Journal of Computational Acoustics, Vol. 9, No. 4 (2001) 1347–1354


# ABSORBING LAYER VIA WAVE-EQUATION SPLITTING*

JOSÉ M. CARCIONE† and FABIO CAVALLINI‡
*Istituto Nazionale di Oceanografia e di Geofisica Sperimentale — OGS,*
*Borgo Grotta Gigante 42/C, I-34010 Sgonico (Trieste), Italy*
†*jcarcione@ogs.trieste.it*
‡*fcavallini@ogs.trieste.it*

Received 1 June 1999
Revised 22 January 2001

Modeling acoustic waves generated by a localized source is always vexed by the nagging problem of spurious reflections and wraparound arising when the wavefront reaches the boundary of the numerical mesh. This difficulty may be circumvented by using a very large computational domain, which is very inefficient, or can be tackled by using some kind of absorbing boundary technique, which has not yet found a universally satisfactory solution. In this work, the wave equation is modified by introducing a term that is nonzero only in a narrow strip near the boundary. Then, a splitting technique permits to compute part of the solution analytically (hence, at no computational cost), while an application of Weyl's formula for the exponential of a matrix leads to a second-order accurate scheme that completes the algorithm. An application to SH seismic wave modeling shows that the performance of the present method is competitive with standard ones. Moreover, there is evidence for a potential application to the modeling of wave propagation in porous media, where stiff differential equations arise.

## 1. Introduction

The problem of absorbing nonphysical reflections or wraparound from the boundaries of a numerical mesh has been extensively investigated. For instance, some authors[1–3] use a Gaussian or an exponential decaying factor in a region surrounding the model. This approach is suggested by the fact that, when all the properties of the medium are constant in space, the solution represents travelling waves that are exponentially attenuated in space, with all frequency components equally attenuated.[4]

The aim of the present work is to justify the choice of an exponential damping function in the light of a splitting method applied to the wave equation, by showing that it is related to a physical solution of the problem. Moreover, the absorbing-layer problem is solved in closed analytical form in the sense that the solution is not affected by the accuracy of the polynomial expansion of the evolution operator within the computational domain. Finally, we apply Weyl's formula to design a numerical algorithm that solves the problem of wave

*Presented at ICTCA'99, the 4th International Conference on Theoretical and Computational Acoustics, May 1999, Trieste, Italy.

propagation in the absorbing strip more accurately than the original technique by Cerjan and coworkers,[1] yet demanding a small increase of computational cost.

## 2. The Wave Equation and the Absorbing Layer

Wave propagation in a linear medium, including anisotropic and attenuation effects, and describing motion in a porous material,[5–7] can be expressed in mathematical form as a first-order vector differential equation in time

$$\frac{\partial \mathbf{u}}{\partial t} = \mathcal{M}\mathbf{u} + \mathbf{f}\,, \tag{2.1}$$

where $\mathbf{u}(t,\,\mathbf{x})$ is the unknown time-dependent $d$-dimensional vector field, $\mathcal{M}$ is a time-independent linear differential operator containing the spatial derivatives and material properties, and $\mathbf{f}(t,\,\mathbf{x})$ is (essentially) the body force vector. The formal solution of Eq. (2.1) is given by

$$\mathbf{u}(t,\,\mathbf{x}) = \mathrm{e}^{(t-t_0)\mathcal{M}}\mathbf{u}(t_0,\,\mathbf{x}) + \int_0^{t-t_0} \mathrm{e}^{\tau\mathcal{M}}\mathbf{f}(t-\tau,\,\mathbf{x})\mathrm{d}\tau\,. \tag{2.2}$$

Putting in the previous equation $t = k\Delta t$, $t_0 = (k-1)\Delta t$ and introducing $\mathbf{u}_k(\mathbf{x}) = \mathbf{u}(k\Delta t,\,\mathbf{x})$, we get the numerical time-marching algorithm

$$\mathbf{u}_k(\mathbf{x}) = \mathrm{e}^{\Delta t\mathcal{M}}\mathbf{u}_{k-1}(\mathbf{x}) + \int_0^{\Delta t} \mathrm{e}^{\tau\mathcal{M}}\mathbf{f}(k\Delta t-\tau,\,\mathbf{x})\mathrm{d}\tau\,. \tag{2.3}$$

