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Constitutive model and wave equations for linear, viscoelastic, anisotropic media

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ABSTRACT

Rocks are far from being isotropic and elastic. Such simplifications in modeling the seismic response of real geological structures may lead to misinterpretations, or even worse, to overlooking useful information. It is useless to develop highly accurate modeling algorithms or to naively use amplitude information in inversion processes if the stress-strain relations are based on simplified rheologies. Thus, an accurate description of wave propagation requires a rheology that accounts for the anisotropic and anelastic behavior of rocks.

This work presents a new constitutive relation and the corresponding time-domain wave equation to model wave propagation in inhomogeneous anisotropic and dissipative media. The rheological equation includes the generalized Hooke's law and Boltzmann's superposition principle to account for anelasticity. The attenuation properties in different directions, associated with the principal axes of the medium, are controlled by four relaxation functions of viscoelastic type. A dissipation model that is consistent with rock properties is the general standard linear solid. This is based on a spectrum of relaxation mechanisms and is suitable for wavefield calculations in the time domain.

One relaxation function describes the anelastic properties of the quasi-dilatational mode and the other three model the anelastic properties of the shear modes. The convolutional relations are avoided by introducing memory variables, six for each dissipation mechanism in the 3-D case, two for the generalized SH-wave equation, and three for the qP - qSVwave equation. Two-dimensional wave equations apply to monoclinic and higher symmetries.

A plane analysis derives expressions for the phase velocity, slowness, attenuation factor, quality factor and energy velocity (wavefront) for homogeneous viscoelastic waves. The analysis shows that the directional properties of the attenuation strongly depend on the values of the elasticities. In addition, the displacement formulation of the 3-D wave equation is solved in the time domain by a spectral technique based on the Fourier method. The examples show simulations in a transversely-isotropic clayshale and phenolic (orthorhombic symmetry).

INTRODUCTION

Modeling waves in a realistic medium involves several aspects. In the first place, the wave equation should be able to properly simulate the body waves and the complete set of waves produced by different types of interface, like for instance, Rayleigh waves at the free surface, Stoneley waves at the solid-solid interface, head waves, and guided waves. These interfaces could have irregular shapes, i.e., the input model should allow arbitrary variations of the physical properties. But these conditions alone are not enough to describe a realistic geological structure, for petrophysical and lithological properties play an important role, particularly in the targets of exploration geophysics, i.e., reservoir rocks.

In the context of reservoir monitoring and evaluation, describing these targets correctly is more important than in the past when an isotropic and elastic rheology was sufficient to model the response of the subsurface on a regional scale. Reservoir rocks can show effective anisotropy in the seismic band, as in the case of cracked limestones. Moreover, fluid-filled cracked rocks and porous sandstones show considerable attenuation properties. In fact, recent experimental work shows that anisotropy of attenuation is more pronounced than anisotropy of elasticity (Hosten et al., 1987; Arts et al., 1992). Thus, a realistic rheology should be able to

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model the anisotropic attenuation characteristic of these rocks.

For long wavelengths, cracked and fracture systems are correctly described by an effective anisotropic rheology. For instance, hexagonal symmetry describes fine layering, and orthorhombic symmetry simulates a finely layered system with vertical fractures. On the other hand, the various dissipation mechanisms can be modeled by a viscoelastic constitutive relation. Since attenuation is caused by a large variety of dissipation mechanisms, it is difficult, if not impossible, to build a general microstructure theory embracing all these mechanisms. When the medium is isotropic, incorporation of attenuation poses no problems since two relaxation functions are enough to describe the anelastic characteristics of the body waves because these modes decouple in a homogeneous medium and attenuation is isotropic. In anisotropic media, the problem is more complex since one has to decide the time (or frequency) dependence of 21 stiffness parameters. However, this does not mean that the medium has to be described by 21 relaxation functions. Recent works by Mehrabadi and Cowin (1990) and Helbig (1993) show that only six of the 21 stiffnesses have an intrinsic physical meaning.

Most applications use the Kelvin-Voigt constitutive law, but this rheology models a particular type of frequency-dependent relaxation matrix (Lamb and Richter, 1966; Auld, 1990). A suitable stress-strain relation should be able to control the quality factors, at least along principal axes of the medium. The rheology presented in this paper is an extension of the 2-D constitutive equation given in Carcione (1990a) and a generalization of the 3-D constitutive equation introduced in Carcione (1990b). Three-dimensional media require particular attention in the properties of the shear modes since quality factors of the slow and fast waves can be different. One relaxation function models the anelastic properties of the quasi-dilatational mode, and three relaxation functions are used to control the quality factors of the shear waves along preferred directions. At the very least, the wave equation should perform correctly for rheologies of cubic to orthorhombic symmetry, for instance, and for rotated versions of these systems.

Modeling wave propagation in isotropic dissipative media has been carried out by several researchers (e.g., Bourbie and Gonzalez-Serrano, 1983; Carcione et al., 1988; Carcione, 1993; Emmerich and Korn, 1987; Kang and McMechan, 1993; Mikhailenko, 1985; Witte and Richards, 1990). Modeling in anisotropic media, including dissipation effects, is more scarce. For instance, Carcione (1990a, b) modeled 2-D wavefields by solving the full wave equation with spectral methods, and Gajewski and Psencik (1992) used a ray method to compute high-frequency seismic wavefields in weakly attenuating media. In the present work, the new rheology is used to compute 3-D snapshots by means of a time-domain spectral technique (Tal-Ezer et al., 1990).

The paper is organized as follows. The first section introduces the 3-D time-domain constitutive relation, the strain memory tensors, and memory variables, and from these, the stress-strain relations. The second section derives the first-order partial differential equations of the memory variables in time. The next two sections consider the alternative 3-D velocity-stress formulation, and the 2-D SH and qP - qSV displacement formulations. The physics of wave propagation is investigated in the last two sections through a plane-wave analysis and simulations, with examples that indicate how to treat typical problems.

CONSTITUTIW RELATION

Notation and convention

Let f and g be scalar time-dependent functions. The Riemann convolution off with g is defined by

$$f * g = \begin{cases} 0: & t < 0 \\ \int_0^t f(\tau) g(t-\tau)\, d\tau: & t \geq 0, \end{cases} \tag{1}$$

where t is the time variable. Hooke's law can be expressed in 3-D space or in 6-D space depending on whether the stress and the strain are tensors or vectors. By convention, we take A as a 6-D vector and A as the corresponding symmetric 3 x 3 tensor. The definition of convolution may be extended easily to include vectors and tensors (or matrices):

$$f * \underset{\sim}{\mathbf{A}} = \int_0^t f(\tau) \underset{\sim}{\mathbf{A}}(t-\tau)\, d\tau, \tag{2a}$$

$$\underset{\sim}{\mathbf{\Psi}} * \mathbf{A} = \int_0^t \underset{\sim}{\mathbf{\Psi}}(\tau) \cdot \mathbf{A}(t-\tau)\, d\tau, \tag{2b}$$

where $\underset{\sim}{\mathbf{\Psi}}$ is a 6 x 6 matrix, and the dot denotes ordinary matrix multiplication.

As a convention, any function $f(t)$ is said to be of Heaviside type if the past history of f up to time t = 0 vanishes. This is,

$$f'(t) = f'(t) H(t), \tag{3}$$

where $H(t)$ is the step function and there is no restriction on f. If f and g are of Heaviside type, the Boltzmann operation (e.g., Leitman and Fisher, 1984) is defined by

$$f \odot g = \mathring{f} g + (\dot{f}' H) * g, \tag{4}$$

where $\mathring{f} = f'(t = 0) = f(t = 0^+)$. The dot denotes time differentiation. Note that the Boltzmann operation is the time derivative of the convolution between f and g.

Let subindices i, j, k, and m take values from 1 to 3 for the three Cartesian coordinates x, y, and z, respectively. In general, the Einstein convention for repeated indices will be used. The symbol *, used as superscript, means complex conjugate. The operators Re[•] and Im[•] take real and imaginary parts, respectively. The trace of $\underset{\sim}{\mathbf{A}}$ is the sum of its diagonal elements and it is denoted by $\mathrm{tr}\underset{\sim}{\mathbf{A}}$. Finally, vectors are written as columns with the superscript "T" denoting the transpose.

