and K_m, respectively, are replaced by complex bulk moduli calculated from a continuous distribution of relaxation mechanisms based on the standard linear solid. The complex moduli obtained by Liu *et al.*, [1976] can be rewritten as

$$K_u^* = K_u \ln^{-1}\left[e\left(\frac{1+i\omega\tau_2}{1+i\omega\tau_1}\right)^{2/\pi\bar{Q}_u}\right], \tag{44}$$

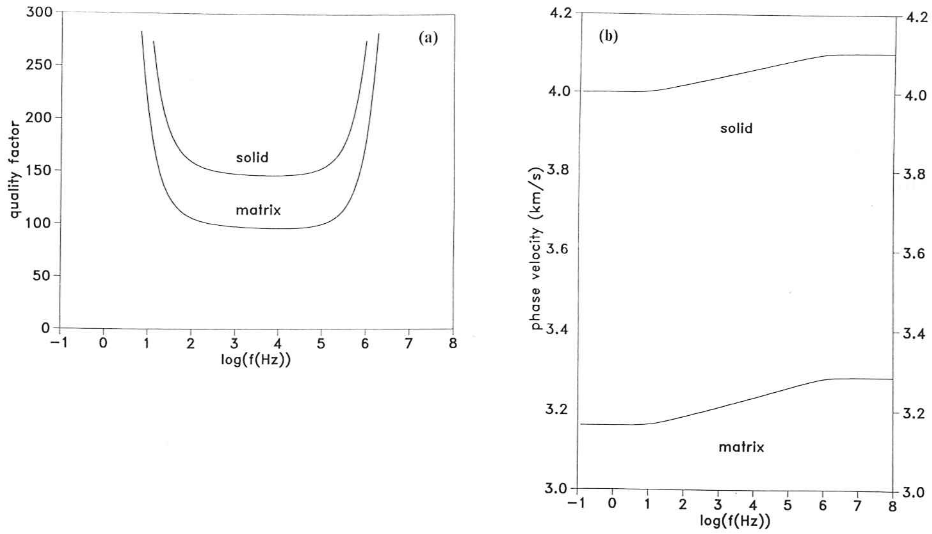

Fig. 2. — (*a*) quality factors, and (*b*) phase velocities of the solid and the dry frame (matrix). The theory used is the continuous relaxation model based on the standard linear solid rheology. The quality factors are nearly constant over part of the sonic band.

where $u=s$, $u=m$ for the solid and the matrix, respectively. τ_1 and τ_2 are time constants, and $\bar{Q}_u$ defines the value of the quality factor which remains nearly constant over the selected frequency range. In (44), e denotes the Euler number. The quality factors are $Q_u = \text{Re}[K_u^*]/\text{Im}[K_u^*]$, and the phase velocities are $c_u = \text{Re}^{-1}(\rho_u/K_u^*)$. Replacing the solid and matrix complex moduli in equations (D 7 *a-c*), the viscoacoustic coefficients A_{vr}, $r=1, \ldots, 3$, are obtained such that

$$(45\,a, b, c)\quad \bar{K}A_{v1} = \frac{1}{K_s^*} - \frac{1}{K_m^*} + \beta\left[\frac{1}{K_s^*} - \frac{1}{K_f}\right], \qquad \bar{K}A_{v2} = \frac{1}{K_m^*} - \frac{1}{K_s^*}, \qquad \bar{K}A_{v3} = \frac{1}{K_m^*},$$

with

$$(46)\qquad \bar{K} = \frac{\beta}{K_m^*}\left[\frac{1}{K_f} - \frac{1}{K_s^*}\right] + \frac{1}{K_s^*}\left[\frac{1}{K_m^*} - \frac{1}{K_s^*}\right].$$

The material properties of the porous medium are given in Table I. The parameters in Eq. (44) are $\tau_1 = 1.5\times10^{-2}$, $\tau_2 = 1.5\times10^{-7}$, $\bar{Q}_s = 150$ and $\bar{Q}_m = 100$. Figure 2 displays (*a*) the quality factors, and (*b*) the phase velocities of the solid and the matrix. As can be seen, the quality factors are nearly constants from 1 to 10 kHz. Calculation of the quality factors and phase velocities of the porous medium is done with Eq. (42) and (43), respectively, and replacing the matrix $\underset{\sim}{B}$ in (40) by the matrix of the viscoacoustic coefficients. This is

$$(47)\qquad \underset{\sim}{B} \rightarrow \begin{bmatrix} A_{v1} & A_{v2} \\ -A_{v2} & A_{v3} \end{bmatrix}.$$

