GEOPHYSICS, VOL. 59, NO. 4 (APRIL 1994); P. 644-657, 7 FIGS., 1 TABLE.

Wavefronts in dissipative anisotropic media

Jose M. Carcione*

ABSTRACT

The purpose of this work is to draw attention to several differences between wave propagation in dissipative anisotropic media and purely elastic anisotropic media. In an elastic medium, the wavefront is defined as the envelope of the family of planes that makes the phase of the plane waves zero. It turns out that this definition coincides with the wavefronts obtained from the group and energy velocities, i.e., the three concepts are equivalent. However, for plane waves traveling in dissipative anisotropic media these concepts are different. Despite these differences, the velocity of the envelope of plane waves closely approximates the energy velocity, and therefore can represent the wavefront from a practical point of view. On the other hand, the group velocity describes the wavefront only when the attenuation is relatively low, i.e., for Q values higher than 100. The values of the different velocities and the shape of the wavefront are considerably influenced by the relative values of the attenuation along the principal axes of the anisotropic medium. This means that the anisotropic coefficients in attenuating anisotropic media may differ substantially from the corresponding elastic coefficients. Moreover, it is shown that the usual orthogonality properties between the slowness surface and energy velocity vector and the wavefront and wavenumber vector does not hold for dissipative anisotropic media.

INTRODUCTION

Waves in anelastic media travel at velocities that are different from those of waves in elastic media, the differences depending on the degree of attenuation that produces more or less velocity dispersion. If the medium is isotropic (and homogeneous) the velocity of the energy coincides with the velocity of the plane waves, i.e., the phase velocity (Carcione, 1990). However, in dissipative anisotropic media the energy velocity is neither the phase velocity nor the group velocity. The question arises if the wavefront (energy velocity times one unit of propagation time) can be described as the envelope of plane waves. In this paper, I investigate this point and the influence of anelasticity on the different physical velocities.

The paper is organized as follows. The first section introduces the frequency-domain constitutive relation for anisotropic-viscoelastic media from a general point of view. In the next sections the phase, energy, group, and envelope velocities for homogeneous viscoelastic plane waves are defined, the latter being the velocity of the envelope of plane waves. Then, the calculation of the different velocities is carried out explicitly for the SH mode in a plane of symmetry of an orthorhombic medium. Finally, I introduce a constitutive equation that models the attenuation properties along preferred directions and analyze the behavior of the different physical velocities.

CONSTITUTIVE RELATION

Hooke's law for elastic anisotropic media can be extended to dissipative media by replacing the elastic constants by appropriate relaxation components (Christensen, 1982; Carcione, 1990). The viscoelastic constitutive relation is Boltzmann's superposition principle which can be expressed as

$$T_I(\mathbf{x}, t) = \dot{\psi}_{IJ}(\mathbf{x}, t) * S_J(\mathbf{x}, t), \qquad I, J = 1, \ldots, 6, \tag{1}$$

where x = (x, y , z) is the position vector and t is the time variable, with the asterisk denoting time convolution;

$$\mathbf{T}^T = [T_1, T_2, T_3, T_4, T_5, T_6] = [\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{xz}, \sigma_{xy}] \tag{2}$$

is the stress vector,

$$\mathbf{S}^T = [S_1, S_2, S_3, S_4, S_5, S_6] = [\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, 2\varepsilon_{yz}, 2\varepsilon_{xz}, 2\varepsilon_{xy}] \tag{3}$$

is the strain vector, and ψ_{IJ} are the components of the relaxation matrix $\mathbf{\Psi}(\mathbf{x}, t)$, such that $\psi_{IJ} = \psi_{JI}$. A dot above

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*Osservatorio Geofisico Sperimentale, P.O. Box 2011 Opicina, 34016 Trieste, Italy.

the relaxation components indicates time differentiation. The notation assumes implicit summation over repeated indices, with vectors written as columns and the superscript T denoting the transpose. A general solution representing viscoelastic plane waves is of the form

$$[\cdot]e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}, \quad \mathbf{k} = \boldsymbol{\kappa} - i\boldsymbol{\alpha}, \tag{4}$$

where ω is the angular frequency, and k is the complex wavenumber vector with κ and $\boldsymbol{\alpha}$ the real wavenumber and attenuation vectors, respectively. They indicate the directions and magnitudes of propagation and attenuation. When the directions of propagation and attenuation coincide, the wave is called homogeneous. Hence,

$$\mathbf{k} = (\kappa - i\alpha)\hat{\boldsymbol{\kappa}} \equiv k\hat{\boldsymbol{\kappa}}, \tag{5}$$

where

$$\hat{\boldsymbol{\kappa}} = \hat{\mathbf{e}}_x \ell_x + \hat{\mathbf{e}}_y \ell_y + \hat{\mathbf{e}}_z \ell_z \tag{6}$$

defines the propagation direction through the direction cosines ℓ_x, ℓ_y, and ℓ_z. Substituting the plane wave (4) into the stress-strain relation (1) yields

$$T_I = p_{IJ} S_J, \quad p_{IJ} = \int_{-\infty}^{\infty} \dot{\psi}_{IJ}(\tau) e^{-i\omega\tau}\, d\tau, \tag{7}$$

where p_{IJ} are the components of the stiffness matrix $\underset{\sim}{\mathbf{p}}(\mathbf{x}, \omega)$. In matrix notation, equation (7) reads

$$\mathbf{T} = \underset{\sim}{\mathbf{p}} \cdot \mathbf{S}, \tag{8}$$

where the dot denotes ordinary matrix multiplication. For anelastic media, the components of $\underset{\sim}{\mathbf{p}}$ are complex and frequency dependent.

SLOWNESS AND PHASE VELOCITY

The dispersion relation for homogeneous viscoelastic plane waves has the form of the elastic dispersion relation, but the quantities involved are complex and frequency dependent. It can be written as (Carcione, 1990),

$$\det [\underset{\sim}{\boldsymbol{\Gamma}} - \rho V^2 \underset{\sim}{\mathbf{I}}] = 0, \tag{9}$$

where

$$\underset{\sim}{\boldsymbol{\Gamma}} = \underset{\sim}{\mathbf{L}} \cdot \underset{\sim}{\mathbf{p}} \cdot \underset{\sim}{\mathbf{L}} \tag{10}$$

is the 3 x 3 Christoffel matrix, with

$$\underset{\sim}{\mathbf{L}} = \begin{bmatrix} \ell_x & 0 & 0 & 0 & \ell_z & \ell_y \\ 0 & \ell_y & 0 & \ell_z & 0 & \ell_x \\ 0 & 0 & \ell_z & \ell_y & \ell_x & 0 \end{bmatrix} \tag{11}$$

the direction cosine matrix. In equation (9), pV is the density and V is the complex velocity (instead of the phase velocity in elastic media) given by

$$V = \frac{\omega}{k}, \tag{12}$$

defined as a complex scalar quantity. The complex velocities of the three wave modes are found by solving equation (9). The phase-velocity vector is defined as the frequency divided by the real wavenumber

$$\mathbf{V}_p = \frac{\omega}{\mathrm{Re}\,[k]}\hat{\boldsymbol{\kappa}} = \left(\mathrm{Re}\left[\frac{1}{V}\right]\right)^{-1}\hat{\boldsymbol{\kappa}}, \tag{13}$$

in virtue of equation (12), with Re [•] taking the real part. The slowness is the reciprocal of the phase velocity. In vector form it is given by

$$\mathbf{s} = \mathrm{Re}\left[\frac{1}{V}\right]\hat{\boldsymbol{\kappa}}, \tag{14}$$

i.e., it is the real part of the complex slowness.

ENERGY VELOCITY AND WAVEFRONT

In the absence of body sources, the complex Umov-Poynting theorem or energy balance equation for homogeneous viscoelastic plane waves in a dissipative medium is given by (Carcione, 1990),

$$2\boldsymbol{\alpha}^T \cdot \mathbf{P} + i\omega[(\varepsilon_s)_{\text{peak}} - (\varepsilon_v)_{\text{peak}}] - \omega(\varepsilon_d)_{\text{AV}} = 0, \tag{15}$$

where P is the complex Umov-Poynting vector defined as

$$\mathbf{P} = -\frac{1}{2}\underset{\sim}{\boldsymbol{\Sigma}} \cdot \mathbf{v}^*, \tag{16}$$

with $\underset{\sim}{\boldsymbol{\Sigma}}$ the stress tensor given by

$$\underset{\sim}{\boldsymbol{\Sigma}} = \begin{bmatrix} T_1 & T_6 & T_5 \\ T_6 & T_2 & T_4 \\ T_5 & T_4 & T_3 \end{bmatrix} \tag{17}$$

The vector v is the particle velocity

$$\mathbf{v} = \dot{\mathbf{u}}, \tag{18}$$

where u is the displacement vector related to the strain vector by S = $-ik\underset{\sim}{\mathbf{L}}^T \cdot$ u. The asterisk used as superscript denotes complex conjugate.

The real part of the Umov-Poynting vector gives the average power flow density over a cycle. The quantities

$$(\varepsilon_s)_{\text{peak}} = \frac{1}{2}\,\mathrm{Re}\,[\mathbf{S}^T \cdot \underset{\sim}{\mathbf{p}} \cdot \mathbf{S}^*], \tag{19}$$

and

$$(\varepsilon_v)_{\text{peak}} = \frac{1}{2}\rho \mathbf{v}^T \cdot \mathbf{v}^* \tag{20}$$

are the peak strain and peak kinetic energy densities, and

$$(\varepsilon_d)_{AV} = \frac{1}{2}\,\mathrm{Im}\,[\mathbf{S}^T \cdot \underset{\sim}{\mathbf{p}} \cdot \mathbf{S}^*], \tag{21}$$

is the dissipated energy density, where Im [•] takes the imaginary part. The average stored energy density is

$$\varepsilon_{AV} = \frac{(\varepsilon_v)_{\text{peak}} + (\varepsilon_s)_{\text{peak}}}{2} = \frac{1}{4}\{\rho \mathbf{v}^T \cdot \mathbf{v}^* + \text{Re}\,[\mathbf{S}^T \cdot \underset{\sim}{\mathbf{p}} \cdot \mathbf{S}^*]\}. \tag{22}$$

In elastic media $(\varepsilon_d)_{AV} = 0$, and also the net energy flow into, or out of a given closed surface S vanishes: $-2\boldsymbol{\alpha}^T \cdot \mathbf{P} = 0$. Thus, the peak kinetic energy is equal to the peak potential energy. As a consequence, the average stored energy is equal to the peak potential energy.

The energy velocity vector is defined as the ratio of the average power flow density to the mean energy density (22). The average power flow density is the real part of the complex Umov-Poynting vector. Hence,

$$\mathbf{V}_e = \frac{2\ \text{Re}\,[\mathbf{P}]}{(\varepsilon_v)_{\text{peak}} + (\varepsilon_s)_{\text{peak}}}. \tag{23}$$

The location of the energy defines the wavefront. Therefore, this is the locus of the end of the energy velocity vector at unit propagation time.

An important relation between the phase velocity and the energy velocity is

$$\hat{\boldsymbol{\kappa}}^T \cdot \mathbf{V}_e = V_p, \tag{24}$$

where V_p is the magnitude of the phase velocity vector (13). Relation (24) is proved by Auld (1990, eq. 7.57) for elastic media. It can be shown that equation (24) also holds for general dissipative anisotropic media and for inhomogeneous viscoelastic plane waves, i.e., for arbitrary directions of propagation and attenuation.

GROUP VELOCITY

I compute the group velocity by generalizing the elastic group velocity

$$\mathbf{V}_g = \hat{\mathbf{e}}_x \frac{\partial \omega}{\partial \kappa_x} + \hat{\mathbf{e}}_y \frac{\partial \omega}{\partial \kappa_y} + \hat{\mathbf{e}}_z \frac{\partial \omega}{\partial \kappa_z}, \tag{25}$$

where the partial derivatives here are taken with respect to the real wavenumber components κ_x, κ_y, and κ_z. Since an explicit real equation of the form $\omega = \Theta^R(\kappa_x, \kappa_y, \kappa_z)$ is not available, equation (25) is not appropriate.

Alternatively, the group velocity can be obtained by implicit differentiation of the dispersion relation (9). For instance, for the x-component

$$\frac{\partial \omega}{\partial \kappa_x} = \left[\frac{\partial \kappa_x}{\partial \omega}\right]^{-1} \tag{26}$$

or, since $\kappa_x = \text{Re}\,[k_x]$,

$$\frac{\partial \omega}{\partial \kappa_x} = \left(\text{Re}\left[\frac{\partial k_x}{\partial \omega}\right]\right)^{-1}. \tag{27}$$

Implicit differentiation of the complex dispersion relation $\Theta(k_x, k_y, k_z, \omega) = 0$ gives

$$\left(\frac{\partial \Theta}{\partial \omega}\delta\omega + \frac{\partial \Theta}{\partial k_x}\delta k_x\right)_{k_y, k_z} = 0. \tag{28}$$

Then,

$$\left(\frac{\partial k_x}{\partial \omega}\right)_{k_y, k_z} = -\frac{\partial \Theta/\partial \omega}{\partial \Theta/\partial k_x}, \tag{29}$$

and similar relations hold for the k_y and k_z components. Replacing the partial derivatives in equation (25), the group velocity can be evaluated as

$$\mathbf{V}_g = -\hat{\mathbf{e}}_x\left(\text{Re}\left[\frac{\partial \Theta/\partial \omega}{\partial \Theta/\partial k_x}\right]\right)^{-1} - \hat{\mathbf{e}}_y\left(\text{Re}\left[\frac{\partial \Theta/\partial \omega}{\partial \Theta/\partial k_y}\right]\right)^{-1} - \hat{\mathbf{e}}_z\left(\text{Re}\left[\frac{\partial \Theta/\partial \omega}{\partial \Theta/\partial k_z}\right]\right)^{-1}, \tag{30}$$

which is a generalization to dissipative media of Auld's (1990) equation (7.78).

ENVELOPE VELOCITY

Let us assume for simplicity that $\ell_y = 0$, i.e.,the propagation is in the (x, z)-plane. Then, the wavenumber direction can be defined by $\ell_x = \sin\theta$ and ℓ_z = cos 8, with θ the angle between the wavenumber vector and the z-axis.

As will be seen in the following demonstration, the wavefront, defined by the energy velocity, is not the envelope of the family of planes (straight lines in this case),

$$x\ell_x + z\ell_z = V_p, \tag{31}$$

which is the usual definition found in most books (Love, 1952, art. 209; Musgrave, 1970,88; Fedorov, 1968,148). The velocity of the envelope of plane waves at unit propagation time, which I call V_{env}, is

$$V_{env}^2 = x^2 + z^2 = V_p^2 + \left(\frac{dV_p}{d\theta}\right)^2. \tag{32}$$

This equation is obtained by deriving equation (31) with respect to θ, squaring it and adding the result to the square of equation (31) (Postma, 1955; Zauderer, 1989, 93).

However, application of the same operations to equation (24) yields

$$V_e^2 = V_{env}^2 - \frac{d\mathbf{V}_e}{d\theta} \cdot \hat{\boldsymbol{\kappa}}\left(\frac{d\mathbf{V}_e}{d\theta} \cdot \hat{\boldsymbol{\kappa}} + 2\frac{d\hat{\boldsymbol{\kappa}}}{d\theta} \cdot \mathbf{V}_e\right). \tag{33}$$

I show in the next section that, in general, the second term on the right-hand side of equation (33) does not vanish for dissipative media, and therefore the wavefront cannot be synthetized as the envelope of plane waves.

SH-WAVES IN ORTHORHOMBIC MEDIA

In this section I explicitly perform the calculation of the different velocities for SH-waves in one of the planes of symmetry of an orthorhombic medium. The dispersion relation (9) in, say the (x, z)-plane, separates into a quadratic factor and a linear factor. The latter is the dispersion relation for viscoelastic SH-waves:

$$\Gamma_{22} - \rho V^2 = 0, \quad \Gamma_{22} = p_{66}\ell_x^2 + p_{44}\ell_z^2. \tag{34}$$

The solution of equation (34) is given by

$$V = \sqrt{\frac{p_{66}\ell_x^2 + p_{44}\ell_z^2}{\rho}}. \tag{35}$$

The displacement field has the following form:

$$U = U_0 \hat{\mathbf{e}}_y e^{-\boldsymbol{\alpha}\cdot\mathbf{x}} e^{i\omega(t - \mathbf{s}\cdot\mathbf{x})}, \tag{36}$$

where U_0 is a complex quantity.

The Umov-Poynting vector and energy densities are calculated in the Appendix. Substitution of these quantities into equation (23) gives the energy velocity for the SH-waves

$$\mathbf{V}_e = \frac{V_p}{\mathrm{Re}\,[V]}\left\{\ell_x\ \mathrm{Re}\left[\frac{p_{66}/\rho}{V}\right]\hat{\mathbf{e}}_x + \ell_z\ \mathrm{Re}\left[\frac{p_{44}/\rho}{V}\right]\hat{\mathbf{e}}_z\right\}. \tag{37}$$

The complex dispersion relation for the SH-waves is from equation (34):

$$\Theta(k_x, k_z, \omega) = p_{66}k_x^2 + p_{44}k_z^2 - \rho\omega^2 = 0. \tag{38}$$

The partial derivatives are given by

$$\frac{\partial\Theta}{\partial k_x} = 2p_{66}k_x, \tag{39}$$

$$\frac{\partial\Theta}{\partial k_z} = 2p_{44}k_z, \tag{40}$$

and

$$\frac{\partial\Theta}{do} = p'_{66}k_x^2 + p'_{44}k_z^2 - 2\rho\omega, \tag{41}$$

where the prime denotes the derivative with respect to the angular frequency. Consequently, substituting these expressions into equation (30) gives the group velocity for homogeneous SH-plane waves as

$$\mathbf{V}_g = -2\hat{\mathbf{e}}_x\ell_x\left(\mathrm{Re}\left[\frac{D}{Vp_{66}}\right]\right)^{-1} - 2\hat{\mathbf{e}}_z\ell_z\left(\mathrm{Re}\left[\frac{D}{Vp_{44}}\right]\right)^{-1}, \tag{42}$$

where

$$D = \omega(p'_{66}\ell_x^2 + p'_{44}\ell_z^2) - 2\rho V^2.$$

Comparison of equations (37) and (42) indicates that the energy velocity does not coincide with the group velocity. The differences for a realistic medium are analyzed in the example of the next section. In dissipative anisotropic media, the wavefronts are given by the energy velocity multiplied by one unit of propagation time. The group velocity has physical meaning only for low-loss media as an approximation to the wavefront. It is straightforward to verify that the two velocities coincide for purely elastic media.