If the forcing term $\mathbf{f}$ does not vary much during a time-step of duration $\Delta t$, then the integrand factor $\mathbf{f}(k\Delta t - \tau,\,\mathbf{x})$ may be approximated with the time-independent vector function $\mathbf{f}_{k-1}(\mathbf{x}) = \mathbf{f}((k-1/2)\Delta t,\,\mathbf{x})$, so that the convolution integral can be computed analytically, and algorithm (2.3) simplifies to

$$\mathbf{u}_k(\mathbf{x}) = \mathrm{e}^{\Delta t\mathcal{M}}(\mathbf{u}_{k-1}(\mathbf{x}) - \mathcal{M}^{-1}\mathbf{f}_{k-1}(\mathbf{x})) - \mathcal{M}^{-1}\mathbf{f}_{k-1}(\mathbf{x})\,. \tag{2.4}$$

From this equation, it is clear that we need to compute a single exponential of an operator once for all and, at each time step, solve a single linear system of first-order differential equations, namely $\mathcal{M}\mathbf{v}(\mathbf{x}) = \mathbf{f}_{k-1}(\mathbf{x})$.

The numerical solution of the problem on a numerical mesh of $N$ points requires the spatial discretization of the model and the calculation of the spatial derivatives with a discrete differential operator. In other words, from now on we represent function $\mathbf{u}_k$ with the $(d \times N)$-dimensional vector $\mathbf{u}_k$ of its sampled values, and operator $\mathcal{M}$ with a matrix $\mathbf{M}$.

In order to eliminate reflections or wraparound from the boundaries of the numerical mesh, an absorbing strip is implemented at the boundaries of the grid. This is achieved by replacing in Eq. (2.1) the operator $\mathcal{M}$ with $\Gamma + \mathcal{M}$, where $\Gamma$ is a space-dependent algebraic operator describing the absorption,[4] i.e. $\Gamma$ is zero everywhere except in a narrow band adjacent to the boundary where it is negative. Accordingly, matrix $\mathbf{M}$ is substituted with $\mathbf{G} + \mathbf{M}$, where $\mathbf{G}$ is the discretized version of operator $\Gamma$, and thus a diagonal matrix whose nonzero entries are the values $\gamma_1, \ldots, \gamma_N$ of a friction coefficient $\gamma$ at the nodes of the

mesh. The recursive algorithm (2.3) then becomes, after discretization and inclusion of the absorbing boundary,

$$\mathbf{u}_k = \mathrm{e}^{\Delta t(\mathbf{G}+\mathbf{M})}\mathbf{u}_{k-1} + \int_0^{\Delta t} \mathrm{e}^{\tau(\mathbf{G}+\mathbf{M})}\mathbf{f}(k\Delta t - \tau)\mathrm{d}\tau\,. \tag{2.5}$$

Thus, a naive implementation of algorithms (2.3) or (2.4) would be to approximate matrix exponentials by means of a truncated Taylor series expansion of the kind

$$\mathrm{e}^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \cdots \tag{2.6}$$

and then to substitute $\mathbf{A}$ with $\Delta t(\mathbf{G}+\mathbf{M})$, or similar, in the expansion. But one can do better, as we shall see in the next section.

