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EQUATIONS AND NUMERICAL METHODS FOR LINEAR WAVE PROPAGATION IN ANELASTIC MEDIA *

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Abstract The equations governing linear wave propagation in viscoelastic media, either single-phase or multiphase, can be written as a single first-order matricial differential equation in time. The formal solution is the evolution operator e^{Mt} acting on the initial condition vector, where M is a linear operator matrix containing the spatial derivatives and medium properties, and t is the time variable. The problem is solved numerically approximating the evolution operator by an optimal polynomial expansion depending on the location of the eigenvalues of M in the complex frequency plane. The eigenvalue analysis is carried out for the anisotropic-viscoelastic and porous viscoacoustic constitutive relations and respective limiting rheologies. For each case an optimal expansion of the evolution operator is identified, which provides highly accurate solutions and fast convergence compared to Taylor expansion or temporal differencing.

I. INTRODUCTION

Linear viscoelasticity provides a general framework for describing the anelastic effects in wave propagation, i.e., the conversion of part of the energy into heat, and the dispersion of the wave field Fourier components with increasing time. A dissipation model which is consistent with real materials is the general standard linear solid which is based on a spectrum of relaxation mechanisms. However, implementation of this rheology in the time-domain is not straightforward due to the presence of convolutional kernels (Bolztmann's superposition principle). To avoid the time convolutions, it is necessary to introduce into the formulation additional variables, called memory variables in virtue of their nature [1]- [5] . The wave equation of the medium can be written as a first-order differential equation in time as

$$\dot{U} = MU + F, \tag{1}$$

where U is a vector whose components are the unknown variables, M is an operator matrix containing the spatial derivatives and material properties, and F is the body force vector.

In (1) and elsewhere, time differentiation is indicated with the dot convention. The differential equation (1) correctly describes the anelastic effects in wave propagation within the framework of linear response theory.

The solution of (1) subject to the initial condition

$$U(t = 0) = U_0 \tag{2}$$

is formally given by

$$U(t) = e^{t\,M}U_0 + \int_0^t e^{\tau\,M}\,F(t-\tau)\,d\tau. \tag{3}$$

In equation (3), $e^{t\,M}$ is called the evolution operator of the system. Solving (3) requires a suitable approximation for the spatial derivatives, which is achieved by the Fourier pseudospectral method [7] . Thus, equations (1), (2) and (3) should be replaced by the discretized equivalent equations.
The numerical solution is obtained by an optimal expansion of the evolution operator as polynomials, whose region of convergence depends on the spatial matrix M, particularly on the location of its eigenvalues in the complex frequency plane. The form of M depends on the rheology and the unknown variables.

Let a plane wave solution to equation (1) be of the form

$$U = U_0 e^{i(\omega_C t - k \cdot x)}, \tag{4}$$

where x is the position variable, ω_C is the complex frequency, and k is the real wavenumber vector. Substituting (4) into (1), and considering constant material properties and zero body forces, yields an eigenvalue equation for the eigenvalues $\lambda = i\omega_C$. The determinant of the system must be zero in order for U_0 to have a non-zero value. Therefore,

$$\det[\tilde{M} - \lambda I] = 0, \tag{5}$$

where $\tilde{M}$ is the spatial Fourier transform of M , and I is the identity matrix. Hereafter, the complex plane of the eigenvalues is called the z -plane. Equation (5) determines the eigenvalues of $\tilde{M}$ in the Fourier method approximation. Actually, the discretized equation should be used, but (5) represents relatively good approximation.

The eigenvalues are analyzed in Section 2 for the following rheologies:

- ANISOTROPIC-VISCOELASTIC
- Isotropic-viscoelastic

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- Anisotropic-elastic
- Isotropic-elastic
- POROUS ISOTROPIC-VISCOACOUSTIC
- Biot-acoustic
- Isotropic-viscoacoustic

The eigenvalue distribution defines the domain where the evolution operator is approximated by a suitable (rapidly converging) polynomial expansion. For each case, a brief review of the numerical integration techniques is given in Section 3. The methods are the following:

- Taylor expansion
- Chebychev Spectral method
- Rapid expansion method
- Polynomial interpolation through conformal mapping
- Polynomial interpolation by residum minimization