Deriving the phase velocity (13) [by using equation (35)], and replacing it in equation (32), gives the magnitude of the envelope velocity:

$$V_{env} = V_p\sqrt{1 + \ell_x\ell_z V_p^2\left(\mathrm{Re}\left[\frac{p_{66} - p_{44}}{\rho V^3}\right]\right)^2}. \tag{43}$$

For elastic media $p_{IJ} = c_{IJ}$ (the elasticities) are real quantities, then, $V_p = V$, and

$$V_{env} = V_e = \frac{1}{\rho V_p}\sqrt{c_{66}\ell_x^2 + c_{44}\ell_z^2}. \tag{44}$$

In elastic anisotropic media, the energy velocity $\mathbf{V}_e$ is always normal to the slowness surface [Auld, 1990, equation (7.71)]. From this fact, and using equation (7.71), Auld [equation (7.74)] also proves that the wavenumber κ is always normal to the wavefront. However, these properties do not apply, in general, to anelastic anisotropic media as will be seen in the following derivations for viscoelastic SH-waves. From the dispersion relation (34), and since the slowness components are $s_x = s\ell_x$, and $s_z = s\ell_z$, the equation for the slowness curve is

$$\Omega(s_x, s_z) = \mathrm{Re}\left[\left(\frac{s_x^2}{\rho/p_{66}} + \frac{s_z^2}{\rho/p_{44}}\right)^{-1/2}\right] - 1 = 0. \tag{45}$$

A vector normal to this curve is given by

$$\nabla_s\Omega = \left(\frac{\partial\Omega}{\partial s_x}, \frac{\partial\Omega}{\partial s_z}\right) = -V_p^2\left\{\ell_x\ \mathrm{Re}\left[\frac{p_{66}/\rho}{V^3}\right]\hat{\mathbf{e}}_x + \ell_z\ \mathrm{Re}\left[\frac{p_{44}/\rho}{V^3}\right]\hat{\mathbf{e}}_z\right\}. \tag{46}$$

It is clear from equations (37) and (46) that $\mathbf{V}_e$ and $\nabla_s\boldsymbol{\Omega}$ are not collinear vectors; thus, the energy velocity is not, in general, normal to the slowness surface.

The other orthogonality property of elastic media (i.e., that the slowness vector must always be normal to the wavefront) is not valid for anelastic media. By using $V_p = \omega/\kappa$, and differentiating equation (24) with respect to θ gives

$$\frac{dtc}{d\hat{e}}\cdot\mathbf{V}_e + \boldsymbol{\kappa}\cdot\frac{d\mathbf{V}_e}{d\theta} = \frac{d\boldsymbol{\kappa}}{d\theta}\cdot\mathbf{V}_e + \gamma\boldsymbol{\kappa}\cdot\frac{d\mathbf{V}_e}{d\phi} = 0, \tag{47}$$

where $\gamma = d\phi/d\theta$, with

$$\tan\phi = \frac{V_{ex}}{V_{ez}} = \frac{\mathrm{Re}\,[p_{66}/V(\theta)]}{\mathrm{Re}\,[p_{44}/V(\theta)]}\tan\theta, \tag{48}$$

defining the energy velocity direction as represented in Figure 1. It can be shown that γ is always different from zero, in particular $\gamma = 1$ for isotropic media. Since $d\kappa/d\theta$ is tangent to the slowness surface, and $\mathbf{V}_e$ is not normal to it, the first term in equation (47) is different from zero. Since $d\mathbf{V}_e/d\phi$ lies on the wavefront, from the second term, equation (47) implies that the wavenumber vector κ is not normal to this surface. In fact, taking into account that $K(o) = [\omega/V_p(\theta)](\hat{\mathbf{e}}_x \sin\theta + \hat{\mathbf{e}}_z\cos\theta)$, calculation of the first term of equation (47) gives

$$\frac{dtc}{d\theta}\cdot\mathbf{V}_e = \omega V_p\ell_x\ell_z\ \mathrm{Re}\left[\frac{(p_{66} - p_{44})}{\rho V}\left(\frac{1}{|V|^2} - \frac{1}{V^2}\right)\right]. \tag{49}$$

For elastic medi .a, V is real and equation (49) is identically zero; in this case, the orthogonality properties are verified:

$$\frac{d\mathbf{V}_e}{d\theta}\cdot\hat{\boldsymbol{\kappa}} = -\frac{1}{\kappa}\frac{d\boldsymbol{\kappa}}{d\theta}\cdot\mathbf{V}_e = 0, \qquad (50)$$

and from equation (33) the envelope velocity equals the energy velocity, and the wavefront is defined by equation (32) in agreement with Postma (1955) and Berryman (1979). In dissipative media this equivalence is no longer valid.

EXAMPLE

A class of constitutive equations for anisotropic-viscoelastic media based on two complex moduli was introduced by Carcione (1990). In the present work, I introduce two additional complex moduli to model in more detail the anelastic properties of the shear modes. The stiffness matrix of this new rheology is

$$\underline{\mathbf{p}} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & c_{14} & c_{15} & c_{16} \\ & p_{22} & p_{23} & c_{24} & c_{25} & c_{26} \\ & & p_{33} & c_{34} & c_{35} & c_{36} \\ & & & c_{44}M_2 & c_{45} & c_{46} \\ & & & & c_{55}M_3 & c_{56} \\ & & & & & c_{66}M_4 \end{bmatrix}, \qquad (51)$$

where

$$p_{I(I)} = c_{I(I)} - D + KM_1 + \frac{4}{3}GM_\delta \qquad \text{for } I = 1, 2, 3, \qquad (52a)$$

$$p_{IJ} = c_{IJ} - D + 2G + KM_1 - \frac{2}{3}GM_\delta$$

$$\text{for } I, J = 1, 2, 3;\ I \neq J. \qquad (52b)$$

Here c_{IJ}, for I, $J = 1, \ldots, 6$ are taken as the low-frequency limit elastic constants, and

$$K = D - \frac{4}{3}G, \qquad (53)$$

where

$$D = \frac{1}{3}(c_{11} + c_{22} + c_{33}), \qquad (54a)$$

and

$$G = \frac{1}{3}(c_{44} + c_{55} + c_{66}). \qquad (54b)$$

M_ν are dimensionless complex moduli, $\nu = 1$ is for the quasi-dilatational mode, and $\nu = 2, 3, 4$ are for the shear waves. In (52a and b), M_δ is a shear modulus with $\delta = 2, 3$, or 4. It can be verified that the mean stress only depends on the first relaxation function involving quasi-dilatational dissipation mechanisms, and that the deviatoric stress components depend on the other relaxation functions, describing quasi-shear mechanisms. Equation (5 1) gives the elasticity matrix of the generalized Hooke's law in the anisotropic-elastic limit when $M_\nu \rightarrow 1$ and the 3-D isotropic-viscoelastic constitutive relation in the isotropic limit.

The stiffness matrix (51) differs from that given in Carcione (1990), since three relaxation functions instead of one are used to describe the anelastic properties of the shear modes. In this way, it is possible to control the quality factor along three preferred directions, such as the principal axes o the anisotropic medium. The choice of the complex moduli depends on the symmetry system. For isotropic, cubic, and hexagonal media, two relaxation functions are necessary and sufficient to model the anelastic properties.

The theory assumes the following form for the complex moduli:

$$M_\nu(\omega) = \frac{1 + i\omega\tau_{\epsilon\nu}}{1 + i\omega\tau_{\sigma\nu}}, \qquad \nu = 1, \ldots, 4, \qquad (55)$$

whose 1-D quality factors are given by

$$Q_\nu(\omega) = \frac{\mathrm{Re}\,[M_\nu]}{\mathrm{Im}\,[M_\nu]} = \frac{1 + \omega^2\tau_{\epsilon\nu}\tau_{\sigma\nu}}{\omega(\tau_{\epsilon\nu} - \tau_{\sigma\nu})}, \qquad (56)$$

where $\tau_{\epsilon\nu}$ and $\tau_{\sigma\nu}$ are relaxation times such that $\tau_{\epsilon\nu} \geq \tau_{\sigma\nu}$. Equation (55) represents the complex modulus of a standard linear solid with maximum attenuation at $\omega_{0\nu} = 1/\sqrt{\tau_{\epsilon\nu}\tau_{\sigma\nu}}$.

The complex velocities are the key to obtain the attenuation properties. These velocities are now given for the symmetry planes of orthorhombic media. In the natural coordinate system, orthorhombic media have $p_{1J} = p_{2J} = p_{3J} = 0$, $J \geq 4$, and $p_{45} = p_{46} = p_{56} = 0$, and nine independent stiffness constants. For these media, the eigenvalues of the Christoffel matrix (10) in the (x, z)-plane of symmetry are as follows:

$$\rho V_{1(2)}^2 = \frac{1}{2}(p_{55} + p_{11}\ell_x^2 + p_{33}\ell_z^2 \pm E), \qquad (57a)$$

and

$$\rho V_3^2 = p_{66}\ell_x^2 + p_{44}\ell_z^2, \qquad (57b)$$

where

$$E = \sqrt{[(p_{33} - p_{55})\ell_z^2 - (p_{11} - p_{55})\ell_x^2]^2 + 4(p_{13} + p_{55})^2\ell_x^2\ell_z^2}.$$

In principle, V_1 is the velocity of the qP wave (+ sign), while V_2 (- sign) and V_3 correspond to the shear waves, with the second one a pure mode. In complex materials this identification does not apply, since along the same wavefront the wave may change from quasi-compressional to quasi-shear or vice versa. Note that, in a weak anisotropic medium, the qSV wave is defined by the velocity V_2.

The physical velocities for the SH-waves are explicitly given in the previous section. The group velocity needs p'_{44} and p'_{66} which from equation (52) involves the calculation of M'_ν, $\nu = 2$ and 4, given by

$$M'_\nu = \frac{i(\tau_{\epsilon\nu} - \tau_{\sigma\nu})}{(1 + i\omega\tau_{\sigma\nu})^2}. \qquad (58)$$

Note that for the particular case $M_2 = M_4 = M$, $V3 = \sqrt{M}\, V_{pe}$ where V_{pe} is the elastic phase velocity. When this happens, the right-hand side of equation (49) vanishes and $V_e = V_{env}$ from equations (33) and (50). Moreover, the orthogonality properties of elastic media are verified. This case corresponds to ***SH*** isotropic attenuation in the (x, z) plane of symmetry.

The phase velocities $\mathbf{V}_{p1}$ and $\mathbf{V}_{p2}$ of the ***qP*** and ***qS*** modes are given by equation (13) by substitution of the corresponding complex velocities given in equation (57a). For completeness, the expressions of the group and energy velocities of the coupled modes are given below:

$$\mathbf{V}_{em} = \frac{V_{pm}}{D_m} \operatorname{Re} \left[\frac{1}{V_m} ([\ell_x(p_{11} + p_{55}|B_m|^2) + \ell_z(p_{13}B_m + p_{55}B_m^*)]\hat{\mathbf{e}}_x + [\ell_x(p_{55}B_m + p_{13}B_m^*) + \ell_z(p_{55} + p_{33}|B_m|^2)]\hat{\mathbf{e}}_z \right], \qquad m = 1, 2 \tag{59}$$

where

$$D_m = \rho(1 + |B_m|^2) \operatorname{Re}[V_m],$$

and

$$B_m = -\frac{p_{11}\ell_x^2 + p_{55}\ell_z^2 - \rho V_m^2}{(p_{13} + p_{55})\ell_x \ell_z}.$$

A careful numerical evaluation of equation (59) should consider the limits when either ℓ_x or $\ell_z \rightarrow 0$. For instance, when $\ell_x \rightarrow 0$ and $\ell_z \rightarrow 1$, $B_1 \rightarrow \infty$ and $B_2 \rightarrow 0$. Taking these limits gives the appropriate energy velocities.

$$\mathrm{V}gm = -2\ell_x\hat{\mathbf{e}}_x\left(\operatorname{Re}\left[\frac{E_m}{V_m}[p_{11}w_m + p_{55}v_m - (p_{13} + p_{55})^2\ell_z^2]^{-1}\right]\right)^{-1} - 2\ell_z\hat{\mathbf{e}}_z\left(\operatorname{Re}\left[\frac{E_m}{V_m}[p_{55}w_m + p_{33}v_m - (p_{13} + p_{55})^2\ell_x^2]^{-1}\right]\right)^{-1},$$

$$m = 1, 2, \tag{60}$$

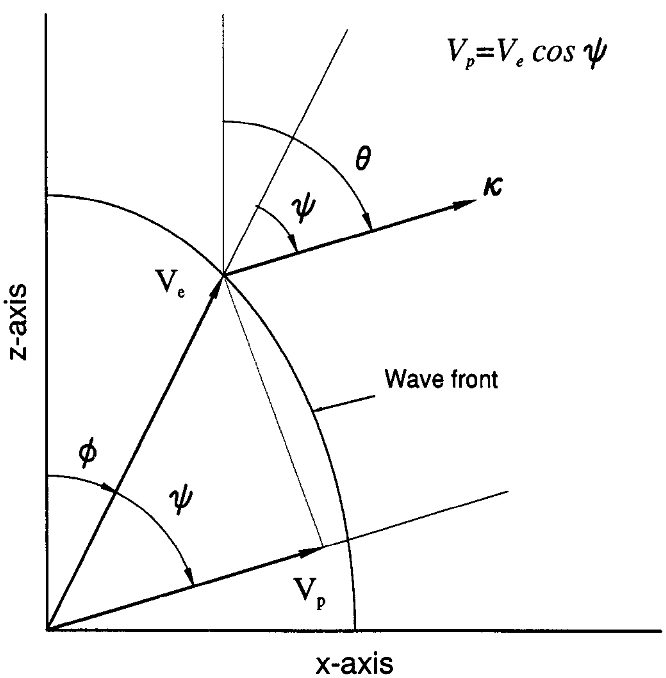


FIG 1. Relation between the energy velocity $\mathbf{V}_e$ and the phase velocity $\mathbf{V}_p$.

where

$$E_m = (\omega\Gamma_{11}' - 2\rho V_m^2)w_m + (\omega\Gamma_{33}' - 2\rho V_m^2)v_m - 2\omega\Gamma_{13}\Gamma_{13}',$$

with

$$v_m = \Gamma_{11} - \rho V_m^2$$

and

$$w_m = \Gamma_{33} - \rho V_m^2,$$

and in terms of the Christoffel components

$$\Gamma_{11} = p_{11}\ell_x^2 + p_{55}\ell_z^2,$$

$$\Gamma_{33} = p_{55}\ell_x^2 + p_{33}\ell_z^2,$$

and

$$\Gamma_{13} = (p_{13} + p_{55})\ell_x\ell_z.$$

These velocities have been obtained in Carcione (1990) for a meridian plane in hexagonal media, which are similar to the expressions for the (x, z)-plane in orthorhombic media. The envelope velocities are computed numerically [i.e., the calculation of $dV_p/d\theta$ in equation (32)].

The properties of the medium are given in Table 1. The relaxation times represent dissipation mechanisms with maximum attenuation at a frequency of $f_0 = 2\pi\omega_0 = 20$ Hz. The 1-D quality factors at ω_0 are indicated below the relaxation times, with $M_\delta = M_2$ for this problem. For a given value of the quality factor $Q_{0\nu}$ at ω_0, the values of the relaxation times are determined by

$$\tau_{\sigma\nu} = (\omega_0 Q_{0\nu})^{-1}[\sqrt{Q_{0\nu}^2 + 1} - 1], \tag{61a}$$

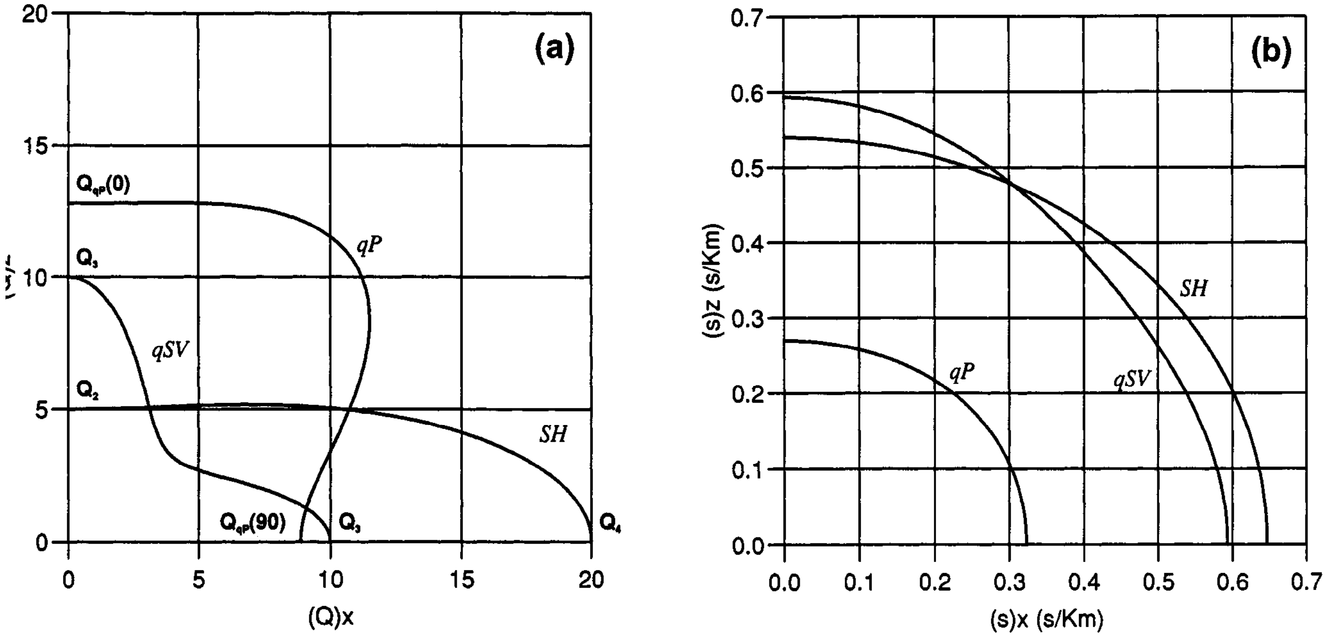
and

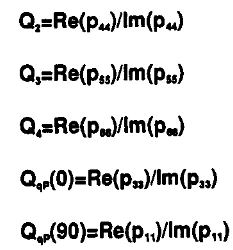
$$\tau_{\epsilon\nu} = \tau_{\sigma\nu} + 2(\omega_0 Q_{0\nu})^{-1}. \tag{61b}$$

The quality factor is defined as the peak strain energy density (19) divided by the dissipated energy density (21). From the appendix, the quality factor for SH-waves is

$$Q_{\mathrm{SH}} = \operatorname{Re}[V_3^2]/\operatorname{Im}[V_3^2], \tag{62}$$

and a similar expression is obtained for the coupled modes (Carcione, 1990). Figure 2a shows a polar representation of the quality factor curves in the (x, z)-plane. Only one quarter of the plane is displayed because of symmetry considerations. The values at the Cartesian axes are defined by the 1-D quality factors Q_ν for the shear waves, and simple functions of the stiffnesses for the ***qP*** wave. The slowness and energy velocity curves are illustrated in Figures 2b and 2c, respectively.





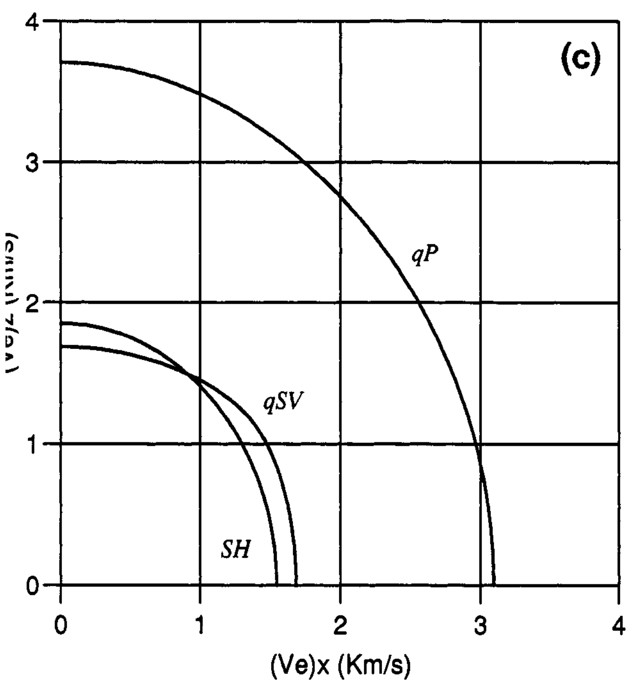


FIG. 2. Polar representations of (a) quality factors, (b) slownesses, and (c) energy velocities of the three propagating modes in the (x, z)-plane of symmetry of the orthorhombic medium defined in Table 1. The values of the quality factors at the Cartesian axes are defined by the quantities $Q_\nu, \nu = 1, 4$ given in the Table.

Table 1. Material properties.

Elasticities (GPa) and density (Kg/m^3)									
c_{11}	c_{12}	c_{13}	c_{22}	c_{23}	c_{33}	c_{44}	c_{55}	c_{66}	ρ
11.7	6.7	7.0	15.4	7.0	17.4	3.8	3.5	3.1	1364.

Relaxation times (s)							
$\tau_{\epsilon 1}$	$\tau_{\sigma 1}$	$\tau_{\epsilon 2}$	$\tau_{\sigma 2}$	$\tau_{\epsilon 3}$	$\tau_{\sigma 3}$	$\tau_{\epsilon 4}$	$\tau_{\sigma 4}$
$8.23\ 10^{-3}$	$7.70\ 10^{-3}$	$9.71\ 10^{-3}$	$6.52\ 10^{-3}$	$8.79\ 10^{-3}$	$7.20\ 10^{-3}$	$8.36\ 10^{-3}$	$7.57\ 10^{-3}$
$Q_1=30$		$Q_2=5$		$Q_3=10$		$Q_4=20$	

The different physical velocities of the *SH* mode versus the propagation angle are represented in Figure 3, with $\theta = 0$ measured from the z-axis. The values at $\theta = 0$ and $\theta = 90$ degrees are, in accordance with Figure 3a, the values of Q_2 and Q_4, respectively. Figure 3a corresponds to the values defined in Table 1, and Figure 3b and Figure 3c represent the low- and high-frequency limits, respectively. Velocity dispersion causes the differences between these two limits, where the energy, group and envelope velocities coincide because of the elastic behavior. The differences in Figure 3a indicate that the group velocity is not a good approximation for the wavefront, and that the envelope velocity closely approaches the energy velocity. On the other hand, by inverting the values of Q_2 and Q_4, the behavior of the velocities changes substantially (see Figure 3d). Surprisingly, the energy and envelope velocities are equal to the phase velocity for this particular case. This implies that anisotropic coefficients for dissipative media depend considerably on the anisotropic dissipation coefficients, such as the ratio Q_2/Q_4 for the pure shear mode. The effects of more dissipation can be appreciated in Figures 3e and 3f, where besides the inversion at the coordinates axes, the differences between the energy and envelope velocities are distinguishable. I demonstrated in the previous section that, unlike elastic media, the energy velocity vector in anelastic media is not normal to the slowness surface. The angle δ between the normal [equation (46)] and the energy velocity vector (37) is represented in Figure 4 versus the propagation angle θ. The deviation is larger in the direction of maximum attenuation, but is not very important from a practical point of view. Figure 5 displays the energy densities (A-7), (A-8), and (A-9) as a function of the propagation angle. They are normalized with respect to the kinetic energy; in elastic media the strain and kinetic energies coincide and do not depend on the propagation direction as in this case.

The analysis of the physical velocities for the coupled modes refers to Figures 6 and 7. It can be shown that the behavior of the *qP* curves at the coordinate axes depends on the relation c_{33}/c_{11}, and cannot be inverted by an appropriate choice of the 1-D quality factors as in the preceding case. The behavior of the *qSV* modes depends mainly on the values of Q_2 and Q_3, the latter defining the values of the attenuation at the coordinate axes. The effects of the choice of these parameters can be appreciated in Figures 7a and 7e where anelasticity strongly influences the shape of the curves at intermediate propagation directions.

An important result of this analysis is that the envelope velocity closely approaches the energy velocity even in the presence of strong attenuation. This means that the envelope velocity at unit propagation time is a good approximation of the wavefront. At least from a practical point of view, this fact indicates that the τp-transform (or plane wave decomposition) of a reflection hyperbola in dissipative anisotropic media can still be a good approximation to the slowness surface of the upper medium (Hake, 1986).

However, we should remember that the analysis was carried out for homogeneous viscoelastic plane waves. These types of waves represent a particular case where the attenuation and propagation vectors have the same direction. This is an exception in the earth since inhomogeneous anelastic waves are generated in the presence of heterogeneities and interfaces. Therefore, a more realistic analysis should consider the case when the propagation and attenuation directions are different. In fact, as pointed out by Winterstein (1987), these effects can be important for Q values less than 15 and for large propagation distances in anelastic heterogeneous media.

CONCLUSIONS

The proper definition of the wavefront should be that given by the location of the energy. For elastic anisotropic media, there is no contradiction since the usual definition of wavefront coincides with the concept of energy velocity. However, this is not the case in dissipative anisotropic media. Moreover, the physical meaning of group velocity breaks down since in the presence of strong attenuation the wave packet spreads considerably. Therefore, for general dissipative media, it is convenient to define the wavefront as the locus of the end of the energy velocity vector. In practice, the envelope velocity closely approaches the energy velocity for all propagation directions, and therefore is a good approximation to the wavefront. This fact has favorable implications for plane-wave decomposition and τp-transform processing techniques, at least for homogeneous anelastic plane waves.

From the analysis, it is seen that the differences in the quality factors along the principal axes strongly influence the velocities and, therefore, the anisotropic coefficients that could be defined from them. On the other hand, the group velocity can be used to describe the wavefront but only for low-loss media involving Q factors of, say, greater than 100. Actually, the group velocity is much simpler to compute than the energy velocity, which involves the calculation of the eigenvectors of the Christoffel equation. In addition, it is proved that, in general, for anelastic media the energy velocity vector is not normal to the slowness surface, and that the propagation vector is not normal to the wavefront. These predictions should be confronted with laboratory experiments. Promising research in this direction was recently reported by Arts and Rasolofosaon (1992) who developed an experimental method for obtaining the slowness and attenuation curves of homogeneous plane waves in general anisotropic viscoelastic rocks.

Future work should consider the case when the propagation and attenuation directions differ, and the numerical analysis of the resulting wavefield with forward modeling techniques (Carcione et al., 1992).

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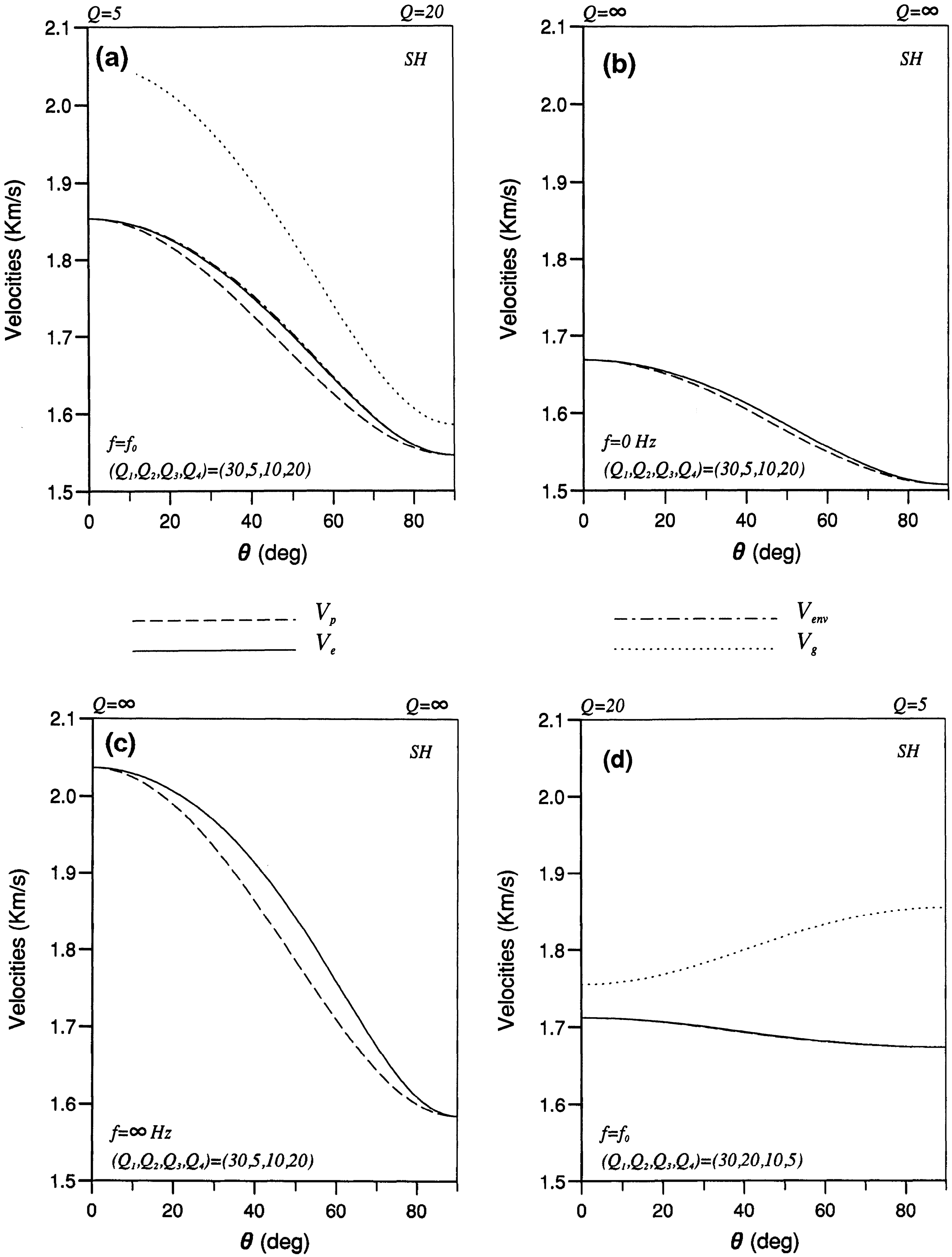

FIG. 3. Physical velocities of the *SH* mode versus the propagation angle θ measured from the z-axis. The values at $\theta = 0$ degrees and $\theta = 90$ degrees in Figure 3a correspond to Q_2 and Q_4 given in Table 1. Note that the envelope velocity V_{env} closely approaches the energy velocity V_e, even for strong dissipation. However, deviations from the elastic case (3b and 3c) are substantial.

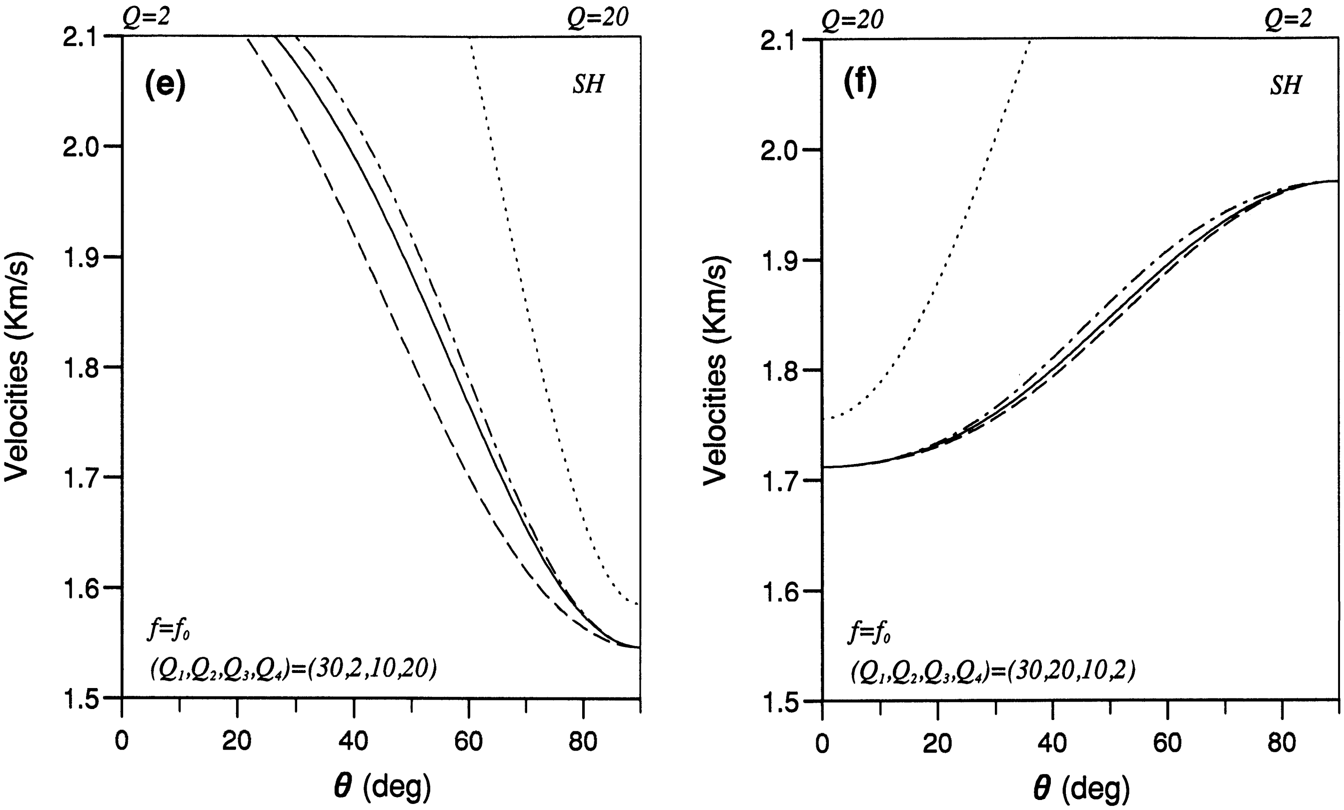

FIG. 3. (continued)

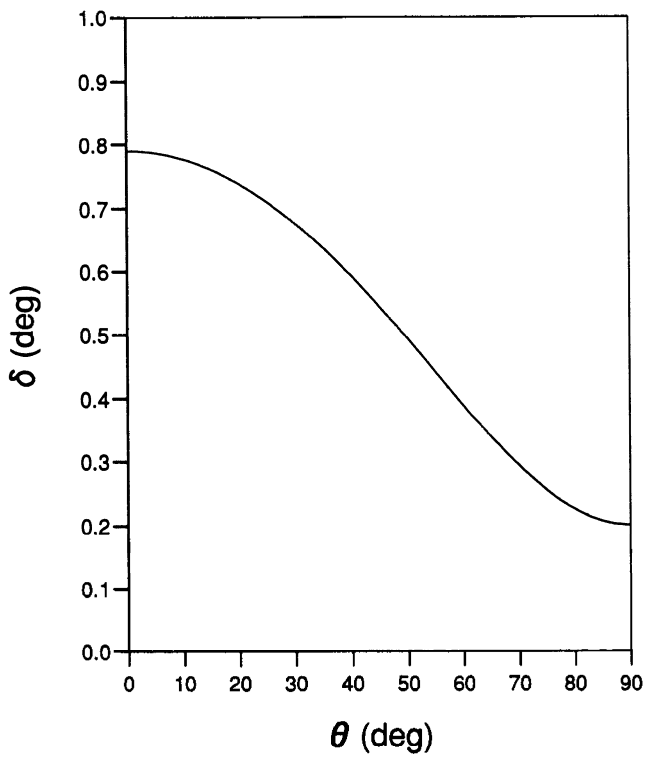

FIG. 4. Deviation of the energy velocity from the normal to the slowness surface versus the propagation angle for the dissipative orthorhombic medium defined in Table 1. In elastic media there is not such deviation.

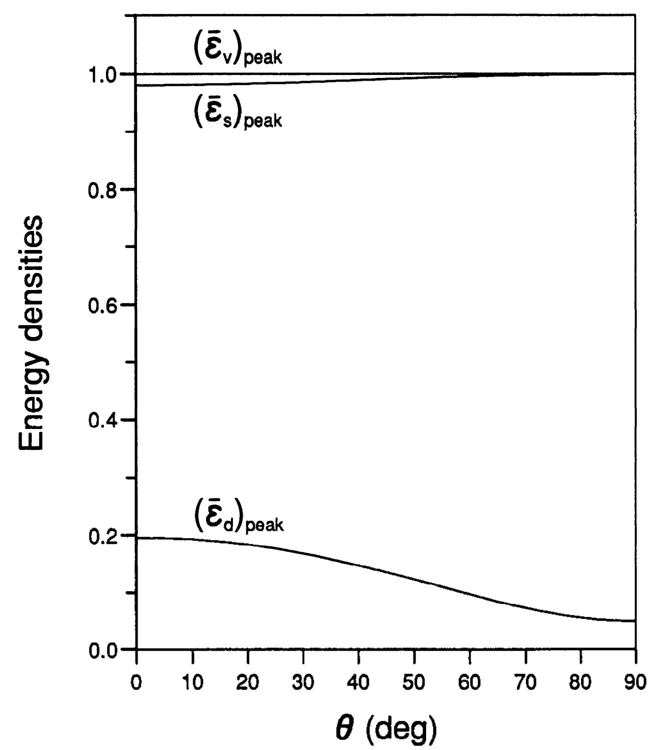

FIG. 5. Normalized energy densities as a function of the propagation angle. The dissipated energy is higher along the direction of maximum attenuation (9 = 0 degrees). In elastic media $(\bar{\varepsilon}_v)_{\text{peak}} = (\bar{\varepsilon}_s)_{\text{peak}} = 1$ for all propagation directions.

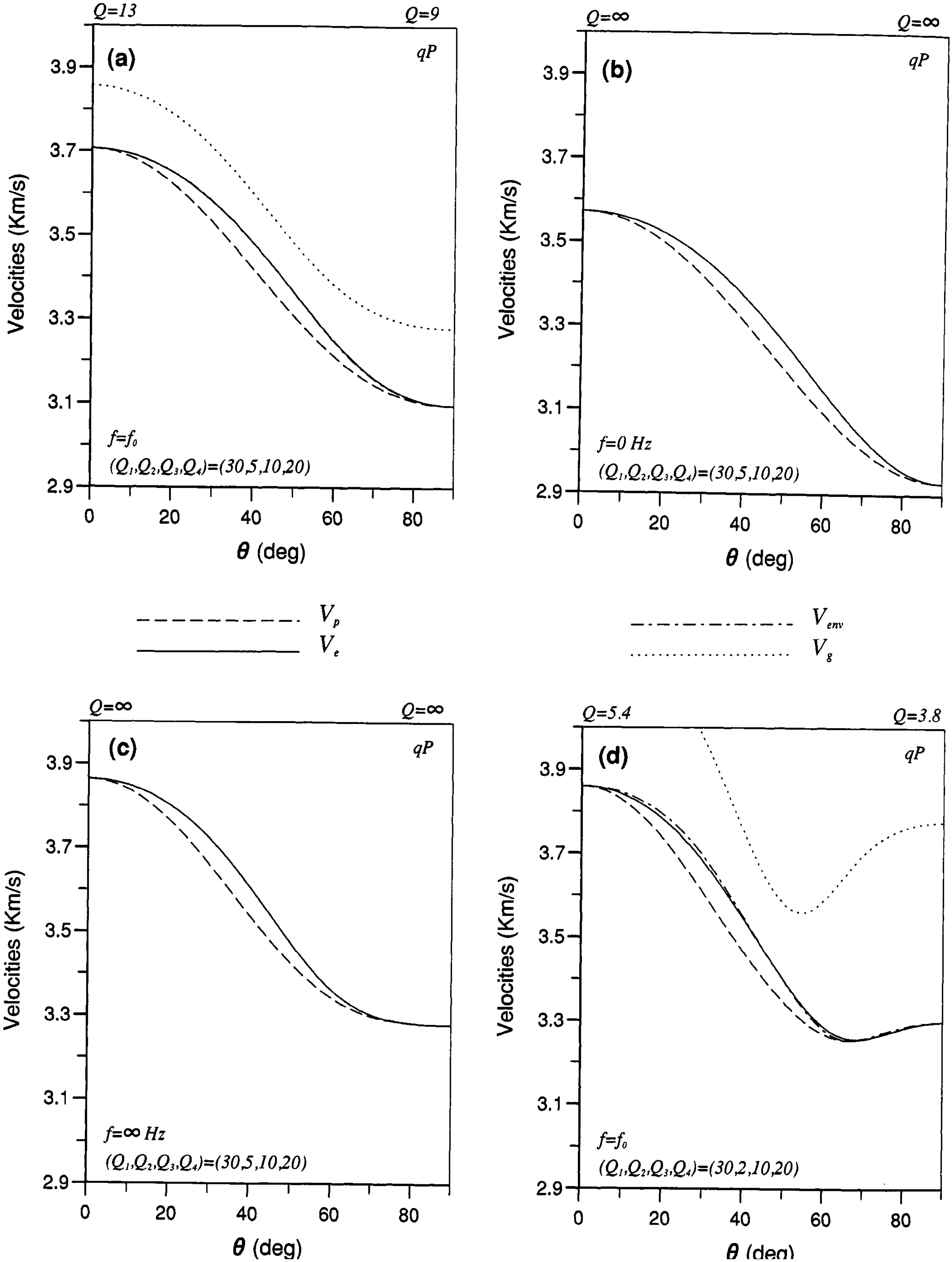

FIG. 6. Physical velocities of the ***qP*** mode versus the propagation angle θ measured from the z-axis. The values at θ = 0 degrees and θ = 90 degrees in Figure 3a correspond to the values indicated in Figure 2a for this wave.

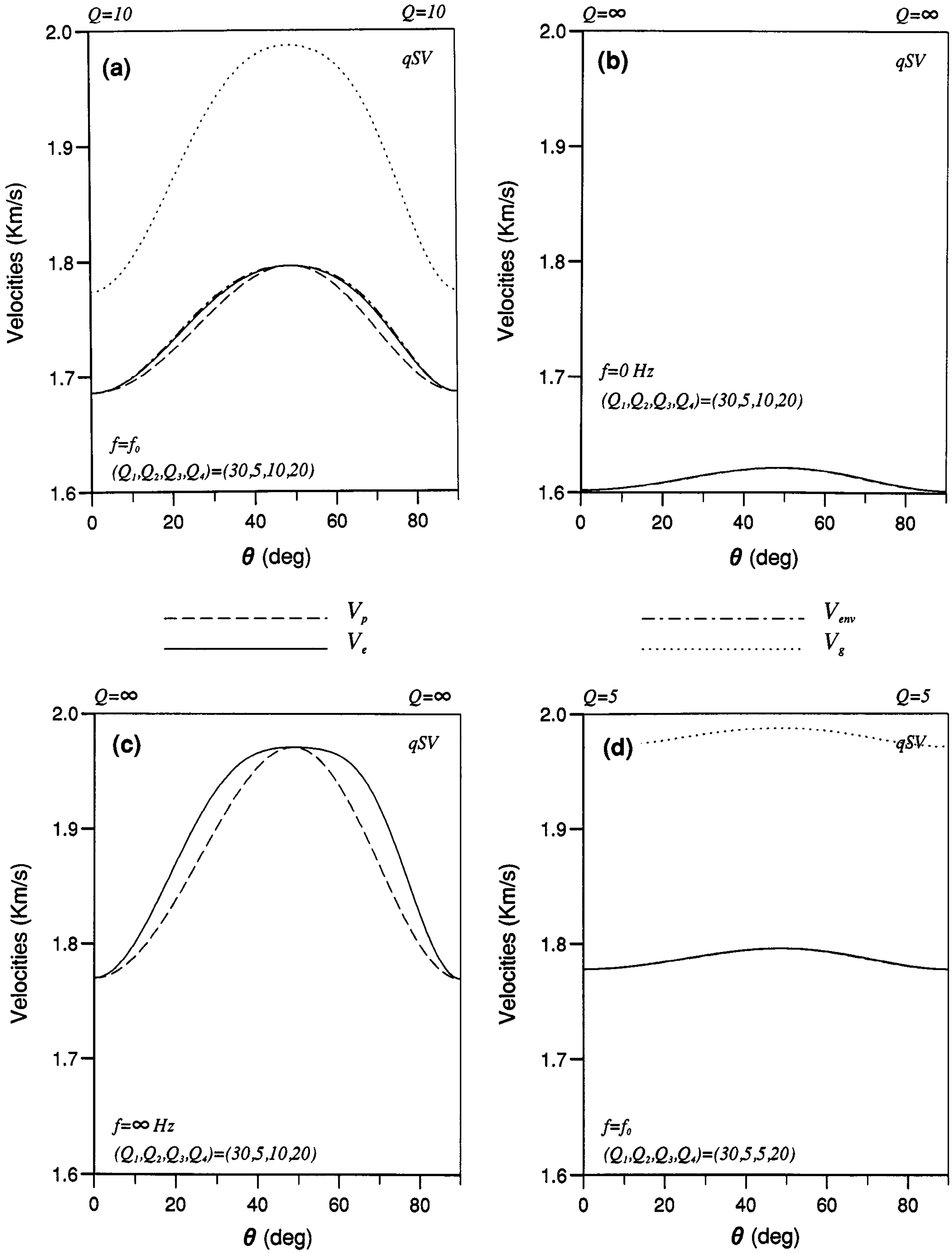

FIG. 7. Physical velocities of the *qSV* mode versus the propagation angle θ measured from the z-axis. The values at θ = 0 degrees and θ = 90 degrees correspond to Q_3 and Q_4 as indicated in Figure 2. Comparison of Figures 6a and 6e shows that the behavior of the velocities is strongly influenced by the choice of different quality factors in different directions.

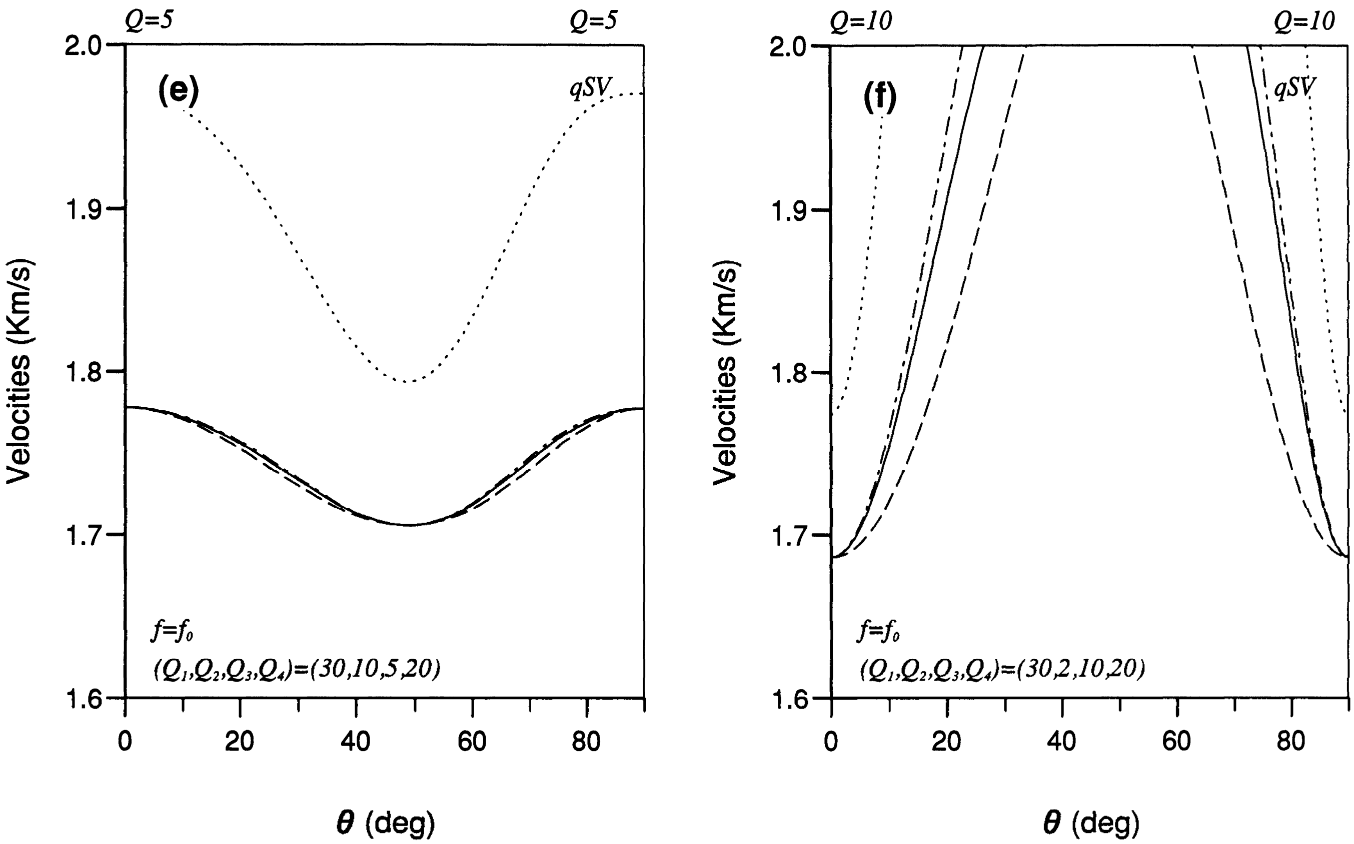

FIG. 7. (continued)

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APPENDIX A

UMOV-POYNTING VECTOR AND ENERGY DENSITIES FOR SH-WAVES IN ORTHORHOMBIC MEDIA

Consider an SH-plane wave propagating in the (x, z)-plane of symmetry. Then, the displacement can be written as

$$\mathrm{U} = U_0 \hat{\mathbf{e}}_y e^{i(\omega t - k_x x - kzz)}. \quad \text{(A-1)}$$

The associated strain components are

$$S_4 = \frac{\mathrm{au}}{\partial z} = -ik_z U_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}, \quad \text{(A-2)}$$

$$S_6 = \frac{\partial u}{\partial x} = -ik_x U_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}. \quad \text{(A-3)}$$

The stress components are

$$T_4 = p_{44} S_4 = -ip_{44} k_z U_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}, \quad \text{(A-4)}$$

$$T_6 = p_{66} S_6 = -ik_x p_{66} U_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}. \quad \text{(A-5)}$$

From equation (16) the Umov-Poynting vector is

$$\mathrm{P} = -\frac{1}{2}\dot{u}^*(T_4\hat{\mathbf{e}}_z + T_6\hat{\mathbf{e}}_x) = \frac{1}{2}\omega^2|U_0|^2 e^{-2\alpha\cdot\mathbf{x}}\frac{1}{V}(\ell_z p_{44}\hat{\mathbf{e}}_z + \ell_x p_{66}\hat{\mathbf{e}}_x). \quad \text{(A-6)}$$

Note that for elastic media the Umov-Poynting vector is real. On the other hand, from the energy balance (15) it can be seen that the imaginary part of the Umov-Poynting vector is closely related to the dissipated energy. The peak kinetic energy density is from equation (20),

$$(\varepsilon_v)_{\text{peak}} = \frac{1}{2}\rho\dot{\mathbf{u}}^{*T}\cdot\dot{\mathbf{u}} = \frac{1}{2}\rho\omega^2|U_0|^2 e^{-2\alpha\cdot\mathbf{x}}, \quad \text{(A-7)}$$

and the peak potential energy density is from equation (19),

$$(\varepsilon_s)_{\text{peak}} = \frac{1}{2}\,\mathrm{Re}\,[p_{44}|S_4|^2 + p_{66}|S_6|^2]$$

$$= \frac{1}{2}\rho\omega^2|U_0|^2 e^{-2\alpha\cdot\mathbf{x}}\frac{\mathrm{Re}\,[V^2]}{|\dot{V}|'^2}, \quad \text{(A-8)}$$

where equations (12) and (35) have been used.

Similarly, the dissipated energy density is

$$(\varepsilon_d)_{AV} = \frac{1}{2}\rho\omega^2|U_0|^2 e^{-2\alpha\cdot\mathbf{x}}\frac{\mathrm{Im}\,[V^2]}{|V|^2}. \quad \text{(A-9)}$$

As can be seen from equations (A-7) and (A-8), the two energy densities are identical for elastic media since V is real. For dissipative media, the difference is given by the factor Re $[V^2]/|V|^2$.