## 3. The Splitting Method

Explicit time integration algorithms imply a polynomial expansion of the discretized evolution operator

$$\exp[\Delta t(\mathbf{G}+\mathbf{M})]\,. \tag{3.1}$$

A Taylor integration algorithm, for instance, is based on expansion (2.6) of the exponential function. Such an operation involves the choice of a given order of truncation of the series. An $n$th-order expansion of the evolution operator (3.1) requires $O(n)$ operations with $\mathbf{G}+\mathbf{M}$. However, the absorbing operation can be solved in closed analytical form by splitting the evolution operator, so that expansion (2.6) is applied to $\mathbf{M}$ alone. Indeed, it is well known[8] that if any two matrices $\mathbf{A}$ and $\mathbf{B}$ both commute with their commutator, then they satisfy Weyl's formula (much used in quantum mechanics):

$$\exp(\mathbf{A})\exp(\mathbf{B}) = \exp(\mathbf{A}+\mathbf{B}+[\mathbf{A},\,\mathbf{B}]/2)\,, \tag{3.2}$$

where $[\mathbf{A},\,\mathbf{B}] = \mathbf{AB}-\mathbf{BA}$ is the commutator. We shall use Weyl's formula in the equivalent form

$$\exp(\mathbf{A}+\mathbf{B}) = \exp(\mathbf{A})\exp(\mathbf{B})\exp\left(\frac{-[\mathbf{A},\,\mathbf{B}]}{2}\right)\,. \tag{3.3}$$

In our case, matrices $\Delta t\mathbf{G}$ and $\Delta t\mathbf{M}$ satisfy the assumption of Weyl's formula up to infinitesimals of order three as the time step $\Delta t$ tends to zero. Indeed we have

$$[\Delta t\mathbf{G},\,[\Delta t\mathbf{G},\,\Delta t\mathbf{M}]] = (\Delta t)^3[\mathbf{G},\,[\mathbf{G},\,\mathbf{M}]] \tag{3.4}$$

$$[\Delta t\mathbf{M},\,[\Delta t\mathbf{G},\,\Delta t\mathbf{M}]] = (\Delta t)^3[\mathbf{M},\,[\mathbf{G},\,\mathbf{M}]]\,. \tag{3.5}$$

Therefore, we may apply (with good approximation) Weyl's formula (3.3) to obtain

$$\exp(\Delta t(\mathbf{G}+\mathbf{M})) \cong \exp(\Delta t\mathbf{G})\exp(\Delta t\mathbf{M})\exp\left(\frac{-(\Delta t)^2[\mathbf{G},\,\mathbf{M}]}{2}\right) \tag{3.6}$$

which is the cornerstone of our improved numerical implementation of the absorbing boundary layer. In fact, using this formula, we may rearrange algorithm (2.5) as

$$\mathbf{u}_k = \exp(\Delta t\mathbf{G})\exp(\Delta t\mathbf{M})\exp(-(\Delta t)^2[\mathbf{G},\,\mathbf{M}])\mathbf{u}_{k-1} + \int_0^{\Delta t}\exp(\tau\mathbf{G})\exp(\tau\mathbf{M})\exp(-\tau^2[\mathbf{G},\,\mathbf{M}])\mathbf{f}(k\Delta t-\tau)\mathrm{d}\tau\,. \tag{3.7}$$

This method borrows from Cerjan *et al.*[1] the heuristic idea of an absorbing boundary, but there are substantial differences in the respective developments. While their approach is essentially a numerical trick that modifies the discretized equations, here a damping term is added to the original partial differential equations: this leads to a clear physical interpretation, allows for a mathematical analysis of the technique, and makes the procedure applicable also to finite elements (besides finite differences and spectral methods). Accordingly, the present algorithm is more akin to that of Kosloff and Kosloff[4] or Sochacki *et al.*,[2,3] in so far as they modify the wave equation by adding a diffusive and, possibly, a damping term in which suitable coefficients are switched to zero off the boundary zone. In the next section, we shall go a step forward by applying the splitting method to such an equation: this will yield a substantial part of the solution analytically. From Eq. (3.7), we recover the procedure of Sochacki *et al.*[2,3] by just disregarding exponentials containing $(\Delta t)^2$ and $\tau^2$. Their first-order algorithm is cheaper than the scheme obtained from Eq. (2.5) because in the former, the effect of the absorbing layer consists in a simple componentwise multiplication by the nonzero scalar factors constituted by the entries $\exp(\Delta t\gamma_1),\ldots,\exp(\Delta t\gamma_N)$ of the diagonal matrix $\exp(\Delta t\mathbf{G})$, and similarly for $\exp(\tau\mathbf{G})$, whereas the nondiagonal matrix exponentials remain the same as in the undamped case.