II. WAVE EQUATIONS AND EIGENVALUES OF $\tilde{M}$

Anisotropic-viscoelastic rheology

In order to implement Boltzmann's principle in the generalized Hooke's law, two relaxation functions based on the standard linear solid rheology are considered. One relaxation function describes the anelastic properties of the quasi-dilatational mode, and the other is related to the quasi-shear mode. This can be done by forcing the mean stress to depend on the first relaxation function, and the deviatoric components on the second (in this case, at least for some coordinate system, and usually along symmetry axes of the material). Moreover, the resulting rheological relation gives Hooke's law in the anisotropic-elastic limit, and the isotropic-viscoelastic rheology in the isotropic-anelastic limit [3], [5]. The equation of motion of a two-dimensional anisotropic-viscoelastic medium is formed with the following equations [1]:

i) The equation of momentum conservation:

$$\nabla \cdot T = \rho \ddot{u} + f, \tag{6}$$

where $T^T = [T_1, T_2, T_3, T_4, T_5, T_6] \equiv [\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{xz}, \sigma_{xy}]$ is the stress vector, with σ_{ij}, $i,j = 1,\ldots,3$ the stress components. Defining the position vector by $x = (x, y, z)$, $u(x, t)$ and $f(x, t)$ denote the displacement and body force vectors, respectively; $\rho(x)$ is the density, and $\nabla \cdot$ is a divergence operator defined by

$$\nabla \cdot \rightarrow \nabla_{iJ} = \begin{bmatrix} \partial/\partial x & 0 & 0 & 0 & \partial/\partial z & \partial/\partial y \\ 0 & \partial/\partial y & 0 & \partial/\partial z & 0 & \partial/\partial x \\ 0 & 0 & \partial/\partial z & \partial/\partial y & \partial/\partial x & 0 \end{bmatrix}.$$

ii) The stress-strain relations:

$$T_I = [A_{IJ} + A_{IJ}^{(\nu)} M_{u\nu}] S_J + A_{IJ}^{(\nu)} \sum_{l=1}^{L_\nu} e_{Jl}^{(\nu)}, \tag{7}$$

where $I, J = 1,\ldots,6$, and $\nu = 1, 2$. $S^T = [S_1, S_2, S_3, S_4, S_5, S_6] \equiv [\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, 2\varepsilon_{yz}, 2\varepsilon_{xz}, 2\varepsilon_{xy}]$ is the strain vector, with ε_{ij}, $i,j = 1,\ldots,3$ the strain components; $e_{Jl}^{(\nu)}$ are memory variables related to the L_ν mechanisms which describe the anelastic characteristics of the quasi-dilatational mode ($\nu = 1$), and quasi-shear modes ($\nu = 2$); and A_{IJ} and $A_{IJ}^{(\nu)}$ are functions of the elasticities c_{IJ}, $I, J = 1,\ldots,6$ of the medium. Finally, $M_{u\nu} = [1 - \sum^{L_\nu} (1 - \tau_{\varepsilon l}^{(\nu)}/\tau_{\sigma l}^{(\nu)})]$, where $\tau_{\sigma l}^{(\nu)}$ and $\tau_{\varepsilon l}^{(\nu)}$ are material relaxation times. Implicit summation over repeated indices is assumed.

iii) the memory variable equations:

$$\dot{e}_{Jl}^{(\nu)} = S_J \phi_{\nu l} - e_{Jl}^{(\nu)}/\tau_{\sigma l}^{(\nu)}, \qquad l = 1, \ldots, L_\nu, \tag{8}$$

where $\phi_{\nu l} = (1 - \tau_{\varepsilon l}^{(\nu)}/\tau_{\sigma l}^{(\nu)})/\tau_{\sigma l}^{(\nu)}$.

Equations (6), (7) and (8) are the basis for the numerical solution algorithm. For simplicity, a two-dimensional transversely-isotropic medium with symmetry axis parallel to the z-axis is considered. Then, c_{11}, c_{33}, c_{13} and c_{55} define the elastic characteristics of the medium. Choosing one relaxation mechanism for each mode ($L_1 = L_2 = 1$), the unknown variable vector is given by

$$U^T = [u_x, u_z, \dot{u}_x, \dot{u}_z, e_1, e_2, e_3], \tag{9}$$

where $e_1 = e_{11}^{(1)} + e_{31}^{(1)}$, $e_2 = e_{11}^{(2)} - e_{31}^{(2)}$, and $e_3 = e_{51}^{(2)}$, in terms of the memory variables. The spatial operator is