Boltzmann law

The most general constitutive relation for an anisotropic and linear viscoelastic medium can be expressed as (Carcione, 1990b)

$$\mathbf{T} = \dot{\underset{\sim}{\mathbf{\Psi}}} * \mathrm{S}, \quad (T_I = \psi_{IJ} * \dot{S}_J,\ I, J = 1, \ldots, 6), \tag{5}$$

where

$$\mathbf{T} = [T_1, T_2, T_3, T_4, T_5, T_6]^{\mathrm{T}} = [\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{xz}, \sigma_{xy}]^{\mathrm{T}} \quad (6)$$

is the stress vector,

$$\mathbf{S} = [S_1, S_2, S_3, S_4, S_5, S_6]^{\mathrm{T}} = [\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}]^{\mathrm{T}} \quad (7)$$

is the strain vector, and $\underline{\Psi}$ is the symmetric relaxation matrix, which in its general form has 21 independent components (Fabrizio and Morro, 1988). Stress, strain, and relaxation matrix depend on the position vector x = (x, y, z) and the time variable, and strain is related to the displacement field u = (u_x, u_y, u_z) by the usual strain-displacement relations (e.g., Auld, 1990), where $\gamma_{ij} = 2\varepsilon_{ij}$, $i \neq j$. Equation (5) is called the Boltzmann superposition principle, or simply, the Boltzmann law.

A class of constitutive equations for anisotropic-viscoelastic media based on two relaxation functions was introduced by Carcione (1990b). In the present work, two additional relaxation functions are included to model in more detail the anelastic properties of the shear modes. The relaxation matrix of this new rheology is

$$\underline{\Psi} = \begin{bmatrix} \psi'_{11} & \psi'_{12} & \psi'_{13} & c_{14} & c_{15} & c_{16} \\ & \psi'_{22} & \psi'_{23} & c_{24} & c_{25} & c_{26} \\ & & \psi'_{33} & c_{34} & c_{35} & c_{36} \\ & & & c_{44}\chi_2 & c_{45} & c_{46} \\ & & & & c_{55}\chi_3 & c_{56} \\ & & & & & c_{66}\chi_4 \end{bmatrix} H, \quad (8)$$

where

$$\psi_{I(I)} = c_{I(I)} - D + K\chi_1 + \frac{4}{3} G\chi_\delta \qquad \text{for } I = 1, 2, 3, \quad (9a)$$

$$\psi'_{IJ} = c_{IJ} - D + 2G + K\chi_1 - \frac{2}{3} G\chi_\delta \qquad \text{for } I, J = 1, 2, 3;\ I \neq J. \quad (9b)$$

Here c_{IJ}, for $I, J = 1, \ldots, 6$ are the low-frequency limit (relaxed) elasticities, and

$$K = D - \frac{4}{3} G, \quad (10a)$$

where

$$D = \frac{1}{3}(c_{11} + c_{22} + c_{33}), \qquad G = \frac{1}{3}(c_{44} + c_{55} + c_{66}). \quad (10b)$$

The quantities χ_ν are dimensionless relaxation functions: V = 1 is for the quasi-dilatational mode, and v = 2, 3, 4 are for the shear waves. In equations (9a) and (9b), χ_δ is a shear relaxation function that can be chosen such that δ = 2, 3, or 4. The choice of the relaxation matrix (8) can be justified on the following grounds: the mean stress Θ_σ = trT/3 can be expressed in terms of the mean strain Θ_ε = trS/3 as

$$\Theta_a = \frac{1}{3}(c_{J1} + c_{J2} + c_{J3})S_J + K\dot{\Theta}_\varepsilon * [(\chi_1 - 1)H], \quad (11)$$

which depends only on the first relaxation function involving quasi-dilatational dissipation mechanisms. The trace of the stress tensor is invariant under transformation of the coordinate system. This fact assures that Θ_σ depends only on χ_1 in any system. Moreover, it can be shown that the deviatoric stresses $T_I - \Theta_\sigma$, $I \leq 3$ and T_I, $I > 3$, solelydependon the relaxation functions associated with the quasi-shear mechanisms, i.e., v > 1. Equation (8) gives the elasticity matrix of the generalized Hooke's law in the anisotropic-elastic limit when $\chi_\nu \rightarrow 1$, and gives the 3-D isotropic-viscoelastic constitutive relation in the isotropic limit (Carcione, 1993).

The relaxation matrix (8) generalizes that given in Carcione (1990b), since here three relaxation functions instead of one are used to describe the anelastic properties of the shear modes. In this way, it is possible to control the quality factor along three preferred directions, like, for instance, the principal axes of the anisotropic medium.

The choice of the relaxation functions depends on the symmetry system, since the attenuation symmetries follow the symmetry of the crystallographic form of the material (Neumann, 1885). For isotropic, cubic, and hexagonal media, two relaxation functions are necessary and sufficient to model the anelastic properties. For instance, for transversely isotropic media with symmetry axis in the z-direction, $\chi_\delta = \chi_2$ = x39 and $\psi_{66} = (\psi_{11} - \psi_{12})/2$ should be taken to preserve transverse isotropy. For azimuthally anisotropic media, the relaxation matrix is rotated conveniently (Carcione, 1990b).

The theory assumes the following form for the relaxation functions:

$$\chi_\nu(t) = 1 - \frac{1}{L_\nu} \sum_{\ell=1}^{L_\nu} \left(1 - \frac{\tau_{\varepsilon\ell}^{(\nu)}}{\tau_{\sigma\ell}^{(\nu)}}\right) e^{-t/\tau_{\sigma\ell}^{(\nu)}}, \qquad \nu = 1, \ldots, 4, \quad (12)$$

where $\tau_{\varepsilon\ell}^{(\nu)}$ and $\tau_{\sigma\ell}^{(\nu)}$ are material relaxation times such that $\tau_{\varepsilon\ell}^{(\nu)} \geq \tau_{\sigma\ell}^{(\nu)}$. Each pair of relaxation times describes a dissipation mechanism. The form (12) corresponds to a rational function of $s = i\omega$ in the frequency domain, where ω is the angular frequency. In fact, the complex modulus is the time Fourier transform of $d[\chi_\nu(t)H(t)]/dt$. It yields

$$M_\nu(s) = \frac{1}{L_\nu} \sum_{\ell=1}^{L_\nu} \frac{1 + s\tau_{\varepsilon\ell}^{(\nu)}}{1 + s\tau_{\sigma\ell}^{(\nu)}}. \quad (13)$$

Equation (12) represents the relaxation function of a generalized standard linear solid consisting of L_ν single elements connected in parallel. Note that $\tau_{\varepsilon\ell}^{(\nu)} \rightarrow \tau_{\sigma\ell}^{(\nu)}$ gives the low-frequency elastic limit where $\chi_\nu \rightarrow 1$ and $M_\nu \rightarrow 1$. The quality factor corresponding to each relaxation function is Q = $\mathrm{Re}[M_\nu]/\mathrm{Im}[M_\nu]$. The relaxation functions (12) are sufficiently general to describe any type of frequency behavior of attenuation and velocity dispersion.

Strain memory variables

Applying the Boltzmann operation (4) to the stress-strain relation (5) gives

$$T_I = (A_{IJ} + A_{IJ}^{(\nu)}\mathring{\chi}_\nu)S_J + A_{IJ}^{(\nu)} \sum_{\ell=1}^{L_\nu} E_{J\ell}^{(\nu)}, \tag{14}$$

where the A's only depend on x through the elasticities c_{IJ}, and

$$E_{J\ell}^{(\nu)} = \phi_{\nu\ell} * S_J, \qquad J = 1, \ldots, 6,$$

$$(\ell = 1, \ldots, L_\nu, \qquad \nu = 1, \ldots, 4) \tag{15}$$

are the components of the strain memory vector $\mathbf{E}_\ell^{(\nu)}$, with

$$\phi'_{\nu\ell}(t) = \frac{1}{L_\nu \tau_{\sigma\ell}^{(\nu)}}\left(1 - \frac{\tau_{\epsilon\ell}^{(\nu)}}{\tau_{\sigma\ell}^{(\nu)}}\right)e^{-t/_{\sigma\ell}^{(\nu)}} \tag{16}$$

being the response function of the ℓth dissipation mechanism.

Note that in 3-D space the strain memory is a symmetric tensor given by

$$\underset{\sim}{\mathbf{E}}_\ell^{(\nu)} = \begin{bmatrix} e_{xx\ell}^{(\nu)} & e_{xy\ell}^{(\nu)} & e_{xz\ell}^{(\nu)} \\ & e_{yy\ell}^{(\nu)} & e_{yz\ell}^{(\nu)} \\ & & e_{zz\ell}^{(\nu)} \end{bmatrix} \equiv \phi_{\nu\ell} * \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ & \varepsilon_{yy} & \gamma_{yz} \\ & & \varepsilon_{zz} \end{bmatrix}$$

$$= \phi_{\nu\ell} * \underset{\sim}{\mathbf{S}}, \tag{17}$$

corresponding to the ℓth dissipation mechanism of the relaxation function χ_ν. This tensor contains the past history of the material due to the ℓ mechanism. In the elastic case, e.g., when $\tau_{\epsilon\ell}^{(\nu)} \to \tau_{\sigma\ell}^{(\nu)}$, the strain memory for this mechanism vanishes, since $\phi'_{\nu\ell} \to 0$. As the strain tensor, the memory tensor $\underset{\sim}{\mathbf{E}}_\ell^{(\nu)}$ possesses the unique decomposition

$$\underset{\sim}{\mathbf{E}}_\ell^{(\nu)} = \underset{\sim}{\mathbf{E}}_{0\ell}^{(\nu)} + \left(\frac{1}{3}\,\mathrm{tr}\underset{\sim}{\mathbf{E}}_\ell^{(\nu)}\right)\underset{\sim}{\mathbf{1}}, \qquad \mathrm{tr}\underset{\sim}{\mathbf{E}}_{0\ell}^{(\nu)} = 0, \tag{18}$$

where the traceless symmetric tensor $\underset{\sim}{\mathbf{E}}_{0\ell}^{(\nu)}$ is the deviatoric strain memory tensor, and $\underset{\sim}{\mathbf{1}}$ denotes the 3 x 3 identity matrix. Then, the dilatational and shear memory variables are defined by

$$e_{1\ell} = \mathrm{tr}\underset{\sim}{\mathbf{E}}_\ell^{(1)}$$

and

$$e_{ij\ell} = [\underset{\sim}{\mathbf{E}}_{0\ell}^{(\nu)}]_{ij}, \tag{19}$$

respectively, where $\nu = \delta$ for $i = j$, $\nu = 2$ for $ij = 23$, $\nu = 3$ for $ij = 13$, and $\nu = 4$ for $ij = 14$. Following a similar procedure given in Carcione (1990b), the stress-strain relations, in terms of the strain components and memory variables, are

$$\sigma_{xx} = \hat{c}_{11}\varepsilon_{xx} + \hat{c}_{12}\varepsilon_{yy} + \hat{c}_{13}\varepsilon_{zz} + c_{14}\gamma_{yz} + c_{15}\gamma_{xz} + c_{16}\gamma_{xy} + K\sum_{\ell=1}^{L_1} e_{1\ell} + 2G\sum_{\ell=1}^{L_\delta} e_{11\ell}, \tag{20a}$$

$$\sigma_{yy} = \hat{c}_{12}\varepsilon_{xx} + \hat{c}_{22}\varepsilon_{yy} + \hat{c}_{23}\varepsilon_{zz} + c_{24}\gamma_{yz} + c_{25}\gamma_{xz} + c_{26}\gamma_{xy} + K\sum_{\ell=1}^{L_1} e_{1\ell} + 2G\sum_{\ell=1}^{L_\delta} e_{22\ell}, \tag{20b}$$

$$\sigma_{zz} = \hat{c}_{13}\varepsilon_{xx} + \hat{c}_{23}\varepsilon_{yy} + \hat{c}_{33}\varepsilon_{zz} + c_{34}\gamma_{yz} + c_{35}\gamma_{xz} + c_{36}\gamma_{xy} + K\sum_{\ell=1}^{L_1} e_{1\ell} - 2G\sum_{\ell=1}^{L_\delta}(e_{11\ell} + e_{22\ell}), \tag{20c}$$

$$\sigma_{yz} = c_{14}\varepsilon_{xx} + c_{24}\varepsilon_{yy} + c_{34}\varepsilon_{zz} + \hat{c}_{44}\gamma_{yz} + c_{44}\sum_{\ell=1}^{L_2} e_{23\ell} + c_{45}\gamma_{xz} + c_{46}\gamma_{xy}, \tag{20d}$$

$$\sigma_{xz} = c_{15}\varepsilon_{xx} + c_{25}\varepsilon_{yy} + c_{35}\varepsilon_{zz} + c_{45}\gamma_{yz} + \hat{c}_{55}\gamma_{yz} + c_{55}\sum_{\ell=1}^{L_3} e_{13\ell} + c_{56}\gamma_{xy}, \tag{20e}$$

$$\sigma_{xy} = c_{16}\varepsilon_{xx} + c_{26}\varepsilon_{yy} + c_{36}\varepsilon_{zz} + c_{46}\gamma_{yz} + c_{56}\gamma_{xz} + \hat{c}_{66}\gamma_{xz} + c_{66}\sum_{\ell=1}^{L_4} e_{12\ell}, \tag{20f}$$

$$\sigma_{xy} = c_{16}\varepsilon_{xx} + c_{26}\varepsilon_{yy} + c_{36}\varepsilon_{zz} + 2c_{46}\varepsilon_{yz} + c_{56}\varepsilon_{xz} + G\sum_{\ell=1}^{L_4} e_{12\ell} + 2[c_{66} + G(\mathring{\chi}_\nu - 1)]\varepsilon_{xy}, \tag{20g}$$

where

$$\hat{c}_{IJ} = \mathring{\psi}_{IJ}, \qquad I, J = 1, \ldots, 6 \tag{21}$$

are the high-frequency limit elasticities, or unrelaxed relaxation components. The second component σ_{xy} corresponds to transversely-isotropic media. The terms containing the strain components describe the instantaneous (unrelaxed) response of the medium, and the terms containing memory variables involve the previous states of deformation. Note that since $\underset{\sim}{\mathbf{E}}_{0\ell}^{(\delta)}$ is traceless, $e_{11\ell} + e_{22\ell} + e_{33\ell} = 0$, and the number of independent variables is six, i.e., the number of independent components of the strain tensor. The nature of the terms can be easily identified: in the diagonal stress components, the dilatational memory variable is multiplied by a generalized incompressibility K, and the shear memory variables are multiplied by a generalized rigidity modulus G.

MEMORY VARIABLES EQUATIONS

Application of the Boltzmann operation (4) to the deviatoric part of equation (17) gives

$$\dot{\underset{\sim}{\mathbf{E}}}_{0\ell}^{(\nu)} = \phi_{\nu\ell}J \odot \underset{\sim}{\mathbf{S}}_0 = \mathring{\phi}_{\nu\ell}\underset{\sim}{\mathbf{S}}_0 + (\mathring{\phi}'_{\nu\ell}H) * \underset{\sim}{\mathbf{S}}_0. \tag{22}$$

where $\underset{\sim}{\mathbf{S}}_0$ is the deviatoric strain tensor. Since $\phi'_{\nu\ell} = \phi'_{\nu\ell}/\tau_{\sigma\ell}^{(\nu)}$, equation (22) becomes

$$\dot{\underset{\sim}{\mathbf{E}}}_{0\ell}^{(\nu)} = \mathring{\phi}_{\nu\ell}\underset{\sim}{\mathbf{S}}_0 - \frac{\phi_{\nu\ell}}{\tau_{\sigma\ell}^{(\nu)}} * \underset{\sim}{\mathbf{S}}_0 = \mathring{\phi}_{\nu\ell}\underset{\sim}{\mathbf{S}}_0 - \frac{\underset{\sim}{\mathbf{E}}_{0\ell}^{(\nu)}}{\tau_{\sigma\ell}^{(\nu)}}. \tag{23}$$