On the other hand, the cost to be paid for the additional accuracy ensured by the second-order algorithm (3.7) is not high, because the exponentials containing the commutator may be computed analytically and exactly. Indeed, straightforward analytical computations show that the commutator between the differential matrix operator $\mathcal{M}$ and the algebraic matrix operator $\Gamma$ is the simple algebraic matrix operator

$$\mathcal{C} = \begin{pmatrix} 0 & \gamma_z/\rho & \gamma_x/\rho \\ \mu\gamma_z & 0 & 0 \\ \mu\gamma_x & 0 & 0 \end{pmatrix}, \tag{3.8}$$

where a subscript denotes partial differentiation. Hence we get for any real number $\xi$, using the Lagrange–Sylvester interpolation polynomial,[9]

$$\exp(\xi\mathcal{C}) = \frac{1}{r^2}\left(\frac{1}{2}\exp(r\xi)(\mathcal{C}+rI)\cdot\mathcal{C} + \frac{1}{2}\exp(-r\xi)(\mathcal{C}-rI)\cdot\mathcal{C} + r^2 I - \mathcal{C}\cdot\mathcal{C}\right) \tag{3.9}$$

where $r=\sqrt{(\mu/\rho)(\gamma_x^2+\gamma_z^2)}$ and $I$ is the identity, and then $\exp(-(\Delta t)^2[\mathbf{G},\,\mathbf{M}]/2)$ follows by substituting $\mathcal{C}$ with its discretized version. Finally, we point out that it is not obvious how to design a second order in time algorithm that produces an efficient absorbing boundary layer in a natural way. For example, the leapfrog scheme seems the least apt to this purpose

as the amplification factor of the Fourier mode used in the von Neumann stability analysis is exactly 1 for the leapfrog scheme: In other words, the leapfrog scheme has no damping whatever $\Delta t$ and $\Delta x$ may be.

## 4. Example

We consider the two-dimensional SH wave equation. Newton's second law (as generalized to continua by Cauchy, Green, and others) and Hooke's law yield the velocity-stress formulation:

$$\frac{\partial v}{\partial t} = \frac{1}{\rho}\left(\frac{\partial \sigma_{zy}}{\partial z} + \frac{\partial \sigma_{xy}}{\partial x}\right) + f + \gamma v\,, \tag{4.1}$$

$$\frac{\partial \sigma_{zy}}{\partial t} = \mu\frac{\partial v}{\partial z} + \gamma\sigma_{zy}\,, \quad \frac{\partial \sigma_{xy}}{\partial t} = \mu\frac{\partial v}{\partial x} + \gamma\sigma_{xy}\,, \tag{4.2}$$

where $v$ is the particle velocity, $\sigma$ denotes stress, $f$ is the body force, and $\gamma(x,\, z) \leq 0$ is the friction coefficient describing the absorbing layer. In matrix form, we obtain equation

$$\frac{\partial \mathbf{u}}{\partial t} = (\Gamma + \mathcal{M})\mathbf{u} + \mathbf{f}\,, \tag{4.3}$$

with

$$\mathbf{u} = [v,\, \sigma_{zy},\, \sigma_{xy}]^\top, \quad \mathbf{f} = [f,\, 0,\, 0]^\top, \tag{4.4}$$

and

$$\mathcal{M} = \begin{pmatrix} 0 & \rho^{-1}\partial_z & \rho^{-1}\partial_x \\ \mu\partial_z & 0 & 0 \\ \mu\partial_x & 0 & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix}. \tag{4.5}$$

The following spatial dependence for $\gamma$ is chosen,[4]

$$\gamma = -\frac{U_0}{\cosh^2(\alpha \cdot n)} \tag{4.6}$$

where $U_0$ is a constant, $\alpha$ is a decay factor and $n$ denotes the distance in number of grid points from the boundary.