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ M_{31} & M_{32} & 0 & 0 & M_{35} & M_{36} & M_{37} \\ M_{41} & M_{42} & 0 & 0 & M_{45} & M_{46} & M_{47} \\ M_{51} & M_{52} & 0 & 0 & M_{55} & 0 & 0 \\ M_{61} & M_{62} & 0 & 0 & 0 & M_{66} & 0 \\ M_{71} & M_{72} & 0 & 0 & 0 & 0 & M_{77} \end{bmatrix}, \tag{10}$$

with

$$\rho M_{31} = \partial/\partial x \; [(c_{11} - D) + (D - c_{55}) M_{u1} + c_{55} M_{u2}] \; \partial/\partial x + \partial/\partial z \; (c_{55} M_{u2}) \; \partial/\partial z,$$

$$\rho M_{32} = \partial/\partial x \; [(c_{13} + 2c_{55} - D) + (D - c_{55}) M_{u1} - c_{55} M_{u2}] \; \partial/\partial z + \partial/\partial z \; (c_{55} M_{u2}) \; \partial/\partial x,$$

$$\rho M_{35} = \partial/\partial x \; (D - c_{55}), \quad \rho M_{36} = \partial/\partial x \; c_{55}, \quad \rho M_{37} = \partial/\partial z \; c_{55},$$

$$\rho M_{41} = \partial/\partial z \; [(c_{13} + 2c_{55} - D) + (D - c_{55}) M_{u1} - c_{55} M_{u2}] \; \partial/\partial x + \partial/\partial x \; (c_{55} M_{u2}) \; \partial/\partial z,$$

$$\rho M_{42} = \partial/\partial z \; [(c_{33} - D) + (D - c_{55}) M_{u1} + c_{55} M_{u2}] \; \partial/\partial z + \partial/\partial x \; (c_{55} M_{u2}) \; \partial/\partial x,$$

$$\rho M_{45} = \partial/\partial z \; (D - c_{55}), \quad \rho M_{46} = -\partial/\partial z \; c_{55},$$

$$\rho M_{47} = \partial/\partial x \; c_{55},$$

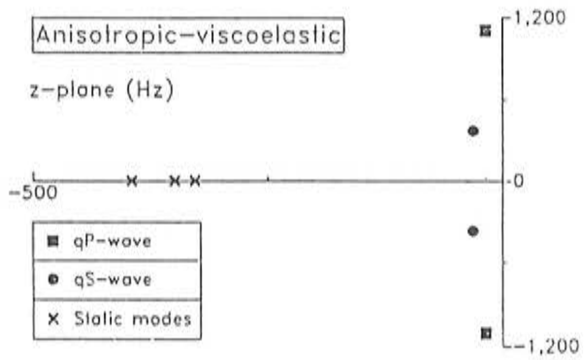
$$M_{51} = \phi_1 \; \partial/\partial x, \quad M_{52} = \phi_1 \; \partial/\partial z, \quad M_{55} = -1/\tau_\sigma^{(1)},$$

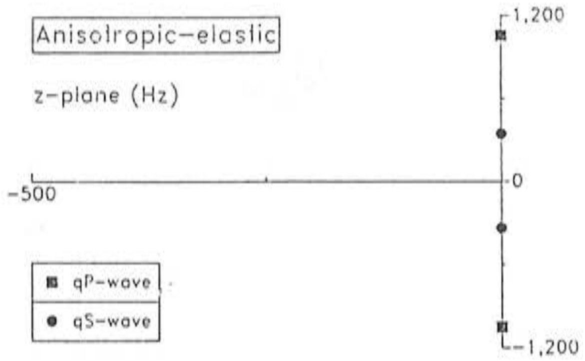
$$M_{61} = \phi_2 \; \partial/\partial x, \quad M_{62} = -\phi_2 \; \partial/\partial z, \quad M_{66} = -1/\tau_\sigma^{(2)},$$

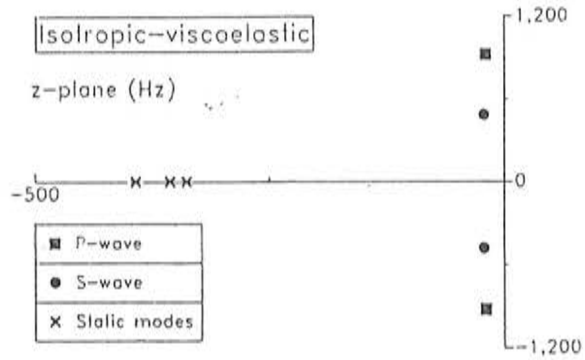
$$M_{71} = \phi_2 \; \partial/\partial z, \quad M_{72} = \phi_2 \; \partial/\partial x, \quad M_{77} = -1/\tau_\sigma^{(2)},$$

where $D = (c_{11} + c_{33})/2$. The subindex l denoting a physical mechanism has been omitted for simplicity. In the anisotropic-elastic limit, i.e., when $\tau_\epsilon^{(\nu)} \to \tau_\sigma^{(\nu)}$, and the memory variables vanish, equation (2.2) become Hooke's law. In the isotropic-viscoelastic limit, c_{11}, $c_{33} \to \lambda + 2\mu$, $c_{13} \to \lambda$ and $c_{55} \to \mu$, with λ and μ the Lame constants, and (7) becomes the isotropic-viscoelastic rheology [5].

The eigenvalues of $\tilde{M}$ are obtained from equation (5), where the following substitution: $\partial/\partial x \to ik_x$, and $\partial/\partial z \to ik_z$, with k_x and k_z the wavenumber components, gives $\tilde{M}$ from M.







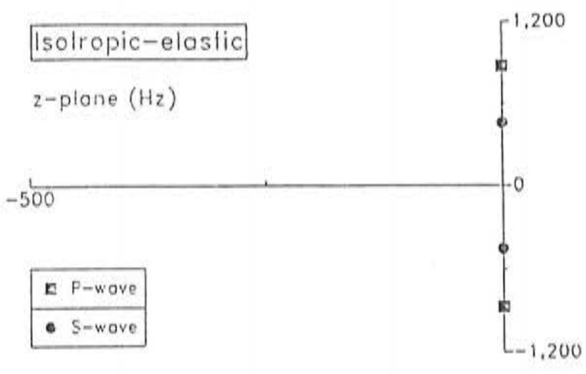


Fig. 1. Eigenvalue distribution of the spatial matrix $\tilde{M}$ in the complex frequency plane for the different rheologies of a single-phase solid.

The eigenvalue distribution for the different rheologies is displayed in Fig. 1. The material is a clayshale having $\tau_\epsilon^{(1)} = \tau_\epsilon^{(2)} = 0.0030\ s^{-1}$, $\tau_\sigma^{(1)} = 0.0027\ s^{-1}$ and $\tau_\sigma^{(2)} = 0.0025\ s^{-1}$, which give highest dissipation around $f = 50$ Hz [1]. The eigenvalues corresponds to $k_x = k_z = 0.16\ m^{-1}$. The negative real part of the propagating modes is a consequence of the anelasticity, stronger for the shear modes. The static modes arise from the fact that the formulation was done in the time-domain; they are grouped approximately around $-1/\tau_\sigma^{(1)}$ and $-1/\tau_\sigma^{(2)}$. The differences are mainly due to anelasticity which introduces the static modes, since anisotropy only produces a shift of the wave mode eigenvalues in the vertical direction. Section 3 analyzes the appropriate methods for each rheology.

Porous isotropic-viscoacoustic rheology

Invoking the correspondence principle, Biot formally obtained a viscoelastic equation of motion which includes all possible dissipation mechanisms. The approach involves the presence of convolutional integrals which arise from the replacement of the elastic coefficients by time operators. When standard linear solid kernels are considered for the time operators, the equation of motion of the isotropic-viscoacoustic porous medium is given by the following equations [2]:

i) Biot equations:

$$\nabla \begin{bmatrix} p \\ p_f \end{bmatrix} = \begin{bmatrix} -\rho & \rho_f \\ -\rho_f & m \end{bmatrix} \begin{bmatrix} \ddot{u} \\ -\ddot{w} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \eta/K \end{bmatrix} \begin{bmatrix} \dot{u} \\ -\dot{w} \end{bmatrix} + \begin{bmatrix} s \\ s_f \end{bmatrix}, \tag{11}$$

where p and p_f are the pressure fields of the matrix-fluid system and fluid, respectively; u is the displacement of the solid; w is a vector representing the flow of the fluid relative to the solid, and s and s_f are body force vectors. The material properties are: ρ, the composite density; ρ_s, the solid density; ρ_f, the fluid density; m, the tortuosity; η, the fluid viscosity; and K, the global permeability.