Similarly, applying the Boltzmann operation to $\mathrm{tr}\underset{\sim}{\mathbf{E}}_\ell^{(1)}$ gives

$$tr\dot{\underset{\sim}{\mathbf{E}}}_{\ell}^{(1)} = \mathring{\phi}_{1\ell}\,\mathrm{tr}\underset{\sim}{\mathbf{S}}_0 - \frac{tr\underset{\sim}{\mathbf{E}}_{\ell}^{(1)}}{\tau_{\sigma\ell}^{(1)}} \quad (24)$$

The explicit equations in terms of the memory variables are

$$\dot{e}_{1\ell} = 3\mathring{\phi}_{1\ell}\Theta_{\varepsilon} - \frac{e_{1\ell}}{\tau_{\sigma\ell}^{(1)}}, \qquad \ell = 1, \cdots, L_1, \quad (25a)$$

$$\dot{e}_{11\ell} = \mathring{\phi}_{\delta\ell}(\varepsilon_{xx} - \Theta_{\varepsilon}) - \frac{e_{11\ell}}{\tau_{\sigma\ell}^{(\delta)}}, \quad \ell = 1, \ldots, \quad L_{\delta}, \quad (25b)$$

$$\dot{e}_{22\ell} = \mathring{\phi}_{\delta\ell}(\varepsilon_{yy} - \Theta_{\varepsilon}) - \frac{e_{22\ell}}{\tau_{\sigma\ell}^{(\delta)}} \qquad \ell = 1, \ldots, L_{\delta}, \quad (25c)$$

$$\dot{e}_{23\ell} = \mathring{\phi}_{2\ell}\gamma_{yz} - \frac{e_{23\ell}}{\tau_{\sigma\ell}^{(2)}}, \qquad \ell = 1, \ldots, L_2, \quad (25d)$$

$$\dot{e}_{13\ell} = \mathring{\phi}_{3\ell}\gamma_{xz} - \frac{e_{13\ell}}{\tau_{\sigma\ell}^{(3)}}, \qquad \ell = 1, \ldots, L_3, \quad (25e)$$

$$\dot{e}_{12\ell} = \mathring{\phi}_{4\ell}\gamma_{xy} - \frac{e_{12\ell}}{\tau_{\sigma\ell}^{(4)}}, \qquad \ell = 1, \ldots, L_4, \quad (25f)$$

where $\Theta_{\varepsilon} = \mathrm{tr}\underset{\sim}{\mathbf{S}}/3$.

WAVE EQUATION FORMULATIONS

Two different formulations of the anisotropic-viscoelastic wave equation can be considered. The first is the displacement formulation, where the unknown variables are the displacement field u and memory variables (19). In this case, the wave equation is formed with the strain-displacement relations, the stress-strain relations (20a)–(20g), the memory variables equations (25a)–(25f), and the equations of momentum conservation (e.g., Auld, 1990):

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} = \rho\ddot{u}_x + f_x, \quad (26a)$$

$$\frac{\partial\sigma_{xy}}{\mathrm{ax}} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z} = \rho\ddot{u}_y + f_y, \quad (26b)$$

$$\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} = \rho\ddot{u}_z + f_z, \quad (26c)$$

where f(x, t) = (f_x, f_y, f_z) is the body force vector and p(x) is the density. When using spectral techniques for computing the spatial derivatives, this formulation is convenient for the Fourier pseudospectral method. The algorithm introduced in Tal-Ezer et al., (1990) uses this method and a spectral time integration technique based on a polynomial expansion of the evolution operator.

On the other hand, in the presence of boundary conditions, like at the surface of the earth, the use of nonperiodic pseudospectral differential operators requires a velocity-stress formulation. For instance, in Carcione (1992), this formulation is used in conjunction with the Chebychev differential operator in order to properly model the surface Rayleigh waves. In this case, the unknown field variables are the particle velocity vector v = $(v_x, v_y, v_z) = \dot{\mathbf{u}}$, the stress components, and the time derivative of the memory variables. The first formulation is second-order in time, while the velocity-stress formulation is first-order in time. Moreover, in this case the material properties are not differentiated explicitly, as in the displacement formulation.

TWO DIMENSIONAL WAVE EQUATIONS

Two-dimensional wave equations are referred to in the geophysical literature as *SH* and *P - SV*. When exciting an isotropic medium with a line source perpendicular to a given plane, one of the shear waves is decoupled from the other two modes and has antiplane motion. This is strictly true if the material properties are uniform in the direction perpendicular to the plane. This situation can be generalized up to monoclinic media provided that the plane of propagation is the plane of symmetry of the medium. In fact, propagation in the plane of mirror symmetry of a monoclinic medium is the most general situation for which pure shear waves exist at all propagation angles.

SH-wave equation

Let us assume that the (x, z) plane is the symmetry plane of a monoclinic medium. The antiplane assumption that displacement u = $u_y(x, z, t)\hat{\mathbf{e}}_\nu$ implies that the only nonzero stress components are $\sigma_{yz}(x, z, t)$ and $\sigma_{xy}(x, z, t)$. Following the same steps to get the 3-D wave equation, the displacement formulation of the SH-wave equation is given by

1) The stress-strain relations

$$\sigma_{yz} = \hat{c}_{44}\gamma_{yz} + c_{46}\gamma_{xy} + c_{44}\sum_{\ell=1}^{L_2} e_{23\ell}, \quad (27a)$$

$$\sigma_{xy} = c_{46}\gamma_{yz} + \hat{c}_{66}\gamma_{xy} + c_{66}\sum_{\ell=1}^{L_4} e_{12\ell}, \quad (27b)$$

2) The memory variables equations (25d) and (25f).
3) Newton's equation (26b) with $\partial/\partial y = 0$.

***qP − qSV*-wave** equation

A line source perpendicular to the (x, z) plane generates a displacement field u = $u_x\hat{\mathbf{e}}_x + u_z\hat{\mathbf{e}}_z$. The relevant elasticities are

$$\begin{bmatrix} c_{11} & c_{13} & c_{15} \\ & c_{33} & c_{35} \\ & & c_{55} \end{bmatrix},$$

where here, instead of equations (10a) and (lob),

$$K = D - G, \quad D = \frac{1}{2}(c_{11} + c_{33}), \quad G = c_{55}. \quad (28)$$

and

$$\psi'_{I(I)} = c_{I(I)} - D + K\chi_1 + G\chi_\delta \qquad \text{for } I = 1 \text{ or } 3, \quad (29a)$$

$$\psi'_{13} = c_{13} - D + 2G + K_{\chi 1} - G_{\chi\delta} \quad (29b)$$

Then, the $qP - qSV$ wave equation involves the following equations:

1) The stress-strain relations

$$\sigma_{xx} = \hat{c}_{11}\varepsilon_{xx} + \hat{c}_{13}\varepsilon_{zz} + c_{15}\gamma_{xz} + K\sum_{\ell=1}^{L_1} e_{1\ell} + G\sum_{\ell=1}^{L_\delta} e_{11\ell}, \quad (30a)$$

$$\sigma_{zz} = \hat{c}_{13}\varepsilon_{xx} + \hat{c}_{33}\varepsilon_{zz} + c_{35}\gamma_{xz} + K\sum_{\ell=1}^{L_1} e_{1\ell} - G\sum_{\ell=1}^{L_\delta} e_{11\ell}, \quad (30b)$$

$$\sigma_{xz} = c_{15}\varepsilon_{xx} + c_{35}\varepsilon_{zz} + \hat{c}_{55}\gamma_{yz} + c_{55}\sum_{\ell=1}^{L_3} e_{13\ell} \quad (30c)$$

2) The memory variables equations (25a), (25b), and (25e).
3) Newton's equations (26a) and (26c) with $\partial/\partial y = 0$.

PLANE-WAVE THEORY

The theory of propagation of viscoelastic waves in isotropic media has been investigated by several researchers, for instance Krebes (1984). However, research into anisotropic media is relatively recent. Carcione (1990b) obtained the expressions of the phase, group, and energy velocities, and quality factors for homogeneous viscoelastic plane waves in a transversely-isotropic medium.

A general solution for the displacement field representing viscoelastic plane waves is of the form

$$\mathbf{U} = \mathbf{U}_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}, \quad (31)$$

where $\mathbf{U}_0$ represents a constant complex vector, and ω is the angular frequency. For homogeneous waves, the wavenumber can be written as

$$\mathbf{k} = (\kappa - i\alpha)\hat{\boldsymbol{\kappa}} \equiv k\hat{\boldsymbol{\kappa}}, \quad (32)$$

where

$$\hat{\boldsymbol{\kappa}} = \ell_x\hat{\mathbf{e}}_x + \ell_y\hat{\mathbf{e}}_y + \ell_z\hat{\mathbf{e}}_z \quad (33)$$

defines the propagation direction through the direction cosines ℓ_x, ℓ_y, and ℓ_z. For this kind of wave, planes of constant phase (planes normal to the propagation vector κ) are parallel to planes of constant amplitude (defined by $\boldsymbol{\alpha}^T \cdot \mathbf{x} = const.$). Substituting the plane wave (31) into the stress-strain relation (5) yields

$$\mathbf{T}(\omega) = \underset{\sim}{\mathbf{p}}(\omega) \cdot \mathbf{S}, \quad (34)$$

where T(o) an.d S are related to stress an.d strain by formulas analogous to equation (31), while the components of the complex stiffness matrix are

$$p_{IJ}(\omega) = \int_{-\infty}^{\infty} \dot{\psi}_{IJ}(t)e^{-i\omega t}\,dt. \quad (35)$$

For homogeneous waves, the dispersion equation takes the following simple form (Carcione, 1990b):

$$\det[\underset{\sim}{\mathbf{L}} \cdot \underset{\sim}{\mathbf{p}} \cdot \underset{\sim}{\mathbf{L}}^T - \rho V^2 \underset{\sim}{\mathbf{I}}] = 0, \quad (36)$$

where

$$\underset{\sim}{\mathbf{L}} = \begin{bmatrix} \ell_x & 0 & 0 & 0 & \ell_z & \ell_y \\ 0 & \ell_y & 0 & \ell_z & 0 & \ell_x \\ 0 & 0 & \ell_z & \ell_y & \ell_x & 0 \end{bmatrix} \quad (37)$$

is the direction cosine matrix, and

$$V = \frac{\omega}{k} \quad (38)$$

is the complex velocity. This is a fundamental quantity since it determines uniquely the kinematical and dynamical quantities describing the wave propagation.

Using equation (38), the slowness and attenuation vectors can be expressed in terms of the complex velocity as

$$\mathbf{s} = \mathrm{Re}\left[\frac{1}{V}\right]\hat{\boldsymbol{\kappa}}, \quad (39)$$

and

$$\mathbf{a} = -\omega\,\mathrm{Im}\left[\frac{1}{V}\right]\hat{\boldsymbol{\kappa}}. \quad (40)$$

The phase velocity is the reciprocal of the slowness. In vector form it is given by

$$\mathbf{V}_p = \left(\mathrm{Re}\left[\frac{1}{V}\right]\right)^{-1}\hat{\boldsymbol{\kappa}}. \quad (41)$$

The quality factor is defined as the ratio of the peak strain energy density to the average loss energy density (Auld, 1990). These can be expressed in function of a complex strain energy (Carcione and Cavallini, 1993)

$$\varepsilon_s = \frac{1}{2}\mathbf{S}^T \cdot \underset{\sim}{\mathbf{p}} \cdot \mathbf{S}^*. \quad (42)$$

The peak strain energy is the real part of ε_s and the average loss energy is the imaginary part of ε_s. Then, the quality factor is

$$Q = \frac{\mathrm{Re}[\varepsilon_s]}{\mathrm{Im}[\varepsilon_s]}. \quad (43)$$

It can be shown that for homogeneous plane waves

$$\varepsilon_s = \frac{1}{2}\rho|k|^2\mathbf{u}^T \cdot \mathbf{u}^* V^2. \quad (44)$$

In consequence, since the matrix product in the right-hand side of equation (44) is real, the quality factor in general anisotropic-viscoelastic media takes the following simple form:

$$Q = \frac{\mathrm{Re}[V^2]}{\mathrm{Im}[V^2]}. \qquad (45)$$

The energy velocity vector is defined as the ratio of the average power flow density to the mean energy density. The average power flow density is the real part of the complex Poynting vector

$$\mathbf{P} = -\frac{1}{2}\,\mathbf{\underline{T}} \cdot \mathbf{v}^*, \qquad (46)$$

where $\mathbf{\underline{T}}$ is the 3 x 3 stress tensor and v is the particle velocity field. The mean energy density is equal to half the sum of the kinetic and strain energy densities, where the kinetic energy is simply

$$(\varepsilon_v)_{peak} = \frac{1}{2}\rho \mathbf{v}^T \cdot \mathbf{v}^*. \qquad (47)$$

Then, the energy velocity vector is

$$\mathbf{V}_e = \frac{2\,\mathrm{Re}[\mathbf{P}]}{(\varepsilon_V)_{peak} + \mathrm{Re}[\varepsilon_s]}. \qquad (48)$$

The wavefront is the locus of the end of the energy velocity vector multiplied by one unit of propagation time.

PLANE-WAVE ANALYSIS AND SIMULATIONS

The components of $\mathbf{p}$ are calculated from equation (35) by using equation (8). It gives

$$p_{I(I)} = c_{I(I)} - D + KM_1 + \frac{4}{3}GM_\delta \qquad \text{for } I = 1, 2, 3, \qquad (49a)$$

$$p_{IJ} = c_{IJ} - D + 2G + KM_1 - \frac{2}{3}GM_\delta \qquad \text{for } I, J = 1, 2, 3; \quad I \neq J, \qquad (49b)$$

$$p_{44} = c_{44}M_2, \qquad p_{55} = c_{55}M_3,$$

and

$$p_{66} = c_{66}M_4, \qquad (49c)$$

where M_ν, $\nu = 1, \ldots, L_\nu$ are given by equation (13). The time-independent off-diagonal components keep the same form.

The wave characteristics are computed here for two media of transversely-isotropic and orthorhombic symmetries (clay shale and phenolic, respectively), whose nonzero elasticities, densities, and relaxation times are given in Table 1. The clay shale elasticities are taken from the article by Thomsen (1986) who collected experimental data for a variety of anisotropic materials, and the phenolic elasticities are taken from the experimental work by Cheadle et al., (1990). Transversely-isotropic media require only two complex moduli, and the choice $p_{66} = (p_{11} - p_{12})/2$. Then, the clay shale has two relaxations functions and the phenolic has four; three of them quantify the attenuation of the shear waves. For clay shale, the symmetry constraints imply $\delta = 2$, while for phenolic $\delta = 2$ is a choice. Each relaxation function has one dissipation mechanism with relaxation peaks at 20 Hz in clay shale and at 250 kHz in phenolic. The minimum value of the 1-D quality factors $Q_\nu = \mathrm{Re}[M_\nu]/\mathrm{Im}[M_\nu]$, for each single mechanism, is indicated in Table 1.

The three complex velocities, given by equation (38), are the key to compute the dynamic properties of the medium. They have the same functional form of the elastic phase velocities where the elasticities are replaced by the corresponding complex stiffness components (correspondence principle). For reference, the elastic velocities, up to orthorhombic symmetry, can be found in Auld (1990). Figure 1 illustrates a polar diagram of quality factor curves [equation (45)] in a symmetry plane of an orthorhombic medium. Only one quadrant of the plane is displayed from symmetry considerations. As can be seen, the values of the quality factors along the directions of the Cartesian axes are determined by the 1-D quality factors Q_ν for the shear waves, and by simple functions of the stiffnesses for the qP wave.

Figure 2 shows sections of the slowness, attenuation, quality factor, and energy velocity surfaces for clay shale across three mutually orthogonal planes with the symmetry

Table 1. Material properties of the anisotropic-viscoelastic media.

Medium	c_{11}	c_{12}	c_{13}	c_{22}	c_{23}	c_{33}	c_{44}	c_{55}	c_{66}	ρ
	Elasticities (GPa) and density (Kg/m^3)									
Clay shale	66.6	19.7	39.4	66.6	39.4	39.9	10.9	10.9	23.4	2590.
Phenolic	11.7	6.7	7.0	15.4	7.0	17.4	3.8	3.5	3.1	1364.

	Relaxation time(s)							
	$\tau_\varepsilon^{(1)}$	$\tau_\sigma^{(1)}$	$\tau_\varepsilon^{(2)}$	$\tau_{\sigma^{(2)}}$	$\tau_\varepsilon^{(3)}$	$\tau_\sigma^{(3)}$	$\tau_\varepsilon^{(4)}$	$\tau_\sigma^{(4)}$
Clay shale	$8.00\ 10^{-3}$	$7.49\ 10^{-3}$	$8.00\ 10^{-3}$	$7.25\ 10^{-3}$	$8.00\ 10^{-3}$	$7.25\ 10^{-3}$	$8.00\ 10^{-3}$	$7.25\ 10^{-3}$
Q	30		20		20		20	
Phenolic	$6.40\ 10^{-7}$	$6.00\ 10^{-7}$	$6.40\ 10^{-7}$	$5.80\ 10^{-7}$	$6.40\ 10^{-7}$	$5.60\ 10^{-7}$	$6.40\ 10^{-7}$	$5.30\ 10^{-7}$
Q	30		20		15		10	

axis along the vertical direction. The dotted line corresponds to the quasi-compressional wave. The polarizations of the different modes are plotted in the energy velocity curves; when not plotted, polarizations are normal to the planes. The quality factors of the *SH* mode is the outer continuous curve. The attenuation of the shear modes along the symmetry axis is determined by the set of shear relaxation times, giving Q = 20 (see Table 1), while in other directions the attenuation depends also on the values of the elastic constants.

As mentioned before, dilatational energy dissipates according to the complex modulus M_1 with quality factor Q_1 = 30, as indicated in Table 1. However, the anelastic characteristics of the quasi-compressional wave depend also on M_2. In fact, the relation between the *qP* quality factors at the symmetry axis and at the horizontal plane is

$$\frac{Q(\ell_z = 1)}{Q(\ell_x = 1)} = \frac{c_{33} - D + \mathrm{Re}[A]}{c_{11} - D + \mathrm{Re}[A]}, \tag{50}$$

where

$$A = KM_1 + \frac{4}{3} GM_2.$$

Defining $a = \mathrm{Re}[A] - D$, it can be verified that $a > 0$ and $a < c_{11}$, $a < c_{33}$, for realistic media ($a = 0$ in the elastic case). This implies that, whatever the ratio c_{33}/c_{11}, the ratio between quality factors given by equation (50) departs from unity more than the ratio c_{33}/c_{11}. It follows that the quality factor gives a better indicator of anisotropy than the elastic constants (as is also verified experimentally). Another important consequence of this analysis is that, when $c_{11} > c_{33}$, the *qP* wave attenuates more along the symmetry axis than in the plane of isotropy.

On the other hand, the quality factor of the *qSV* wave along the three axes is uniquely determined by the complex modulus M_2 since it can be seen that

$$Q = \frac{\mathrm{Re}[p_{55}]}{\mathrm{Im}[p_{55}]} = Q_2. \tag{51}$$

Similarly, the *SH* mode has Q = Q_2 along the symmetry axis, and

$$Q = \frac{\mathrm{Re}[p_{66}]}{\mathrm{Im}[p_{66}]},$$

where

$$p_{66} = c_{66} + G(M_2 - 1), \tag{52}$$

in the horizontal plane. For this wave,

$$\frac{Q(\ell_z = 1)}{Q(\ell_x = 1)} = \frac{G\ \mathrm{Re}[M_2]}{c_{66} + G(\mathrm{Re}[M_2] - 1)}, \tag{53}$$

where G is given by equation (10b). Since $\mathrm{Re}[M_2] > 1$, it is straightforward to prove that when $c_{66} > c_{55}$, this ratio is less than 1, and the attenuation is higher along the symmetry axis. This is the case for clay shale, as can be seen in Table 1. The figures show the anisotropic properties of wave propagation. The behavior of these curves with frequency gives zero attenuation (infinite Q factors) at the low- and high-frequency limits, and increasing energy velocities from low to high frequencies.

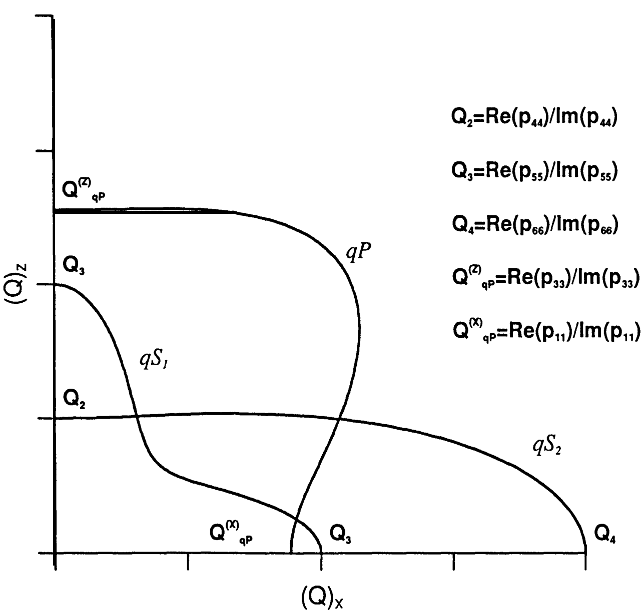

FIG. 1. Polar diagram of quality factor curves in a symmetry plane of an orthorhombic medium. Only one quarter of the plane is displayed from symmetry considerations. The values of the quality factors along the directions of the Cartesian axes are determined by the 1-D quality factors $Q_\nu = \mathrm{Re}[M_\nu]/\mathrm{Im}[M_\nu]$ for the shear waves, and simple functions of the stiffnesses for the *qP*-wave.

Figure 3 shows sections of the slowness, attenuation, quality factor, and energy velocity surfaces for phenolic across three mutually orthogonal planes. As can be seen, the shear waves (continuous curves) form one slowness surface. This behavior is also observed in the quality factor curves. The attenuation along the principal axes of the orthorhombic medium is governed by the four sets of relaxation times whose quality factors are indicated in Table 1. For instance, along the vertical axis, the quality factors for the *qP*, *qS*1, and *qS*2 waves are 30, 20, and 15, confirming the values of Table 1.

The displacement formulation of the wave equation is solved with the method given in Tal-Ezer et al., (1990), which consists of a polynomial expansion of the evolution operator that has spectral accuracy. The unknown variables are the displacements and the memory variables. The grid size used for the simulation in clay shale is 81 x 81 x 81 with a spacing of 20 m in each direction. The source is a pure shear torque located at grid point (40, 40, 40), whose potential vector is directed along the symmetry axis. The source time function is a Ricker wavelet with a central frequency of 20 Hz, i.e., at the relaxation peaks. Figure 4 displays snapshots at 0.24 s across the three orthogonal planes; only one octant is represented because of symmetry properties. The elastic component corresponds to the low frequency-limit response. As can be seen, the source produces mainly shear energy of the *SH* type, which corresponds to the intermediate wavefront envelope in Figure 2d.

As expected, the anelastic wavefronts are attenuated and travel faster compared to the elastic wavefronts since the elastic behavior is taken in the low-frequency limit.

The numerical experiment in phenolic has the same mesh as the previous example with a grid spacing of 1.5 mm, i.e., a cube of 12-cm edge size. The source is a vertical load directed along the z-axis and located at grid point (40, 40, 40). The source time function is a Ricker wavelet with a central frequency of 250 kHz. Snapshots of the wavefield at 0.02 ms propagation time are represented in Figure 5. As before, the elastic component is the low-frequency limit response; The vertical force generates energy mainly in the u_z-component. The shear wave observed in the pictures is coupled with the *qP*-wave. The quality factor curves for this shear mode have the property that, for a given plane, the values along principal axes are similar. For instance, the quality factor has

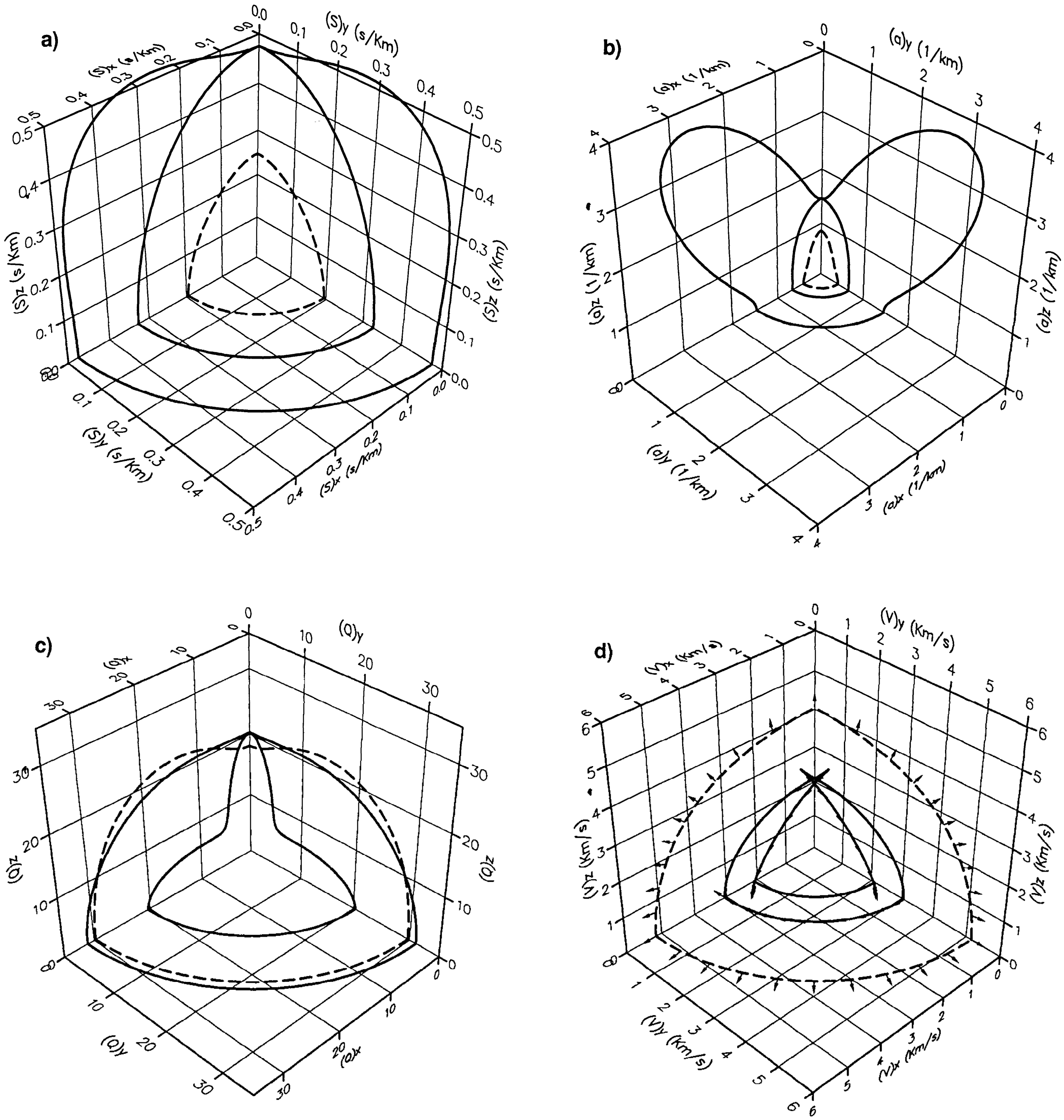

FIG. 2. Sections of the (a) slowness, (b) attenuation, (c) quality factor, and (d) energy velocity surfaces at 20 Hz for clay shale across three mutually perpendicular planes where the symmetry axis coincides with the vertical axis The dotted line corresponds to the quasi-compressional wave. Only one octant of the sections is displayed from symmetry considerations.

the value Q = 20 along the coordinate axes of theyz-plane (see Figure 3c).

CONCLUSIONS

The anisotropic, linear viscoelastic, rheological relation closely models the petrophysical properties of the subsurface. Rocks can be considered as general triclinic systems, and viscoelasticity describes a wide variety of dissipation mechanisms that can be of a mechanical, chemical, electrical, or thermoelastic nature. The proposed constitutive law is based on four relaxation functions, where the different dissipation mechanisms are modeled by pairs of relaxation times. One relaxation function models the anelastic properties of the dilatational field and the other three describe the behavior of the shear waves. This allows more flexibility in

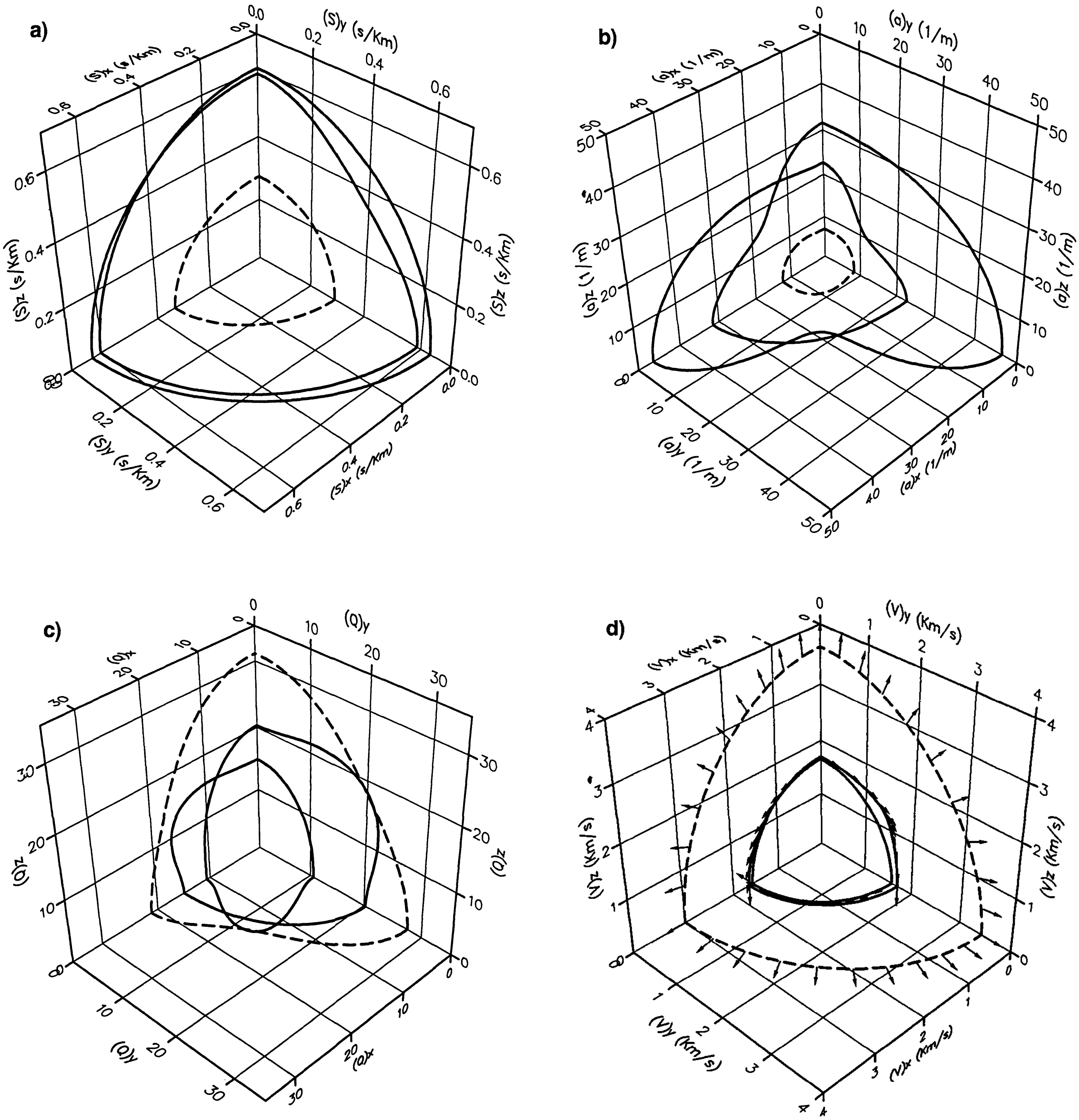

FIG. 3. Sections of (a) slowness, (b) attenuation, (c) quality factor, and (d) energy velocity surfaces at 250 kHz for phenolic across three mutually perpendicular planes. The medium has orthorhombic symmetry. The dotted line corresponds to the quasi-compressional wave. Only one octant of the sections is displayed from symmetry considerations’.