The model is illustrated in Fig. 1, where the wave velocities and density are indicated. They are related to the rigidity modulus $\mu$ by the relation $\rho V^2 = \mu$. The calculations use a grid size of $81 \times 81$ with $dx = dz = 20\,\mathrm{m}$. The point source (indicated by a star in Fig. 1) is a Ricker wavelet with a central frequency of 25 Hz. The spatial derivatives are solved by the Fourier method.[5] Optimal absorbing parameters are $n_{\max} = 18$, $\alpha = 0.11\,\mathrm{m}^{-1}$, and

$$U_0 = \frac{AV_{\max}}{d_{\min}}\,, \tag{4.7}$$

where $A = 0.35$, $V_{\max}$ is the maximum velocity in the mesh, and $d_{\min}$ is the minimum grid size. The need of a thick absorbing layer is in qualitative agreement with what happens

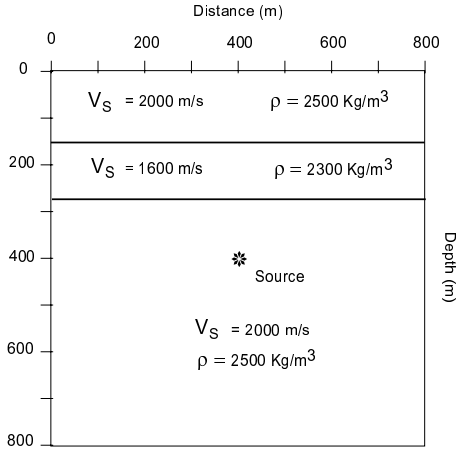


Fig. 1. Computational domain and material properties of the medium under study.

with other methods: For example, it has been pointed out[10] that the method of the Perfectly Matched Layer (PML) also requires relatively large absorbing layers for accuracy at lower frequencies. Unfortunately, a thorough theoretical comparison of the present method with the PML method can hardly be accomplished, because the PML method introduces auxiliary variables (with no physical meaning) that substantially alter the governing equations.[10] On the other hand, resorting to extensive numerical experimentation with both methods about different benchmark problems on different computing environments would require a huge amount of work: As such, it is outside the scope of this preliminary study.

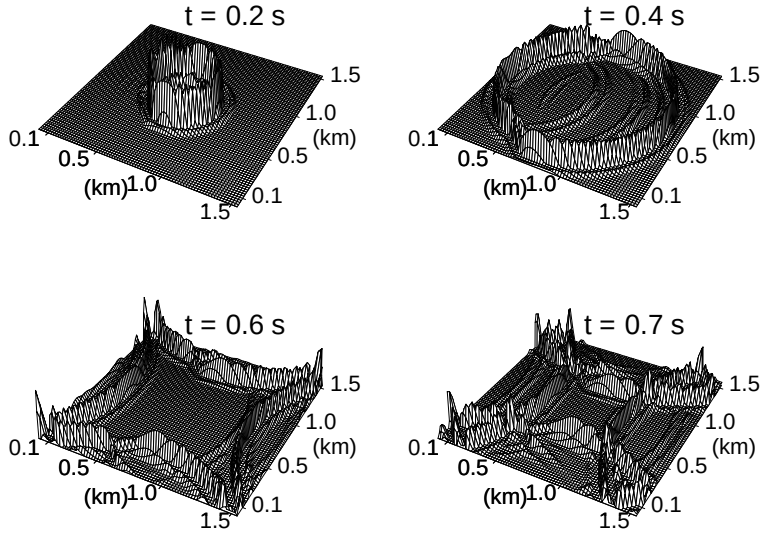


Fig. 2. Snapshots of the wavefield computed without any nonreflecting boundary condition. Reflections from the boundary of the computational domain are quite evident.