Figure 3 shows the quality factors (*a*), and the phase velocities (*b*) of the porous medium. The dotted line corresponds to the elastic porous material, *i.e.* no viscoelasticity in the solid. As can be appreciated, the solid anelasticity contributes mainly to the fast wave. That the slow wave is not much affected comes from the fact that this mode is of a poroelastic nature. This result is in agreement with that of Rasolofosaon [1991]. Wave field computations in the time-domain require the calculation of the relaxed moduli A_{Rr}, and relaxation times $\tau_l^{(r)}$ and $\theta_l^{(r)}$, $r=1, \ldots, 3$. To get them, the matrix of the viscoacoustic coefficients should be fitted to the complex moduli matrix $\underset{\sim}{\mathbf{B}}$. Alternatively, the curve fitting can be done with the quality factors and phase velocities.

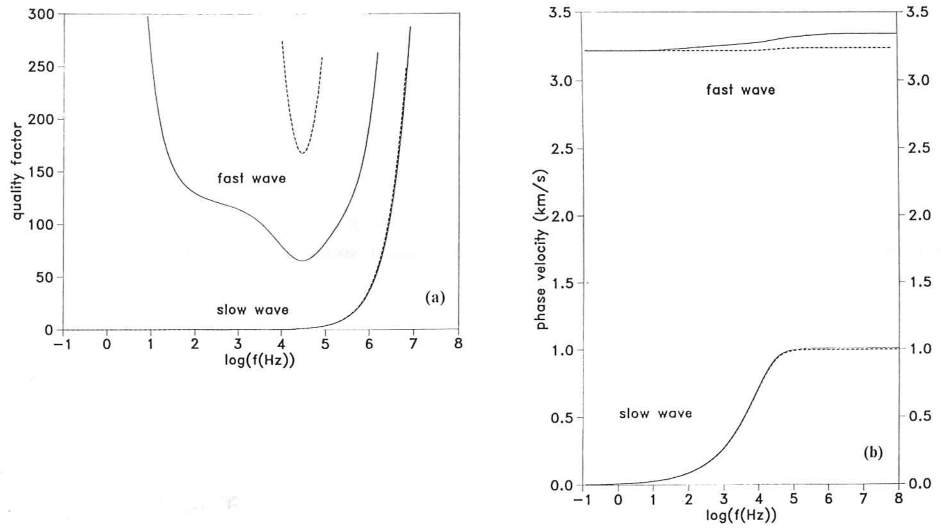

Fig. 3. — (*a*) quality factors and (*b*) phase velocities of the porous medium with internal dissipation in the solid. The dotted lines correspond to the elastic porous material. The anelasticity contributes mainly to the fast wave.

7. Conclusion

Many earth materials can be considered as multiphase solids with fluid-filled pores. Attenuation and velocity dispersion are strongly dependent on the characteristics of the porous medium, which include intrinsic properties of the solid and the fluid, and complex interations of the solid-fluid system. This work proposes a phenomenological model for describing wave propagation in viscoacoustic porous media where any dissipation mechanism can be included within the framework of the linear response theory. The theory requires knowledge of the relaxation functions, whose complex moduli should have a rational form in the frequency-domain. The relaxation times can be obtained by curve fitting provided that one knows the quality factor and phase velocity dispersion curves. The theory allows the modeling of the anelastic properties of dilatational waves observed experimentally in porous rocks, and efficient wave field computations by direct grid methods since the wave equation is expressed in differential form in the time-domain.

Acknowledgments

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APPENDIX A

Constitutive relations and equation of motion

The frequency-domain stress-strain relations for a linear viscoelastic and isotropic porous solid are given by [Biot, 1956, 1962]

(A1 *a*) $$\tilde{\tau}_{xx}=H^*\tilde{e}-2\mu^*(\tilde{e}_{yy}+\tilde{e}_{zz})-C^*\tilde{\zeta},$$

(A1 *b*) $$\tilde{\tau}_{yy}=H^*\tilde{e}-2\mu^*(\tilde{e}_{xx}+\tilde{e}_{zz})-C^*\tilde{\zeta},$$

(A1 *c*) $$\tilde{\tau}_{zz}=H^*\tilde{e}-2\mu^*(\tilde{e}_{xx}+\tilde{e}_{yy})-C^*\tilde{\zeta},$$

(A1 *d, e, f*) $$\tilde{\tau}_{yz}=2\mu^*\tilde{e}_{yz},\qquad \tilde{\tau}_{xz}=2\mu^*\tilde{e}_{xz},\qquad \tilde{\tau}_{xy}=2\mu^*\tilde{e}_{xy},$$

(A1 *g*) $$\tilde{p}_f=-C^*\tilde{e}+M^*\tilde{\zeta},$$

where $\tilde{\tau}_{ij}$ are the stress components of the bulk material, $\tilde{e}_{ij}$ are the strain components of the solid matrix, $\tilde{e}$ and $\tilde{\zeta}$ are the dilatations of the bulk material and fluid phase, respectively, and $\tilde{p}_f$ is the fluid pressure. The qualities H^*, μ^*, C^* and M^* are frequency-dependent material properties. The model is restricted to the viscoacoustic case, which implies that no shear deformations can take place. This is done by taking $\mu^*=0$. Applying the convolution theorem to Eq. (A1 *a-g*) yields

(A2 *a, b*) $$-p=H*e-C*\zeta, \qquad \text{and} \qquad p_f=-C*e+M*\zeta,$$

where p is the bulk pressure, and symbol $*$ denotes time convolution.

Expressed in terms of relaxation functions, and using the properties of convolution, Eq. (A2 *a-b*) can be written as

(A3) $$\begin{bmatrix} p \\ p_f \end{bmatrix}=\begin{bmatrix} \psi_1 & \psi_2 \\ -\psi_2 & \psi_3 \end{bmatrix}*\begin{bmatrix} \dot{e} \\ \dot{\zeta} \end{bmatrix},$$

such that $\dot{\psi}_1=-H$, $\dot{\psi}_2=C$, $\dot{\psi}_3=M$, where a dot above a variable denotes time differentiation. Expressions for the relaxation functions are given in Appendix B. As stated by Biot [1962], the symmetry of the operational matrix of Eq. (A 3) is a consequence of the laws of the thermodynamics of irreversible processes, particularly Onsager's theorem.

The dynamical equations for a saturated poroelastic solid were obtained by Biot using a macroscopic approach, and verified by Burridge & Keller [1981] and de la Cruz &

Spanos [1986] starting from the detailed microstructure of the system. For zero body forces, and restricted to the viscoacoustic case, the equations can be expressed [Biot, 1962] as

$$(\text{A } 4\,a,\, b) \qquad -\mathbf{grad}\, p = \rho\, \ddot{\mathbf{u}} + \rho_f\, \ddot{\mathbf{w}}, \qquad \text{and} \qquad -\mathbf{grad}\, p_f = \rho_f\, \ddot{\mathbf{u}} + m\, \ddot{\mathbf{w}} + \frac{\eta}{\mathrm{K}}\, \dot{\mathbf{w}},$$

where $\mathbf{u}$ is the average displacement of the solid, and $\mathbf{w}$ is an average vector giving the flow of the fluid relative to the solid such that $e = \operatorname{div} \mathbf{u}$, and $\zeta = -\operatorname{div} \mathbf{w}$. The composite density of the saturated material is $\rho = (1-\beta)\,\rho_s + \beta\rho_f$, where ρ_s is the density of the solid, ρ_f is the density of the fluid, and β is the effective porosity; $m = \alpha\rho_f\,\beta$, with α the tortuosity, a dimensionless parameter that is dependent on the pore geometry [*see*, for instance, Johnson *et al.*, 1982]. Finally, η is the fluid viscosity and K is the global permeability of the porous medium. The validity of the theory is given by the condition imposed by Poiseuille flow, which implies a high-frequency limit. For higher frequencies, the viscosity and the density parameters in Eqs. (A 4 *a*, *b*) are time-dependent, and the right-hand side terms become convolutional integrals. Scattering dissipation is not explicitely considered (since the wavelength is assumed larger than the pore size) although, in principle, its contribution could be simulated with the relaxation functions.

Eqs. (A 4 *a*, *b*) can be alternatively written as

$$(\text{A } 5\,a,\, b) \qquad \mathbf{grad}\, \mathbf{P} = \underset{\sim}{\Gamma}\, \ddot{\mathbf{U}} + \underset{\sim}{\mathrm{H}}\, \dot{\mathbf{U}}, \qquad \text{where} \quad \mathbf{U} = \begin{bmatrix} \mathbf{u} \\ -\mathbf{w} \end{bmatrix}, \quad \text{and}$$

$$(\text{A } 6\,a,\, b) \qquad \underset{\sim}{\Gamma} \equiv \begin{bmatrix} -\rho & \rho_f \\ -\rho_f & m \end{bmatrix}, \qquad \text{and} \qquad \underset{\sim}{\mathrm{H}} = \begin{bmatrix} 0 & 0 \\ 0 & \eta/\mathrm{K} \end{bmatrix},$$

are the mass and damping matrices, respectively.

APPENDIX B

Relaxation functions

A relaxation function appropriate for wave field computations in the time-domain can be expressed as

$$(\text{B } 1) \qquad \psi(t) = \left[\mathrm{A_R} + \sum_{l=1}^{\mathrm{L}} \mathrm{A}_l\, e^{-t/\tau_l} \right] \mathrm{H}(t),$$

with A_l, $\mathrm{A_R}$, and τ_l, space-dependent functions, and $\mathrm{H}(t)$ the step function. Fourier transforming the time derivative of the relaxation function gives the complex modulus

[Ben-Menahem & Singh, 1981], which can be written as

$$\text{(B 2}\,a, b, c) \qquad \left\{ \begin{aligned} & \mathrm{M}(\omega)=\dot{\tilde{\psi}}(t)=\frac{\mathrm{A_R}}{\mathrm{L}}\sum_{l=1}^{\mathrm{L}}\frac{1+i\omega\theta_l}{1+i\omega\tau_l}, \\ & \theta_l=\left(1+\mathrm{L}\frac{\mathrm{A}_l}{\mathrm{A_R}}\right)\tau_l, \qquad \mathrm{A}_l=\frac{\mathrm{A_R}}{\mathrm{L}}\left(\frac{\theta_l}{\tau_l}-1\right), \end{aligned} \right.$$

with ω the angular frequency. θ_l, and τ_l are relaxation times, and the tilde indicates time Fourier transformation. Eq. (B 2 *a*) is the expression of a general rational function in the variable $i\omega$. As special cases, the general standard linear solid [Carcione *et al.*, 1988 *a*] and the generalized Maxwell body [Emmerich & Korn, 1987] can be represented by the complex modulus (B 2 *a*).

A parallel connection of L single standard linear elements, each with constants $\mathrm{M_R}/\mathrm{L}$, $\tau_{\varepsilon l}$ and $\tau_{\sigma l}$, $l=1, \ldots$ L, has a complex modulus of the form (B 2 *a*), with

$$\text{(B 3}\,a, b, c) \qquad \mathrm{A_R}=\mathrm{M_R}, \qquad \tau_l=\tau_{\sigma l}, \qquad \theta_l=\tau_{\varepsilon l}.$$

Similarly, a parallel connection of L Maxwell elements, each with constants k_l and τ_l, $l=1, \ldots,$ L, plus a spring of constant $\mathrm{M_R}$, gives

$$\text{(B 4}\,a, b, c) \qquad \mathrm{A_R}=\mathrm{M_R}, \qquad \tau_l\equiv\tau_l, \qquad \theta_l=\left(1+\mathrm{L}\frac{k_l}{\mathrm{M_R}}\right)\tau_l.$$

Expressions for the complex moduli and relaxation functions of single standard linear and Maxwell elements can be found, for instance, in Ben-Menahem and Singh [1981].

For $\omega\to 0$ in (B 2), or $t\to\infty$ in (B 1),

(B 5) $\mathrm{M}(0)\to\psi(\infty)\to\mathrm{A_R}$, the relaxed modulus associated with the long-term behaviour of the system. For $\omega\to\infty$, or $t\to 0$,

(B 6) $\mathrm{M}(\infty)\to\psi(0)\to\mathrm{A}_u\equiv\mathrm{A_R}+\sum_{l=1}^{\mathrm{L}}\mathrm{A}_l=\frac{\mathrm{A_R}}{\mathrm{L}}\sum_{l=1}^{\mathrm{L}}\frac{\theta_l}{\tau_l}$, the unrelaxed modulus, which characterizes the instantaneous response.

APPENDIX C

Eigenvalues of the propagation matrix $\underset{\sim}{\mathrm{M}}$

Let a one-dimensional plane wave solution to Eq. (21) be of the form

$$\text{(C 1)} \qquad \mathbf{D}=\mathbf{D}_0\, e^{i(\omega_c t-kx)},$$

where t is the time, x is the position variable, ω_C is the complex frequency, and k is the real wavenumber. In the 1-D case $\mathbf{D}_0$ has nine components:

$$\mathbf{D}_0 = [e_0, \zeta_0, \dot{e}_0, \dot{\zeta}_0, -\dot{w}_0, e_{10}, \zeta_{30}, e_{20}, \zeta_{20}]. \tag{C 2}$$

Replacing (C 1) into (21), and considering constant material properties and zero body forces, yields

$$i\omega_C \mathbf{D}_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ M_{31} & M_{32} & 0 & 0 & M_{35} & M_{36} & M_{37} & M_{38} & M_{39} \\ M_{41} & M_{42} & 0 & 0 & M_{45} & M_{46} & M_{47} & M_{48} & M_{49} \\ M_{51} & M_{52} & 0 & 0 & M_{55} & M_{56} & M_{57} & M_{58} & M_{59} \\ M_{61} & 0 & 0 & 0 & 0 & M_{66} & 0 & 0 & 0 \\ 0 & M_{72} & 0 & 0 & 0 & 0 & M_{77} & 0 & 0 \\ M_{81} & 0 & 0 & 0 & 0 & 0 & 0 & M_{88} & 0 \\ 0 & M_{92} & 0 & 0 & 0 & 0 & 0 & 0 & M_{99} \end{bmatrix} \mathbf{D}_0, \tag{C 3}$$

where

$$\det \underset{\sim}{\Gamma} M_{31} = -k^2 [\gamma_{22} \psi_1(0) + \gamma_{12} \psi_2(0)], \quad \det \underset{\sim}{\Gamma} M_{32} = -k^2 [\gamma_{22} \psi_2(0) - \gamma_{12} \psi_3(0)],$$

$$\det \underset{\sim}{\Gamma} M_{35} = ik\gamma_{12} \eta/K, \quad \det \underset{\sim}{\Gamma} M_{36} = -k^2 \gamma_{22}, \quad \det \underset{\sim}{\Gamma} M_{37} = k^2 \gamma_{12},$$

$$\det \underset{\sim}{\Gamma} M_{38} = k^2 \gamma_{12}, \quad \det \underset{\sim}{\Gamma} M_{39} = -k^2 \gamma_{22},$$

$$\det \underset{\sim}{\Gamma} M_{41} = -k^2 [\gamma_{12} \psi_1(0) - \gamma_{11} \psi_2(0)],$$

$$\det \underset{\sim}{\Gamma} M_{42} = -k^2 [\gamma_{12} \psi_2(0) + \gamma_{11} \psi_3(0)], \quad \det \underset{\sim}{\Gamma} M_{45} = -ik\gamma_{11} \eta/K,$$

$$\det \underset{\sim}{\Gamma} M_{46} = -k^2 \gamma_{12}, \quad \det \underset{\sim}{\Gamma} M_{47} = -k^2 \gamma_{11},$$

$$\det \underset{\sim}{\Gamma} M_{48} = -k^2 \gamma_{11}, \quad \det \underset{\sim}{\Gamma} M_{49} = -k^2 \gamma_{12},$$

$$\det \underset{\sim}{\Gamma} M_{51} = ik [\gamma_{12} \psi_1(0) - \gamma_{11} \psi_2(0)], \quad \det \underset{\sim}{\Gamma} M_{52} = ik [\gamma_{12} \psi_2(0) + \gamma_{11} \psi_3(0)],$$

$$\det \underset{\sim}{\Gamma} M_{55} = -\gamma_{11} \eta/K, \quad \det \underset{\sim}{\Gamma} M_{56} = ik\gamma_{12}, \quad \det \underset{\sim}{\Gamma} M_{57} = ik\gamma_{11},$$

$$\det \underset{\sim}{\Gamma} M_{58} = ik\gamma_{11}, \quad \det \underset{\sim}{\Gamma} M_{59} = ik\gamma_{12},$$

$$M_{61} = \varphi_1(0), \quad M_{66} = -1/\tau^{(1)}, \quad M_{72} = \varphi_3(0), \quad M_{77} = -1/\tau^{(3)},$$

$$M_{81} = -\varphi_2(0), \quad M_{88} = -1/\tau^{(2)}, \quad M_{92} = -\varphi_2(0), \quad M_{99} = -1/\tau^{(2)},$$

Eq. (C 3) is an eigenvalue equation for the eigenvalues $\lambda = i\omega_C$. The following example clarifies the physical meanning of the eigenvalues. The material properties of the porous solid are given in Table 1, and the relaxation times are: $\tau^{(1)} = 1.600 \times 10^{-2}$, $\theta^{(1)} = 1.785 \times 10^{-2}$; $\tau^{(2)} = 1.600 \times 10^{-4}$, $\theta^{(2)} = 1.860 \times 10^{-4}$; and $\tau^{(3)} = 1.600 \times 10^{-6}$, $\theta^{(3)} = 1.740 \times 10^{-6}$. For $k = 100\ m^{-1}$, the eigenvalues are

$$\lambda_1 = (-762 + i\,341258)\,s^{-1}, \quad \lambda_2 = \lambda_1^*, \quad \lambda_3 = (-84223 + i\,50865)\,s^{-1}, \quad \lambda_4 = \lambda_3^*$$

$$\lambda_5 = 0, \quad \lambda_6 = -55\,s^{-1}, \quad \lambda_7 = -622861\,s^{-1}, \quad \lambda_8 = -5997\,s^{-1}, \quad \lambda_9 = -6855\,s^{-1},$$

where here the symbol * means the complex conjugate. As can be seen the eigenvalues have negative real parts. The complex frequency is given by $\omega_C \equiv \omega + i\,\omega_I = -i\lambda$. Then, the temporal quality factor is defined by [Pilant, 1979].

(C 4) $$Q_t(\omega) = \frac{\omega}{2\,\omega_I} = -\frac{\mathrm{Im}\,[\lambda]}{2\,\mathrm{Re}\,[\lambda]},$$ and the phase velocity is

(C 5) $$c(\omega) = \frac{\omega}{k} = \frac{\mathrm{Im}\,[\lambda]}{k},$$

which should be the same as that given by Eq. (42). Re and Im take real and imaginary parts, respectively. λ_1 and λ_2 correspond to the fast wave, giving $f = 54\ k\mathrm{Hz}$, $Q_t = 223$, and $c = 3413$ m/s., while λ_3 and λ_4 correspond to the slow wave, with $f = 8\ k\mathrm{Hz}$, $Q_t = 0.3$, and $c = 508$ m/s. The zero eigenvalue arises from the fact that the fourth and fifth rows in the propagation matrix are linearly dependent. The other eigenvalues are attenuating static modes which are given by $\lambda_6 \simeq -1/\tau^{(1)}$, $\lambda_7 \simeq -1/\tau^{(3)}$, and $\lambda_8 \simeq \lambda_9 \simeq -1/\tau^{(2)}$. This analysis confirms the stability of the system represented by Eq. (21).

APPENDIX D

Acoustic coefficients

A real material behaves elastically at both very high and very low frequencies. As shown in Appendix B, the complex modulus becomes real when $\omega \to \infty$, or $\omega \to 0$, giving the unrelaxed and relaxed moduli, respectively. In such cases, it can be seen that the quality factors (43) are equal to infinity, indicating elastic behaviour. The reaction of the medium to a propagating wave represents the instantaneous response, and therefore corresponds to the more usual interpretation of elastic behaviour. However, observed discrepancies between the static and dynamic moduli are not caused by a frequency difference but by a difference in strain amplitudes [Cheng & Johnston, 1981; Winkler, Nur & Gladwin, 1979]. Therefore, the present model allows the possibility of choosing the elastic matrix $\underset{\sim}{\Psi}_e$ as

(D 1 *a*, *b*)
$$\left\{\begin{aligned} &\underset{\sim}{\Psi}_e(t) \equiv \lim_{t\to 0} \underset{\sim}{\Psi} = \begin{bmatrix} A_{u1} & A_{u2} \\ -A_{u2} & A_{u3} \end{bmatrix} H(t), \\ &\text{or} \\ &\underset{\sim}{\Psi}_e(t) \equiv \lim_{t\to \infty} \underset{\sim}{\Psi} = \begin{bmatrix} A_{R1} & A_{R2} \\ -A_{R2} & A_{R3} \end{bmatrix} H(t), \end{aligned}\right.$$

depending on whether the elastic limit is the high or low frequency range, respectively. Then, replacing (D 1) into (1) yields the elastic constitutive relation:

$$\text{(D 2)} \qquad \begin{bmatrix} p \\ p_f \end{bmatrix} = \underset{\sim}{\Psi}_e \begin{bmatrix} e \\ \zeta \end{bmatrix} \equiv \begin{bmatrix} A_{e1} & A_{e2} \\ -A_{e2} & A_{e3} \end{bmatrix} \begin{bmatrix} e \\ \zeta \end{bmatrix}.$$

where A_{er}, $r=1, \ldots, 3$ are the acoustic coefficients. These can be obtained as functions of the solid and fluid properties, either experimentally of from Biot theory [1957]. Biot [1955] expressed the stress-strain relations for uniform porosity as

$$\text{(D 3}\,a, b) \qquad -(p-\beta p_f) = A e + Q \varepsilon, \qquad \text{and} \qquad -\beta p_f = Q e + R \varepsilon,$$

where $\varepsilon = e - \zeta/\beta$, with zero shear modulus. A, R and Q are elastic coefficients which can be expressed as functions of the matrix (dry rock), solid and fluid properties. Following Plona & Johnson [1980], after Biot & Willis [1957], these coefficients are

$$\text{(D 4}\,a, b, c) \qquad \bar{K} A = (1-\beta)\left[\frac{(1-\beta)}{K_m} - \frac{1}{K_s}\right] + \frac{\beta}{K_f}, \qquad \bar{K} R = \frac{\beta^2}{K_m},$$

$$\bar{K} Q = \beta\left[\frac{(1-\beta)}{K_m} - \frac{1}{K_s}\right], \quad \text{with}$$

$$\text{(D 5)} \quad \bar{K} = \frac{\beta}{K_m}\left[\frac{1}{K_f} - \frac{1}{K_s}\right] + \frac{1}{K_s}\left[\frac{1}{K_m} - \frac{1}{K_s}\right],$$ where K_s, K_m and K_f are the bulk moduli of the solid, matrix and fluid, respectively.

Comparison of Eqs. (D 2) and (D 3 *a*, *b*), yields the acoustic coefficients:

$$\text{(D 6}\,a, b, c) \qquad A_{e1} = -(A+R+2Q), \qquad A_{e2} = (Q+R)/\beta, \qquad A_{e3} = R/\beta^2.$$

By substitution of (D 5 *a-c*), they are

$$\text{(D 7}\,a, b, c) \qquad \bar{K} A_{e1} = \frac{1}{K_s} - \frac{1}{K_m} + \beta\left[\frac{1}{K_s} - \frac{1}{K_f}\right],$$

$$\bar{K} A_{e2} = \frac{1}{K_m} - \frac{1}{K_s}, \qquad \bar{K} A_{e3} = \frac{1}{K_m}.$$

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