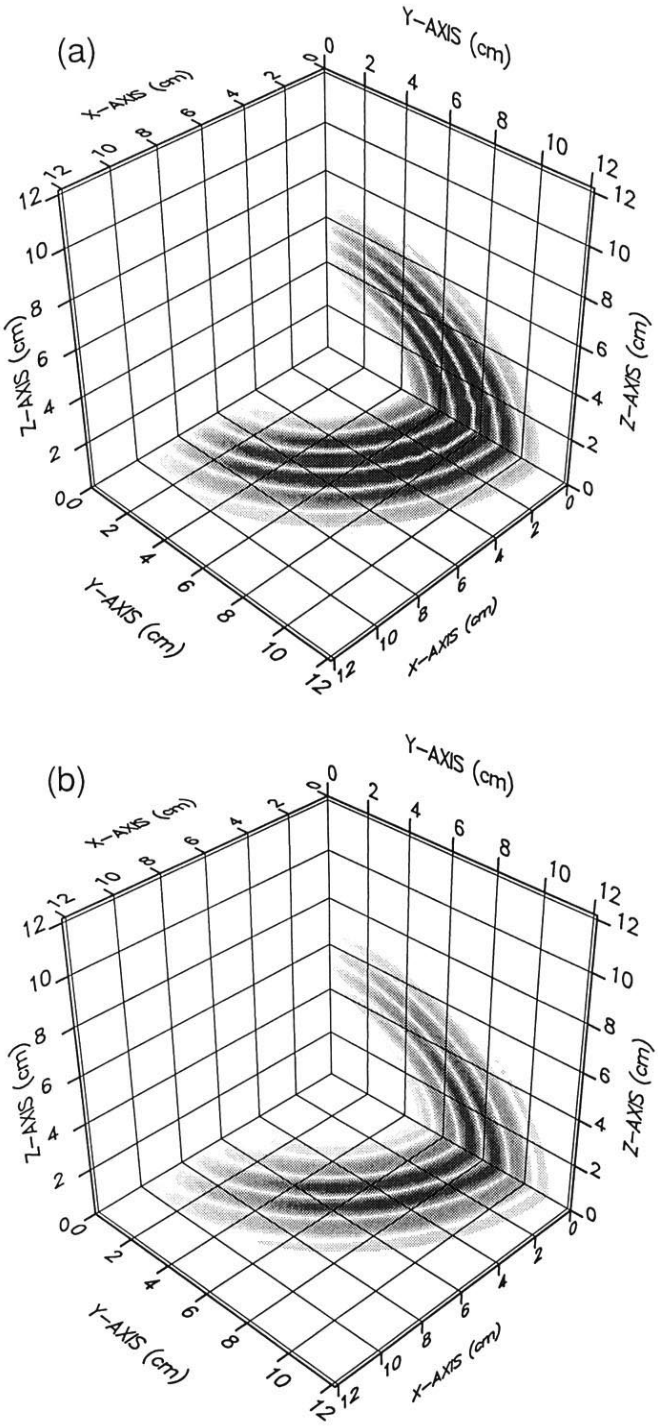

FIG. 4. Snapshot sections of the (a) elastic and (b) anelastic u_x -component of the 3-D wavefield across three orthogonal planes in clay shale (one octant is displayed). The source is a pure shear torque applied at the origin and confined to the horizontal plane. Pure *SH* energy is observed.

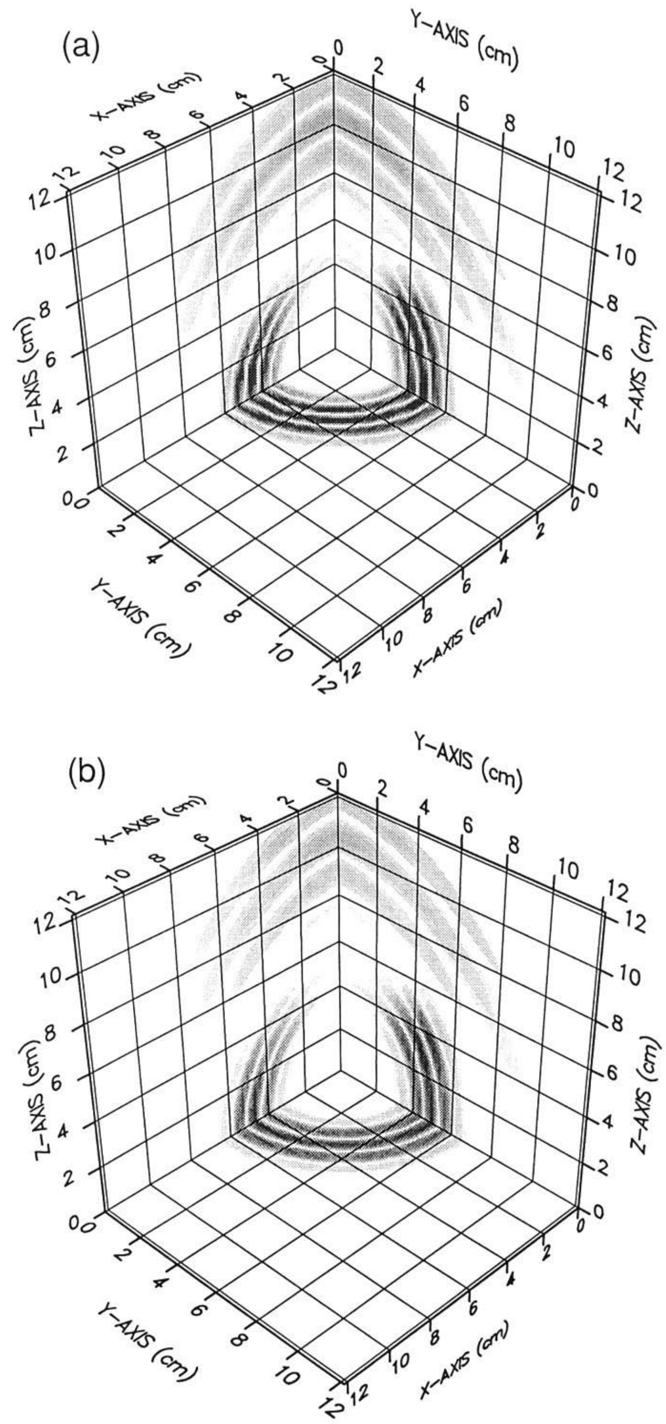

FIG. 5. Snapshot sections of the (a) elastic and (b) anelastic u_z -component of the 3-D wavefield across three orthogonal panes in phenolic (one octant is displayed). The source is a vertical force applied at the origin. The shear wave observed in the pictures is that coupled with the *qP* mode.

defining the attenuation characteristics of the shear modes. For instance, in the orthorhombic example, one relaxation function is assigned to one shear wave and two are assigned to the other shear wave. The constitutive model provides an arbitrary frequency dependence and gives elastic behavior at the low- and high-frequency limits. Moreover, the stress-strain relation can be introduced into Newton's equations to yield a time-domain wave equation based on memory variables. The theory can be formulated either in terms of the displacement field or in terms of the particle velocity and stress components. Two-dimensional wave equations are obtained for the plane of mirror symmetry of a monoclinic medium.

A plane-wave analysis, based on homogeneous harmonic waves, gives the expressions of measurable quantities, like the slowness, attenuation, and energy velocity in terms of the material properties. Since the values of the attenuation can be defined along preferred directions, the theory can be used either for matching experimental data or for characterization of anisotropic media. The characteristics of the transient wavefield are confirmed by comparison of elastic and anelastic snapshots computed with a spectral modeling technique.

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