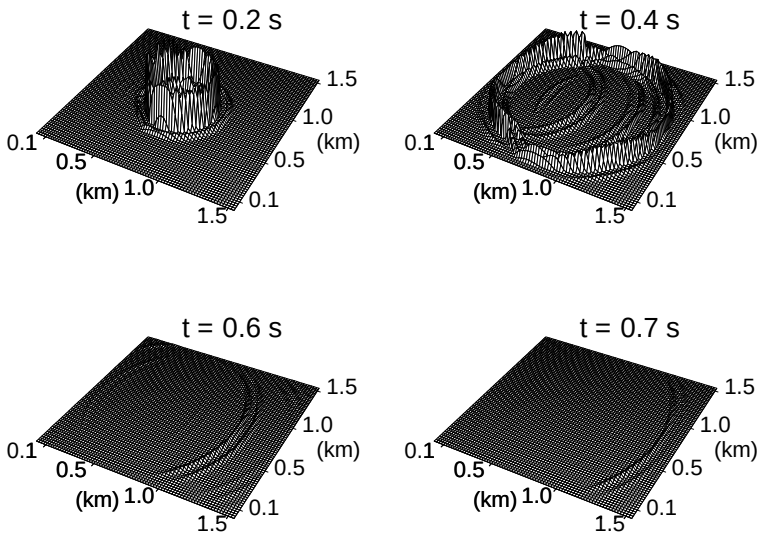


Fig. 3. Snapshots of the wavefield computed with our absorbing-layer technique. Reflections from the boundary of the computational domain are completely absent.

Figure 2 shows amplitude snapshots, at progressive times, of a simulation in which no absorbing boundary or layer has been implemented. In this case, spurious reflections and substantial wraparound effects are visible. For comparison, Fig. 3 shows the corresponding snapshots for a model in which we implemented our absorbing-layer technique, thus suppressing all spurious reflections.

## 5. Conclusions

The splitting method applied to wave equation can be used to solve the absorbing layer problem without any further approximation. The exponential decay function, extensively used as a damper of nonphysical reflections and wraparound, naturally arises as a physical solution of the evolution problem.

When using an $n$th-order expansion of the evolution operator, the absorbing operation requires $n \times N$ multiplications, where $N$ is the number of grid points inside the absorbing layer, and additional memory storage for the absorbing strips. The splitting method reduces the number of operations to $N$. There is no contradiction in using a high-order approximation of the evolution matrix (3.1) in the interior of the computational domain and a second-order (or even a first-order) scheme in the boundary layer: While we want to know the solution accurately in the interior, any gross approximation (provided it decays without reflection) is good enough in the absorbing layer. However, in principle, such a change of approximation order might generate spurious reflections and hence decrease the efficiency of the absorbing layer; but, for a fine enough mesh, this effect is not noticed.

An example, based on the two-dimensional SH wave equation, illustrates the effectiveness of this technique.

Finally, we argue that Weyl's formula may also be used for solving stiff wave propagation problems as, for instance, modeling of Biot-type waves. In this case, the eigenvalues of the propagation matrix have negative real parts and differ greatly in magnitude. The solution to be computed is slowly varying, but perturbations exist that are rapidly damped. The perturbation is the slow wave which, in the presence of fluid viscosity, presents a diffusive character. The resulting algorithm will have, in general, second-order accuracy.

## References

1. C. Cerjan, D. Kosloff, R. Kosloff, and M. Reshef, "A nonreflecting boundary condition for discrete acoustic and elastic wave equations," *Geophysics* **50**, 705 (1985).
2. J. S. Sochacki, R. Kubichek, J. H. George, W. R. Fletcher, and S. Smithson, "Absorbing boundary conditions and surface waves," *Geophysics* **52**, 60 (1987).
3. J. S. Sochacki, J. H. George, R. E. Ewing, and S. B. Smithson, "Interface conditions for acoustic and elastic wave propagation," *Geophysics* **56**, 168 (1991).
4. R. Kosloff and D. Kosloff, "Absorbing boundaries for wave propagation problems," *J. Comput. Phys.* **63**, 363 (1986).
5. J. M. Carcione, D. Kosloff, and R. