On energy definition in electromagnetism: An analogy with viscoelasticity

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Electromagnetic media as, for instance, imperfect dielectrics, semiconductors, and materials with magnetic losses, have time- (and frequency-) dependent permittivity, magnetic permeability, and electric conductivity, and, therefore, energy dissipation and pulse distortion occurs. The electromagnetic Umov–Poynting's theorem is reinterpreted in light of the theory of viscoelasticity in order to define the stored and dissipated energy densities in the time domain. A simple dielectric relaxation model equivalent to a viscoelastic mechanical model illustrates the analogy that identifies electric field with stress, electric induction with strain, dielectric permittivity with reciprocal bulk modulus, and resistance with viscosity.
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INTRODUCTION

Energy-balance equations are important for characterizing the energy stored and the transport properties in a field. However, the definition of stored (free) energy and energy dissipation rate is controversial, both in electromagnetism[1] and viscoelasticity.[2] The problem is particularly intriguing in the time domain, since different definitions may give the same time-average value for harmonic fields. This ambiguity is not present when the constitutive relation can be described in terms of springs and dashpots (e.g., Refs. 3 and 4). That is, when the system can be defined in terms of internal variables and the relaxation function has an exponential form.[5] Christensen[6] and Golden and Graham[7] give a general expression of the viscoelastic energy densities which is consistent with the mechanical model description.

Formal analogies, in the mathematical sense, exist between electromagnetism and other fields like mechanics[8] and viscoelasticity. In this case, Carcione and Cavallini,[9] for instance, show an analogy for vector wavefields and material properties that allows the acoustic and electromagnetic problems to be solved with the same analytical methodology. In this work, the stored electric and magnetic energies are defined in terms of the viscoelastic expression by using the analogy. The theory is applied to a simple dielectric relaxation process that is mathematically equivalent to the Zener model or standard linear solid viscoelastic rheology.[10]

It is important to point out that the theory cannot be applied to the whole range of electromagnetic (e.m.) problems, since a mathematical analogy may not necessarily imply a physical analogy. In viscoelasticity, the real part of the complex moduli (describing pure deformation modes) is positive, and the presence of intrinsic dissipation implies velocity dispersion and vice versa (Kramers–Krönig relations[11,12]). These properties preclude the use of the theory for an ionized gas whose complex permittivity may, in some cases, be negative (e.g., Ref. 13), or for a lossless dispersive dielectric that does not satisfy the Kramers–Krönig relations.

The definition of energy is important in a large number of applications where it is necessary to know how the energy transferred by the e.m. field is related to the strength of the field. This context involves the whole electrical, radio, and optical engineering, where the medium can be assumed dielectrically and magnetically linear. In particular, the Debye model has been applied to bioelectromagnetics[14,15] in the analysis of the response of biological tissues, and to geophysics, in the simulation of ground-penetrating radar wave propagation through wet soils.[16,17,18]

I. MAXWELL'S EQUATIONS AND CONSTITUTIVE RELATIONS

Maxwell's equations for isotropic dispersive media, in the absence of external electric and magnetic currents, are

$$\nabla \times \mathscr{E} = -\frac{\partial \mathscr{B}}{\partial t}, \tag{1}$$

$$\nabla \times \mathscr{H} = \frac{\partial \mathscr{D}}{\partial t} + \mathscr{J}, \tag{2}$$

where $\mathscr{E}$ is the electric field, $\mathscr{H}$ is the magnetic field, $\mathscr{D}$ is the electric induction, $\mathscr{B}$ is the magnetic induction, and $\mathscr{J}$ is the conduction current. The symbol $\times$ denotes the vector product.

For time-harmonic fields with time dependence $\exp(i\omega t)$, where ω is the angular frequency, Eqs. (1) and (2) read

$$\nabla \times \mathbf{E} = -i\omega \mathbf{B}, \tag{3}$$

$$\nabla \times \mathbf{H} = i\omega \mathbf{D} + \mathbf{J}, \tag{4}$$

respectively, where **E**, **D**, **H**, and **B** are the corresponding time-harmonic fields.

We consider constitutive relations $\mathscr{D}(\mathscr{E})$ and $\mathscr{B}(\mathscr{H})$ of the form

$$\mathscr{D} = \epsilon * \frac{\partial \mathscr{E}}{\partial t} \tag{5}$$

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and

$$\mathscr{B}=\mu * \frac{\partial \mathscr{H}}{\partial t} \tag{6}$$

(i.e., nonmoving media), where ϵ and μ are the dielectric permittivity and magnetic permeability functions, and * denotes time convolution. Similarly, generalized Ohm's law can be written as

$$\mathscr{J}=\sigma * \frac{\partial \mathscr{E}}{\partial t}, \tag{7}$$

where σ is the conductivity function.