ii) The stress-strain relations:

$$\begin{bmatrix} p \\ p_f \end{bmatrix} = \begin{bmatrix} \psi_1 & \psi_2 \\ -\psi_2 & \psi_3 \end{bmatrix} \begin{bmatrix} e \\ \zeta \end{bmatrix} + \sum_{l=1}^{L} \left\{ \begin{bmatrix} e_{1l} \\ \zeta_{3l} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e_{2l} \\ \zeta_{2l} \end{bmatrix} \right\}, \quad (12)$$

where e and ζ are the dilatation fields of the solid matrix, and fluid relative to the solid, respectively; and e_{1l}, e_{2l}, ζ_{2l}, and ζ_{3l} are memory variables. $\psi_1 = -(A + R + 2Q)$, $\psi_2 = (Q + R)/\beta$ and $\psi_3 = R/\beta^2$, where A, R, and Q are the classical Biot elastic coefficients, and β is the porosity.

iii) The memory variable equations:

$$\begin{bmatrix} \dot{e}_{1l} \\ \dot{\zeta}_{3l} \end{bmatrix} = \begin{bmatrix} \phi_{1l} & 0 \\ 0 & \phi_{3l} \end{bmatrix} \begin{bmatrix} e \\ \zeta \end{bmatrix} - \begin{bmatrix} 1/\tau_{\sigma l}^{(1)} & 0 \\ 0 & 1/\tau_{\sigma l}^{(3)} \end{bmatrix} \begin{bmatrix} e_{1l} \\ \zeta_{3l} \end{bmatrix}, \quad (13a)$$

$$\begin{bmatrix} \dot{e}_{2l} \\ \dot{\zeta}_{2l} \end{bmatrix} = \begin{bmatrix} -\phi_{2l} & 0 \\ 0 & \phi_{2l} \end{bmatrix} \begin{bmatrix} e \\ \zeta \end{bmatrix} - \frac{1}{\tau_{\sigma l}^{(2)}} I \begin{bmatrix} e_{2l} \\ \zeta_{2l} \end{bmatrix}, \quad (13b)$$

for $l = 1, \ldots, L$, where $\phi_{rl} = -\psi_r L^{-1}(1 - \tau_{\epsilon l}^{(r)}/\tau_{\sigma l}^{(r)})/\tau_{\sigma l}^{(r)}$, $r = 1, 3$, with $\tau_{\sigma l}^{(r)}$ and $\tau_{\epsilon l}^{(r)}$ relaxation times.
In the one-dimensional case with $L = 1$, the unknown vector U has nine components,

$$U^T = [e, \zeta, \dot{e}, \dot{\zeta}, -\dot{w}, e_1, \zeta_3, e_2, \zeta_2]. \quad (14)$$

The spatial matrix M for constant material properties is given by

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ M_{31} & M_{32} & 0 & 0 & M_{35} & M_{36} & M_{37} & M_{38} & M_{39} \\ M_{41} & M_{42} & 0 & 0 & M_{45} & M_{46} & M_{47} & M_{48} & M_{49} \\ M_{51} & M_{52} & 0 & 0 & M_{55} & M_{56} & M_{57} & M_{58} & M_{59} \\ M_{61} & 0 & 0 & 0 & 0 & M_{66} & 0 & 0 & 0 \\ 0 & M_{72} & 0 & 0 & 0 & 0 & M_{77} & 0 & 0 \\ M_{81} & 0 & 0 & 0 & 0 & 0 & 0 & M_{88} & 0 \\ 0 & M_{92} & 0 & 0 & 0 & 0 & 0 & 0 & M_{99} \end{bmatrix},$$

(eq. (15)), where

$\gamma M_{31} = [m\psi_1 + \rho_f\psi_2]\Delta$, $\gamma M_{32} = [m\psi_2 - \rho_f\psi_3]\Delta$,

$\gamma M_{35} = (\rho_f\eta/K)\partial/\partial x$,

$\gamma M_{36} = m\Delta$, $\gamma M_{37} = -\rho_f\Delta$, $\gamma M_{38} = -\rho_f\Delta$,

$\gamma M_{39} = m\Delta$,

$\gamma M_{41} = [\rho_f\psi_1 + \rho\psi_2]\Delta$, $\gamma M_{42} = [\rho_f\psi_2 - \rho\psi_3]\Delta$,

$\gamma M_{45} = (\rho\eta/K)\,\partial/\partial x$,

$\gamma M_{46} = \rho_f\Delta$, $\gamma M_{47} = -\rho\Delta$, $\gamma M_{48} = -\rho\Delta$, $\gamma M_{49} = \rho_f\Delta$,

$\gamma M_{51} = [\rho_f\psi_1 + \rho\psi_2]\,\partial/\partial x$, $\gamma M_{52} = [\rho_f\psi_2 - \rho\psi_3]\,\partial/\partial x$,

$\gamma M_{55} = \rho\eta/K$,

$\gamma M_{56} = \rho_f\,\partial/\partial x$, $\gamma M_{57} = -\rho\,\partial/\partial x$, $\gamma M_{58} = -\rho\,\partial/\partial x$,

$\gamma M_{59} = \rho_f\,\partial/\partial x$,

$M_{61} = \phi_1$, $M_{66} = -1/\tau_\sigma^{(1)}$, $M_{72} = \phi_3$, $M_{77} = -1/\tau_\sigma^{(3)}$,

$M_{81} = -\phi_2$, $M_{88} = -1/\tau_\sigma^{(2)}$, $M_{92} = -\phi_2$, $M_{99} = -1/\tau_\sigma^{(2)}$,

with $\Delta = \partial^2/\partial x^2$ and $\gamma = \rho_f^2 - \rho m$. Biot poroelastic equations are obtained by taking $\tau_\sigma^{(r)} = \tau_\epsilon^{(r)}$, $r = 1,3$. Then, the memory variables vanish and the unknown vector becomes $U^T = [e, \zeta, \dot{e}, \dot{\zeta}, -\dot{w}]$. The equation for a viscoacoustic single-phase solid is obtained with $\rho_f = 0$ and $\phi_2 = \phi_3 = 0$; only one set of relaxation times remains, corresponding to the solid phase (ϕ_1). The unknown vector in this case is $U^T = [e, \dot{e}, e_1]$.

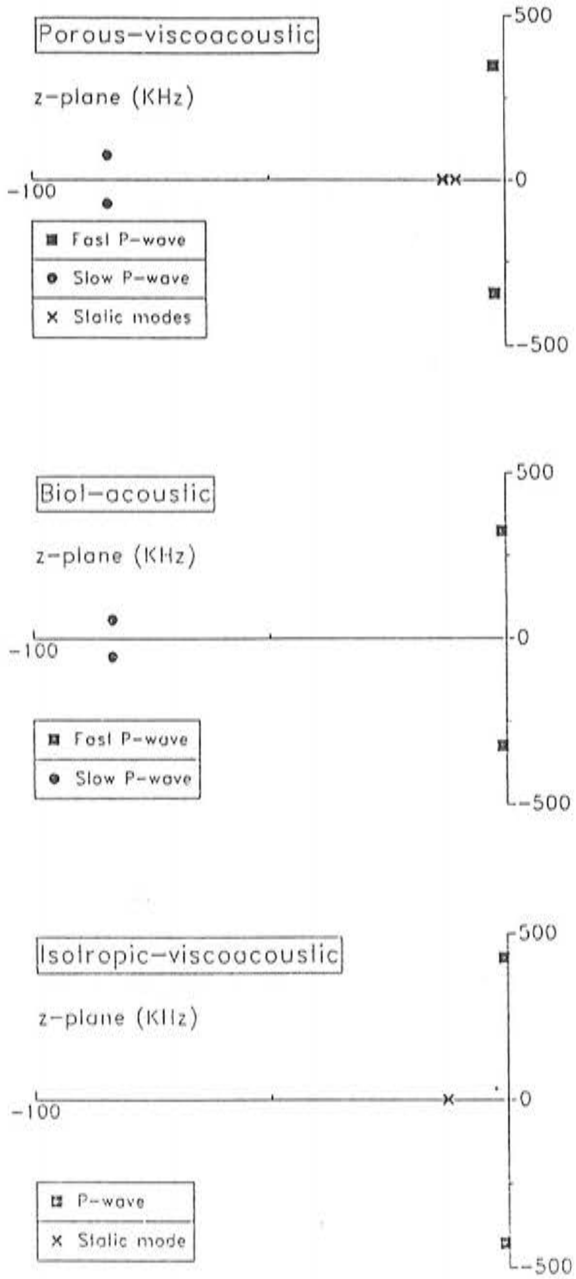


Fig. 2. Eigenvalue distribution of $\tilde{M}$ in the complex frequency plane for a porous viscoacoustic medium and limiting rheologies.

Substitution of $\partial/\partial x$ by ik gives the transformed matrix $\tilde{M}$. Fig. 2 shows the eigenvalue distribution. One of them is zero (not plotted) since the fourth and fifth rows of $\tilde{M}$ are linearly dependent. The slow modes present a quite diffusive behaviour due to the Biot mechanism. They are not present in the single-phase medium, whose attenuation characteristics are viscoelastic.

III. NUMERICAL INTEGRATION METHODS

To illustrate the different techniques, a zero source term is considered for simplicity in equation (1). A detailed formulation with source can be found in the respective references. The formal solution to the system is then given by

$$U(t) = e^{tM}U_0, \tag{16}$$

The numerical solution for general inhomogeneous media requires a polynomial representation of the evolution operator. The different methods are:

<u>Taylor expansion</u> A Taylor expansion of the evolution operator up to the second order is

$$e^{tM} = I + Mt + \frac{1}{2}M^2t^2. \tag{17}$$

Replacing (17) into (16), and substracting $U(-t)$ from $U(t)$ gives

$$U(t) = U(-t) + 2tMU_0. \tag{18}$$

This formula basically gives the equations for second-order temporal differencing valid for small t [7]. Although the region of convergence of the Taylor expansion is the whole z-plane, in order to have high accuracy, the time step should be very small; more precisely, $\Delta t = O(N^{-2})$, using finite-order explicit schemes, where N is the number of grid points.

<u>Chebychev spectral method</u> This technique makes use of the following expansion of e^z [11]:

$$e^z \simeq \sum_{k=0}^{K} C_k J_k(tR) Q_k\left[\frac{z}{tR}\right], \tag{19}$$

where $|z| \le tR$, and z lies close to the imaginary axis. $C_0 = 1$ and $C_k = 2$ for $k \ge 1$, J_k is the Bessel function of order k , and Q_k are modified Chebychev polynomials which satisfy the recurrence relation

$$Q_{k+1}(s) = 2sQ_k(s) + Q_{k-1}(s), \quad Q_0 = 1, \quad Q_1 = s. \tag{20}$$

Substituting tM for z in (19), equation (16) becomes

$$U(t) \simeq \sum_{k=1}^{K} C_k J_k(tR) Q_k\left[\frac{M}{R}\right] U_0, \tag{21}$$

The series has a rapid convergence for $K > tR$, with $K = O(N)$. The value of R should be chosen larger than the range of the eigenvalues of tM. Since this expansion converges for the imaginary axis of the z-plane, it is appropriate for the elastic case [7]. Anelastic problems can be solved with less efficiency using a slight modification [4].

<u>Rapid expansion method</u> In the elastic case where no first time derivatives of the displacements and memory variables are present, the wave equation of the system can be expressed as

$$\ddot{u} = -L^2u + f, \tag{22}$$

where u is the displacement vector, f is the body force vector, and $-L^2$ is a linear matrix operator similar to M [9]. For zero body forces the formal solution to (22) is

$$u(t) = \cos Lt\, u(0) + \frac{\sin Lt}{L}\dot{u}(0). \tag{23}$$

Adding solutions (23) for times t and −t, the displacement time derivative can be eliminated, and the displacement at time t becomes

$$u(t) = -u(-t) + 2\cos(Lt)\,u(0). \tag{24}$$

The method uses the following expansion:

$$\cos Lt \simeq \sum_{k=0}^{K/2} C_{2k} J_{2k}(tR) Q_{2k}\left[\frac{iL}{R}\right], \tag{25}$$

This expansion represents an improvement over the Chebychev spectral method since it contains only even order functions Q_{2k}, however, it can be used only for elastic problems [8].

<u>Polynomial interpolation through conformal mapping</u> As shown in the previous section, in a single-phase anelastic solid, the eigenvalues of M lie on a T-shaped domain D which includes the negative real axis and the imaginary axis. This approach is based on a polynomial interpolation of the exponential function in the complex domain D, on a set of points which is known to have maximal properties. This set, known as Fejer points, is found through a conformal mapping between the unit disc and the domain of the eigenvalues D. In this way, the interpolating polynomial is "almost best" [10].

Getting the Fejer points is as follows: Let $\chi(u)$ be a conformal mapping from the u-plane to z-space, which maps the complement of a disc of radius δ to the complement of D, where δ is the logarithmic capacity of D, given by the limit $\delta = |\chi'(\infty)|$, the prime denoting derivative with respect to the argument. The analytic expression for $\chi(u)$ corresponding to the domain D can be found in [10]. The same function $\chi(u\delta)$ maps the complement of the unit disc to the complement of the domain D.

Then, the Fejer points are $z_j = \chi(u_j)$, $j = 0, \ldots, m-1$ where u_j are the m roots of the equation $u^m = \delta$, with m the degree of the polynomial. The set $[z_j]$, $j = 0, \ldots, m-1$ has maximal properties of convergence. Then, the sequence of polynomials $P_m(z)$ of degree m found by interpolation to an arbitrary function f(z), analytic on D at the points z_j, converge maximally to f(z) on D. The interpolating polynomial in Newton form is

$$P_m(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)(z - z_1) + \ldots$$

$$+ a_m(z - z_0)\ldots(z - z_{m-1}), \tag{26}$$

where $a_j = f[z_0, \ldots, z_j]$, $j = 0, \ldots, m-1$, are the divided differences. The approximating polynomial is given by $P_m(Mt)$ with $f(z) = e^z$.

Polynomial interpolation by residum minimization The preceding method requires a conformal mapping from the unit disc to the domain of the eigenvalues of M to find the interpolating points. This new technique avoids the conformal mapping by finding the interpolating points automatically in an optimal way [12]. Therefore, the method can be applied for any general matrix M, no matter what the domain D.

The idea is to find the interpolating points by minimizing the L_2 -norm of the error. It is well known that the error of the interpolation is

$$E_m(z) = f(z) - P_m(z) = \frac{f^{(m)}(s)}{m!} R_m, \qquad (27a)$$

with

$$R_m(z) = \prod_{i=1}^{m} (z - z_{i-1}) = \sum_{k=0}^{m-1} \alpha_k z^k + z^m \qquad (27b)$$

and s the value for which $f(s) - P_m(s) - E_m(s) = 0$. The superindex (m) denotes the $(m)^{th}$ derivative. Substituting Mt for z in (27a) and using (16), the error of the algorithm is

$$E_m = \frac{f^{(m)}(s)}{m!} \Sigma_m, \quad \text{where } \Sigma_m = R_m(Mt) U_0. \qquad (28a-b)$$

Minimizing the L_2-norm $\|\Sigma_m\|^2 = (\Sigma_m, \Sigma_m)$ is achieved by solving the following set of m linear equations:

$$\frac{\partial}{\partial \alpha_i} \|\Sigma_m\|^2 = 0, \qquad i = 0, \ldots, m-1. \qquad (29)$$

This is equivalent to solve the following system:

$$DA = B, \qquad A = [\alpha_0, \ \ldots \alpha_{m-1}]^T, \qquad (30)$$

where

$$D_{ij} = \left((Mt)^{i-1} U_0, \ (Mt)^{j-1} U_0\right), \qquad (31a)$$

$$B_i = -\left((Mt)^{i-1} U_0, \ (Mt)^m U_0\right), \qquad 1 \le i, j \le m. \qquad (31b)$$

After solving for A, the interpolating points are obtained from the roots of $R_m(z)$. The approximating polynomial is given by (26) with $f(z) = e^z$. Further research is required to determine whether this technique improves the efficiency when solving anelastic wave propagation problems.

IV. CONCLUSIONS

This work briefly reviews some of the theories and algorithms for solving wave propagation problems in linear viscoelastic media. The methods use spectral techniques and solve the wave equation in the time-domain. A consistent introduction of Boltzmann's after-effect principle in the time-domain, for anisotropic and porous media, is achieved by the introduction of memory variables. Some additional assumptions are required in the anisotropic case for the determination of the constitutive relations. The eigenvalue analysis for each rheology indicates that spectral Chebychev methods are suitable for elastic problems, and that polynomial interpolation techniques are required when the medium is anelastic.

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