For harmonic fields, the constitutive relations read

$$\mathbf{D}=\mathscr{F}\left(\frac{\partial \epsilon}{\partial t}\right)\mathbf{E}, \quad \mathbf{B}=\mathscr{F}\left(\frac{\partial \mu}{\partial t}\right)\mathbf{H}, \tag{8}$$

and

$$\mathbf{J}=\mathscr{F}\left(\frac{\partial \sigma}{\partial t}\right)\mathbf{E}, \tag{9}$$

where $\mathscr{F}(\cdot)$ is the Fourier transform operator. For convenience, the medium properties are denoted by the same symbols, in both the time and the frequency domains, i.e.,

$$\mathscr{F}\left(\frac{\partial \epsilon}{\partial t}\right) \rightarrow \epsilon, \ldots, \text{etc} \ldots$$

II. UMOV–POYNTING'S THEOREM FOR HARMONIC FIELDS

The scalar product of the complex conjugate of Eq. (4) with $\mathbf{E}$, use of $\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = (\nabla \times \mathbf{E}) \cdot \mathbf{H}^* - \mathbf{E} \cdot (\nabla \times \mathbf{H}^*)$, and substitution of Eq. (3) gives Umov–Poynting's theorem for harmonic fields,

$$-\nabla \cdot \mathbf{P} = \tfrac{1}{2}\, \mathbf{J}^* \cdot \mathbf{E} - 2i\omega(u_e - u_m), \tag{10}$$

where

$$\mathbf{P} = \tfrac{1}{2}\mathbf{E} \times \mathbf{H}^* \tag{11}$$

is the complex Umov–Poynting vector, and

$$u_e = \tfrac{1}{4}\mathbf{E} \cdot \mathbf{D}^*, \quad u_m = \tfrac{1}{4}\mathbf{B} \cdot \mathbf{H}^*, \tag{12}$$

are the harmonic (complex) (di)electric and magnetic energy densities. The symbol * denotes complex conjugate.

Substitution of the constitutive relations (8) and (9) into Eq. (10) yields

$$2i\nabla \cdot \mathbf{P} = -\omega(\epsilon_T^* |\mathbf{E}|^2 - \mu |\mathbf{H}|^2), \tag{13}$$

where

$$\epsilon_T = \epsilon - \frac{i}{\omega}\, \sigma. \tag{14}$$

Taking real and imaginary parts of (13) gives

$$2\,\mathrm{Im}(\nabla \cdot \mathbf{P}) = \omega[\mathrm{Re}(\epsilon_T)|\mathbf{E}|^2 - \mathrm{Re}(\mu)|\mathbf{H}|^2] = \text{power energy density}, \tag{15}$$

$$2\,\mathrm{Re}(\nabla \cdot \mathbf{P}) = \omega[\mathrm{Im}(\epsilon_T)|\mathbf{E}|^2 + \mathrm{Im}(\mu)|\mathbf{H}|^2] = \text{rate of dissipated energy}, \tag{16}$$

respectively. The time-average energy densities are such that

$$\begin{aligned}
&\tfrac{1}{4}\,\mathrm{Re}(\epsilon)|\mathbf{E}|^2 = \mathrm{Re}(u_e) \quad \text{stored (di)electric}\\
&-\tfrac{1}{2}\,\omega^{-1}\,\mathrm{Im}(\sigma)|\mathbf{E}|^2 = \mathrm{Im}(u_\sigma) \quad \text{stored electric}\\
&\tfrac{1}{4}\,\mathrm{Re}(\mu)|\mathbf{H}|^2 = \mathrm{Re}(u_m) \quad \text{stored magnetic}\\
&\tfrac{1}{4}\,\mathrm{Im}(\epsilon)|\mathbf{E}|^2 = \mathrm{Im}(u_e) \quad \text{dissipated (di)electric}\\
&\tfrac{1}{2}\,\omega^{-1}\,\mathrm{Re}(\sigma)|\mathbf{E}|^2 = \mathrm{Re}(u_\sigma) \quad \text{dissipated electric}\\
&\tfrac{1}{4}\,\mathrm{Im}(\mu)|\mathbf{H}|^2 = \mathrm{Im}(u_m) \quad \text{dissipated magnetic,}
\end{aligned} \tag{17}$$

where

$$u_\sigma = \tfrac{1}{2}\omega^{-1}\mathbf{J}^* \cdot \mathbf{E} = \tfrac{1}{2}\omega^{-1}\sigma^* |\mathbf{E}|^2 \tag{18}$$

is the (complex) electric energy density.

III. ENERGY DEFINITION IN ELECTROMAGNETISM

Time-average energies for harmonic fields are precisely defined. Let us consider, for instance, the formulation of the energy balance equation given in Oughstun and Sherman.[1] Upon taking the scalar product of Eq. (1) with $\mathscr{H}$ and Eq. (2) with $\mathscr{E}$ and taking the difference, Oughstun and Sherman (Ref. 1, Eq. 2.2.5) obtain

$$-\nabla \cdot \mathscr{P} = \mathscr{J} \cdot \mathscr{E} + \frac{\partial U}{\partial t}, \tag{19}$$

where $\mathscr{P} = \mathscr{E} \times \mathscr{H}$ is the Umov–Poynting vector and U is the total energy per unit volume given by

$$U = U_e + U_m, \tag{20}$$

with

$$\frac{\partial U_e}{\partial t} = \mathscr{E} \cdot \frac{\partial \mathscr{D}}{\partial t}, \tag{21}$$

the time rate of (di) electric energy density, and

$$\frac{\partial U_m}{\partial t} = \mathscr{H} \cdot \frac{\partial \mathscr{B}}{\partial t}, \tag{22}$$

the time rate of magnetic energy density.

The time average of the scalar product of two harmonic vector fields, with the same oscillation frequency, is given by

$$\langle \mathscr{A} \cdot \mathscr{B} \rangle = \tfrac{1}{2}\,\mathrm{Re}(\mathbf{A} \cdot \mathbf{B}^*). \tag{23}$$

Note that taking the time average of Eqs. (19) yields

$$-\langle \nabla \cdot \mathscr{P} \rangle = \tfrac{1}{2}\,\mathrm{Re}(\mathbf{J}^* \cdot \mathbf{E}) + 2\omega[\mathrm{Im}(u_e) - \mathrm{Im}(u_m)], \tag{24}$$

which can be obtained from Eq. (10) by taking its real part. Equation (24) gives the balance of the rate of dissipated energy density and is equivalent to Eq. (16).

We now review some of the statements discussed by Oughstun and Sherman.[1]

(1) Poynting's theorem *provides a mathematically consistent formulation of energy flow* (p. 24). This does not preclude the existence of an alternative formulation. For instance, Jeffreys[19] gives an alternative energy balance,

implying a new interpretation of the Poynting vector (see also the interesting discussion in Robinson[20] and Jeffreys[21]).

(2) ... *it cannot be definitely concluded that the time rate of energy flow at a point is uniquely given by the value of the Poynting vector at that point, for one may add to the Poynting vector any solenoidal vector field without affecting the statement of conservation of energy...* (p. 26). In fact, as the authors state, there is no strictly valid justification for the accepted interpretation of the Poynting vector.

(3) ... *one cannot, in general, express the electric energy density and the dissipation separately in terms of the dielectric permittivity and electrical conductivity of a dispersive medium* (p. 31). In the general case [i.e., the time-domain equation (19)], it is not possible to separate the stored energies from the dissipated energies. Relating u_e and U_e gives [see Eq. (2.2.38)],

$$\mathrm{Im}(u_e)=\frac{1}{2\omega}\left\langle\frac{\partial U_e}{\partial t}\right\rangle, \tag{25}$$

and no relationship of this type for the real part of u_e. The same reasoning applies to the magnetic energy. We present in the next section an alternative definition where energy can be, in principle, separated between stored and dissipated.

(4) For time-harmonic fields, the separation is shown in Eq. (17). However, there is no relation between the time-average energies defined in Eq. (17) and the time averages of U_e, U_m (p. 36). In the next section, a link between harmonic energy densities and transient energy densities is obtained.

Note that Oughstun and Sherman [Ref. 1, Eq. (2.1.19)] use an $\exp(-i\omega t)$ time dependence.

IV. UMOV–POYNTING'S THEOREM FOR TRANSIENT FIELDS

Poynting's theorem (19), omitting Oughstun and Sherman's interpretation of the energies, is

$$-\boldsymbol{\nabla}\cdot\mathscr{P}=\mathscr{J}\cdot\mathscr{E}+\mathscr{E}\cdot\frac{\partial\mathscr{D}}{\partial t}+\mathscr{H}\cdot\frac{\partial\mathscr{B}}{\partial t}. \tag{26}$$

Let us consider a stored (di)electric (free) energy density of the form

$$W_e(t)=\frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t}K(t-\tau_1,t-\tau_2)\times\mathscr{D}'(\tau_1)\cdot\mathscr{D}'(\tau_2)d\tau_1\,d\tau_2, \tag{27}$$

where the prime denotes the first-order derivative with respect to the argument. Hunter (Ref. 3, p. 545) and Golden and Graham (Ref. 7, p. 12) define a similar form for the linear viscoelastic case. The underlying assumptions are that the dielectric properties of the medium do not vary with time (nonaging material), and, as in the lossless case, the energy density is quadratic in the electric field. Moreover, the expression includes a dependence on the history of the electric field. However, it is important to note that the above assumption as to the structure of the formula for the free energy density is by no means the only possible one (see Rabotnov,[22] p. 72). Moreover, as we shall see below, the general expression of the free energy is not uniquely determined by the relaxation function.

Differentiating W_e yields

$$\frac{\partial W_e}{\partial t}=\frac{\partial\mathscr{D}}{\partial t}\cdot\int_{-\infty}^{t}K(t-\tau_2,0)\mathscr{D}'(\tau_2)d\tau_2+\frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial}{\partial t}K(t-\tau_1,t-\tau_2)\mathscr{D}'(\tau_1)\cdot\mathscr{D}'(gt_2)d\tau_1\,d\tau_2. \tag{28}$$

The constitutive relation (5) can be rewritten as

$$\mathscr{E}=\beta*\frac{\partial\mathscr{D}}{\partial t}, \tag{29}$$

where $\beta(t)$ is the dielectric impermeability function, satisfying

$$\frac{\partial\epsilon}{\partial t}*\frac{\partial\beta}{\partial t}=\delta(t),\quad \epsilon_\infty\beta_\infty=\epsilon_0\beta_0=1,\quad \epsilon(\omega)\beta(\omega)=1, \tag{30}$$

with the subindices ∞ and 0 corresponding to the limits $t\rightarrow 0$ and $t\rightarrow\infty$, respectively. If

$$\beta(t)=K(t,0)H(t), \tag{31}$$

where $H(t)$ is the Heaviside function, then,

$$\int_{-\infty}^{t}K(t-\tau_2,0)\mathscr{D}'(\tau_2)d\tau_2=\mathscr{E}(t), \tag{32}$$

and (28) becomes

$$\mathscr{E}\cdot\frac{\partial\mathscr{D}}{\partial t}=\frac{\partial W_e}{\partial t}+D_e, \tag{33}$$

where

$$D_e(t)=-\frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial}{\partial t}K(t-\tau_1,t-\tau_2)\times\mathscr{D}'(\tau_1)\cdot\mathscr{D}'(\tau_2)d\tau_1\,d\tau_2 \tag{34}$$

is the rate of dissipation of (di) electric energy density. Note that the relation (31) does not determine the stored energy, i.e., this can not be obtained from the constitutive relation. However, if we assume that

$$K(t,\tau_1)=G(t+\tau_1), \tag{35}$$

such that

$$\beta(t)=G(t)H(t), \tag{36}$$

this choice will suffice to determine K, and

$$W_e(t)=\frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t}G(2t-\tau_1-\tau_2)\mathscr{D}'(\tau_1)\cdot\mathscr{D}'(\tau_2)d\tau_1\,d\tau_2, \tag{37}$$

$$D_e(t)=-\int_{-\infty}^{t}\int_{-\infty}^{t}G'(2t-\tau_1-\tau_2)\mathscr{D}'(\tau_1)\cdot\mathscr{D}'(\tau_2)d\tau_1\,d\tau_2. \tag{38}$$

Equation (35) is consistent with the corresponding theory implied by mechanical models.[6] Breuer and Onat[23] discuss some realistic requirements from which $K(t,\tau_1)$ must have the reduced from $G(t+\tau_1)$.

Let us calculate the time average of the stored energy density for monochromatic fields. Although $\mathscr{D}(-\infty)$ does not vanish, the transient contained in (37) vanishes for sufficiently large times, and this equation can be used to compute the average of time-harmonic fields. The change of variables $\tau_1\rightarrow t-\tau_1$ and $\tau_2\rightarrow t-\tau_2$ yields

$$W_e(t)=\frac{1}{2}\int_0^{\infty}\int_0^{\infty}G(\tau_1+\tau_2)\mathscr{D}'(t-\tau_1)\cdot\mathscr{D}'(t-\tau_2)d\tau_1\,d\tau_2. \tag{39}$$

Using (23), the time average of Eq. (39) is

$$\langle W_e\rangle=\tfrac{1}{4}\,\omega^2|\mathbf{D}|^2\times\int_0^{\infty}\int_0^{\infty}G(\tau_1+\tau_2)\cos[\omega(\tau_1-\tau_2)]d\tau_1\,d\tau_2. \tag{40}$$

A new change of variables $u=\tau_1+\tau_2$ and $v=\tau_1-\tau_2$ gives

$$\langle W_e\rangle=\frac{1}{8}\,\omega^2|\mathbf{D}|^2\int_0^{\infty}\int_{-u}^{u}G(u)\cos(\omega v)du\,dv=\frac{1}{4}\,\omega|\mathbf{D}|^2\int_0^{\infty}G(u)\sin(\omega u)du. \tag{41}$$

From Eq. (36), and using integration by parts, we have that

$$\mathrm{Re}\left[\mathscr{F}\left(\frac{\partial\beta}{\partial t}\right)\right]=\mathrm{Re}[\beta(\omega)]=G(\infty)+\omega\int_0^{\infty}[G(t)-G(\infty)]\sin(\omega t)dt. \tag{42}$$

Using the property

$$\omega\int_0^{\infty}\sin(\omega t)dt=1, \tag{43}$$

we obtain

$$\mathrm{Re}[\beta(\omega)]=\omega\int_0^{\infty}G(t)\sin(\omega t)dt. \tag{44}$$

Substituting Eq. (44) into Eq. (41), and since $\mathbf{E}=\beta(\omega)\mathbf{D}$, we finally get

$$\langle W_e\rangle=\tfrac{1}{4}|\mathbf{D}|^2\ \mathrm{Re}[\beta(\omega)]=\mathrm{Re}(u_e). \tag{45}$$

A similar calculation shows that $\langle D_e\rangle=2\omega\ \mathrm{Im}(u_e)$.

Similarly, the magnetic term on the right-hand side r.h.s. of Eq. (26) can be recasted as

$$\mathscr{H}\cdot\frac{\partial\mathscr{B}}{\partial t}=\frac{\partial W_m}{\partial t}+D_m, \tag{46}$$

where

$$W_m(t)=\frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t}F(2t-\tau_1-\tau_2)\mathscr{B}'(\tau_1)\cdot\mathscr{B}'(\tau_2)d\tau_1\,d\tau_2, \tag{47}$$

$$D_m(t)=-\int_{-\infty}^{t}\int_{-\infty}^{t}F'(2t-\tau_1-\tau_2)\mathscr{B}'(\tau_1)\cdot\mathscr{B}'(\tau_2)d\tau_1\,d\tau_2, \tag{48}$$

are the stored magnetic energy density and rate of dissipation of magnetic energy density, respectively, such that

$$\mathscr{H}=\gamma*\frac{\partial\mathscr{B}}{\partial t},\quad \gamma(t)=F(t)H(t), \tag{49}$$

with γ the magnetic impermeability function.

The rate of dissipated electric energy density can be defined as

$$D_\sigma(t)=-\int_{-\infty}^{t}\int_{-\infty}^{t}\Sigma(2t-\tau_1-\tau_2)\mathscr{E}'(\tau_1)\cdot\mathscr{E}'(\tau_2)d\tau_1\,d\tau_2, \tag{50}$$

where

$$\mathscr{J}=\sigma*\frac{\partial\mathscr{E}}{\partial t},\quad \sigma(t)=\Sigma(t)H(t). \tag{51}$$

Formally, the stored energy density due to the electric currents out of phase with the electric field, satisfies

$$\frac{\partial W_\sigma}{\partial t}=\mathscr{J}\cdot\mathscr{E}-D_\sigma. \tag{52}$$

In terms of the energy densities, Eq. (26) becomes

$$-\nabla\cdot\mathscr{P}=\frac{\partial}{\partial t}(W_\sigma+W_e+W_m)+D_\sigma+D_e+D_m, \tag{53}$$

and the correspondences with the averaged time-harmonic values are

$$\begin{aligned}&\langle W_e\rangle=\mathrm{Re}(u_e),\quad \langle W_m\rangle=\mathrm{Re}(u_m),\\&\langle D_e\rangle=2\omega\ \mathrm{Im}(u_e),\\&\langle D_\sigma\rangle=2\omega\ \mathrm{Re}(u_\sigma),\\&\langle D_m\rangle=-2\omega\ \mathrm{Im}(u_m).\end{aligned} \tag{54}$$

Note that $\langle\mathscr{J}\cdot\mathscr{E}\rangle$ is equal to the rate of dissipated energy density $\langle D_\sigma\rangle$, and that

$$\left\langle\frac{\partial W_e}{\partial t}\right\rangle=0, \tag{55}$$

in contrast with Oughstun and Sherman's result (25). The same property holds for the stored electric and magnetic energy densities.

There are other alternative time-domain expressions for the energy densities whose time-average values coincide

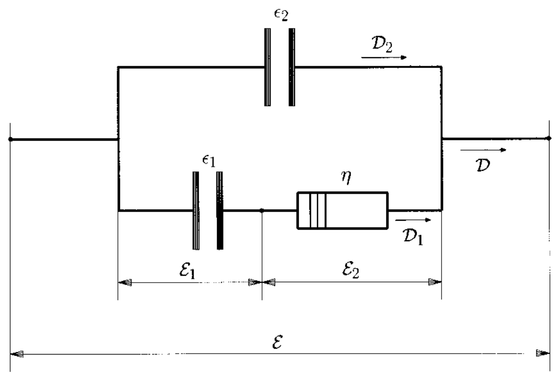

FIG. 1. This electric circuit is equivalent to a purely dielectric relaxation process, where ϵ_1 and ϵ_2 are the capacitances, η is a resistance, $\mathcal{E}$ is the electric field, and $\mathcal{D}$ is the electric induction.

with those given in Eqs. (54), but fail to match the energy in the time domain. For instance, the following definition,

$$W_e' = \tfrac{1}{2}\mathcal{E}\cdot\mathcal{D}, \tag{56}$$

as the stored (di) electric energy density, and

$$D_e' = \frac{1}{2}\left(\mathcal{E}\cdot \frac{\partial \mathcal{D}}{\partial t} - \mathcal{D}\cdot \frac{\partial \mathcal{E}}{\partial t}\right), \tag{57}$$

as the rate of dissipation, satisfy Eq. (53) and $\langle W_e' \rangle = \langle W_e \rangle$ and $\langle D_e' \rangle = \langle D_e \rangle$. However, W_e' is not equal to the energy stored in the capacitors for the Debye model given in the next section [see Eqs. (69) and (71)].

V. EXAMPLE

It is well known that the Debye model used to describe the behavior of dielectric materials[24] is mathematically equivalent to the Zener or standard linear solid model used in viscoelasticity. The following example uses this model to illustrate the concepts presented in the previous section.

A. Debye-type dielectric model

Let us consider a capacitor C_2 in parallel with a series connection between a capacitor C_1 and a resistance R. This circuit obeys the following differential equation:

$$U + \tau_U \frac{\partial U}{\partial t} = \frac{1}{C}\left(I + \tau_I \frac{\partial I}{\partial t}\right), \tag{58}$$

where $U = \partial V/\partial t$, I is the current, V is the voltage, and

$$C = C_1 + C_2, \quad \tau_U = R\left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}, \quad \tau_I = C_1 R. \tag{59}$$

From the point of view of a pure dielectric process, we identify U with $\mathcal{E}$ and I with $\mathcal{D}$ (see Fig. 1). Hence, the dielectric relaxation model is

$$\mathcal{E} + \tau_{\mathcal{E}} \frac{\partial \mathcal{E}}{\partial t} = \frac{1}{\epsilon_0}\left(\mathcal{D} + \tau_{\mathcal{D}} \frac{\partial \mathcal{D}}{\partial t}\right), \tag{60}$$

where

$$\epsilon_0 = \epsilon_1 + \epsilon_2, \quad \tau_{\mathcal{E}} = \frac{1}{\eta}\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right)^{-1}, \quad \tau_{\mathcal{D}} = \epsilon_1/\eta, \tag{61}$$

with η a parameter that introduces the dissipation. Note that ϵ_0 is the static (low-frequency) permittivity and $\epsilon_\infty = \epsilon_0 \tau_{\mathcal{E}}/\tau_{\mathcal{D}} = \epsilon_2 < \epsilon_0$ is the optical (high-frequency) permittivity.

We have that

$$\epsilon(t) = \epsilon_0\left[1 - \left(1 - \frac{\tau_{\mathcal{E}}}{\tau_{\mathcal{D}}}\right)\exp(-t/\tau_{\mathcal{D}})\right]H(t), \tag{62}$$

$$\beta(t) = G(t)H(t) = \frac{1}{\epsilon_0}\left[1 - \left(1 - \frac{\tau_{\mathcal{D}}}{\tau_{\mathcal{E}}}\right)\exp(-t/\tau_{\mathcal{E}})\right]H(t) \tag{63}$$

and

$$\epsilon(\omega) = \beta^{-1}(\omega) = \epsilon_0\left(\frac{1 + i\omega\tau_{\mathcal{E}}}{1 + i\omega\tau_{\mathcal{D}}}\right). \tag{64}$$

Equation (64) can be rewritten as

$$\epsilon(\omega) = \epsilon_\infty + \frac{\epsilon_0 - \epsilon_\infty}{1 + i\omega\tau_{\mathcal{D}}}. \tag{65}$$

For instance, the permittivity (65) describes the response of polar molecules, such as water, to the e.m. field.[16,25]

Substituting Eq. (63) into Eq. (29) and defining the internal variable

$$\xi(t) = \phi \exp(-t/\tau_{\mathcal{E}})H(t) * \mathcal{D}, \quad \phi = \frac{1}{\epsilon_0 \tau_{\mathcal{E}}}\left(1 - \frac{\tau_{\mathcal{D}}}{\tau_{\mathcal{E}}}\right), \tag{66}$$

yields

$$\mathcal{E} = \frac{1}{\epsilon_\infty}\mathcal{D} + \xi, \tag{67}$$

where ξ satisfies

$$\frac{\partial \xi}{\partial t} = \phi D - \frac{\xi}{\tau_{\mathcal{E}}}. \tag{68}$$

The (di)electric energy density is that stored in the capacitors,

$$W_e = \frac{1}{2\epsilon_1}\mathcal{D}_1\cdot\mathcal{D}_1 + \frac{1}{2\epsilon_2}\mathcal{D}_2\cdot\mathcal{D}_2, \tag{69}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are the respective electric inductions. Since $\mathcal{D}_2 = \epsilon_2\mathcal{E}$, $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$, and $\epsilon_\infty = \epsilon_2$, we obtain

$$\mathcal{D}_1 = -\epsilon_\infty \xi, \tag{70}$$

where Eq. (67) has been used. Note that the internal variable is closely related to the electric field acting on the capacitor in series with the dissipation element. Substitution of the electric fields into Eq. (69), and after some calculations, yields

$$W_e = \frac{\epsilon_\infty}{2}\left[\left(\frac{\epsilon_\infty}{\epsilon_0 - \epsilon_\infty}\right)\xi\cdot\xi + \mathcal{E}\cdot\mathcal{E}\right]. \tag{71}$$

Let us verify that Eq. (37) is in agreement with Eq. (71). From Eqs. (63) and (66) we have

$$G(t) = \epsilon_0^{-1} - \phi\tau_{\mathcal{E}}\exp(-t/\tau_{\mathcal{E}}). \tag{72}$$

Replacing Eq. (72) into Eq. (37), and after some algebra, yields

$$W_e = \frac{1}{2\epsilon_0}\,\mathscr{D}\cdot\mathscr{D} - \frac{1}{2}\,\phi\tau_{\mathscr{E}}\left[\exp\left(-\frac{t}{\tau_{\mathscr{E}}}\right)H(t)*\frac{\partial}{\partial t}\,\mathscr{D}(t)\right]^2, \tag{73}$$

where the exponent 2 means the scalar product. Using Eqs. (66) and (68) gives

$$W_e = \frac{1}{2\epsilon_0}\,\mathscr{D}\cdot\mathscr{D} - \frac{1}{2\phi\tau_{\mathscr{E}}}\,(\phi\tau_{\mathscr{E}}\mathscr{D}-\xi)\cdot(\phi\tau_{\mathscr{E}}\mathscr{D}-\xi). \tag{74}$$

Using $\epsilon_\infty\tau_{\mathscr{D}} = \epsilon_0\tau_{\mathscr{E}}$, and a few calculations, shows that the expression in (74) is equal to the stored energy density (71). This equivalence can also be obtained by avoiding the use of internal variables. However, the introduction of these variables is a requirement to obtaining a complete differential formulation of the e.m. equations. This formulation is the basis of most simulation algorithms.[17,18]

The rate of dissipated energy density is

$$D_e = \frac{1}{\eta}\,\frac{\partial\mathscr{D}_1}{\partial t}\cdot\frac{\partial\mathscr{D}_1}{\partial t}, \tag{75}$$

which from Eqs. (70) and (68) becomes

$$D_e = \frac{1}{\eta}\left(\frac{\epsilon_\infty}{\tau_{\mathscr{E}}}\right)^2(\phi\tau_{\mathscr{E}}\mathscr{D}-\xi)\cdot(\phi\tau_{\mathscr{E}}\mathscr{D}-\xi). \tag{76}$$

Taking into account the previous calculations, it is easy to show that substitution of Eq. (72) into Eq. (38) gives Eq. (76).

B. Zener viscoelastic model

Fabrizio and Morro (Ref. 5, p. 42) define a viscoelastic solid with internal variables, for which the free (stored) energy density can be uniquely determined. For simplicity, we only consider dilatational deformations, since this choice does not affect the analysis of the problem. If T is the hydrostatic stress, E is the dilatation (trace of the strain tensor) and ξ is the internal variable, the constitutive relation and growth equation are

$$T = \beta_0 E + \xi \tag{77}$$

and

$$\frac{\partial\xi}{\partial t} = -\alpha\xi - \beta E, \tag{78}$$

respectively, where α, β_0, and β are real and positive constants. Integration of (78) yields

$$\xi(t) = -\beta\,\exp(-\alpha t)H(t)*E(t). \tag{79}$$

It can be easily shown that Eqs. (77) and (78) correspond to a Zener mechanical model consisting of a spring of constant $k_1 = \beta_0(-1+\alpha\beta_0/\beta)$ in parallel with a dashpot of viscosity $\nu = \beta_0^2/\beta$, together in series connection with a spring of constant $k_2 = \beta_0$ (Fig. 2).

The relaxed and unrelaxed moduli are

$$G_\infty = \beta_0 - \beta/\alpha, \quad G_0 = \beta_0, \tag{80}$$

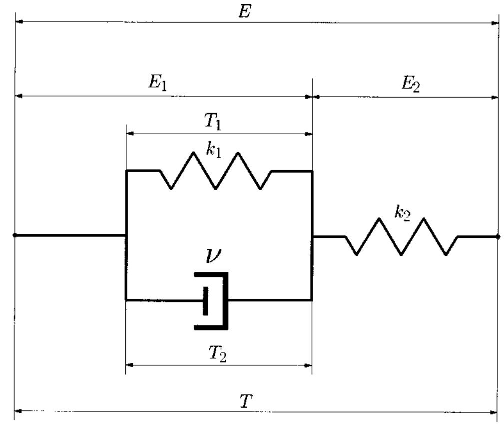

FIG. 2. The Zener viscoelastic model is mathematically equivalent to the electric circuit represented in Fig. 1. k_1 and k_2 are the springs constants, ν is the viscosity of the dashpot, E is the strain, and T is the stress.

respectively, and the relaxation times are

$$\tau_E = \beta_0(\alpha\beta_0-\beta)^{-1}, \quad \tau_T = 1/\alpha \quad (\tau_E > \tau_T). \tag{81}$$

The relaxation function is

$$G(t) = \alpha^{-1}[\alpha\beta_0 - \beta + \beta\,\exp(-\alpha t)]. \tag{82}$$

Assuming that the potential energy is stored in the springs, we have that

$$W_e = \tfrac{1}{2}(k_1E_1^2 + k_2E_2^2), \tag{83}$$

where E_1 and E_2 are the dilatations of the springs. Since $T = k_2E_2$ and $E = E_1 + E_2$, and using (77), we obtain

$$E_1 = -\xi/\beta_0, \quad E_2 = E + \xi/\beta_0. \tag{84}$$

Note that the internal variable is closely related to the dilatation on the spring that is in parallel with the dashpot. Substitution of the dilatations into Eq. (83) yields

$$W_e = \frac{1}{2}\left[\left(\frac{\alpha}{\beta} - \frac{1}{\beta_0}\right)\xi^2 + \frac{1}{\beta_0}\,(\beta_0E+\xi)^2\right]. \tag{85}$$

For isothermal processes, the free energy coincides with the stored energy. In general, the free energy is not unique (Ref. 5, p. 57). However, for exponential relaxation functions having one internal variable, like the Zener model, the free energy is unique.[26]

Day's free energy for Zener systems (Ref. 5, p. 61) is

$$\psi_D = \frac{1}{2}\,G_\infty E^2 + \frac{1}{2}\left\{(G_0-G_\infty)^{-1/2} \times \int_0^\infty G'(s)[E(t)-E(t-s)]ds\right\}^2, \tag{86}$$

where G' is the first-order derivative of the relaxation function. Since

$$\int_0^\infty G'(s)ds = -\beta/\alpha, \quad \int_0^\infty G'(s)E(t-s)ds = \xi,$$

and using Eq. (80), Eq. (86) becomes

$$\psi_D = \frac{1}{2}\left(\beta_0 - \frac{\beta}{\alpha}\right)E^2 + \frac{\beta}{2\alpha}\left(E + \frac{\alpha}{\beta}\,\xi\right)^2 = W_e. \tag{87}$$

This equation differs from the particular form obtained by Fabrizio and Morro (Ref. 5, p. 61) when only dilatational deformations are considered. The cause is a notation error in the book's equation.[27]

The rate of energy density dissipated in the dashpot is

$$D_e = \nu \left(\frac{\partial E_1}{\partial t} \right)^2, \tag{88}$$

which from Eqs. (84) and (78) becomes

$$D_e = \frac{1}{\beta} (\alpha \xi + \beta E)^2. \tag{89}$$

C. Mathematical analogy

The mathematics of the viscoelastic problem is the same as for the dielectric relaxation model previously introduced, since the mathematical equivalence identifies $\mathscr{E}$ with stress T and $\mathscr{D}$ with strain E. Then, from the analogy between stress and electric field, strain and electric induction, internal variables, and bulk modulus and dielectric impermeability, Day's free energy (86) in the electromagnetic case reads

$$\psi_D = \frac{1}{2\epsilon_\infty} \mathscr{D} \cdot \mathscr{D} + \frac{1}{2} \left\{ \left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon_\infty} \right)^{-1/2} \times \int_0^\infty G'(s) [\mathscr{D}(t) - \mathscr{D}(t-s)] ds \right\}^2, \tag{90}$$

where the exponent 2 means the scalar product.

Comparing Eqs. (72) and (82) we obtain the equivalences

$$\frac{\beta}{\alpha} = \frac{1}{\epsilon_\infty} - \frac{1}{\epsilon_0}, \quad \beta_0 = \frac{1}{\epsilon_\infty}, \quad \alpha = \frac{1}{\tau_{\mathscr{E}}}. \tag{91}$$

Substituting these equivalences into Eq. (85), the electromagnetic stored energy density (71) is obtained.

On the other hand, applying the analogy and substituting the equivalences (91) into Eq. (89), yields the electromagnetic rate of dissipated energy density (76).

The complete correspondence between the dielectric and the viscoelastic models is

$$\begin{array}{ccccccc} & \text{fields} & & & \text{properties} & \\ \mathscr{E} & \leftrightarrow & T & \epsilon_\infty & \leftrightarrow & \beta_0^{-1} \\ \mathscr{D} & \leftrightarrow & E & \tau_{\mathscr{D}} & \leftrightarrow & \tau_E \\ \mathscr{E}_1 & \leftrightarrow & T_1 & \tau_{\mathscr{E}} & \leftrightarrow & \tau_T, \\ \mathscr{E}_2 & \leftrightarrow & T_2 & \eta & \leftrightarrow & \nu^{-1} \\ \mathscr{D}_1 & \leftrightarrow & E_1 & \epsilon_1 & \leftrightarrow & k_1^{-1} \\ \mathscr{D}_2 & \leftrightarrow & E_2 & \epsilon_2 & \leftrightarrow & k_2^{-1} \\ \xi & \leftrightarrow & \xi & & & \end{array} \tag{92}$$

where the symbols can be identified in Figs. 1 and 2.

VI. REMARKS

It is important to keep in mind that Eq. (37) holds when $\mathscr{D}(t \to \infty) = 0$, an assumption that is violated by time-harmonic fields. However, after the transient regime (large times after application of the field), the time average of W_e gives $\langle u_e \rangle$, and, for Debye-type dielectric processes (exponential relaxation functions), W_e is the energy stored in the capacitors.

Day's free energy [Eq. (90)] is unique for exponential dielectric functions with one internal variable. Actually, Day's free energy is a particular form of Eq. (37). Both energies coincide for Debye-type dielectric processes.

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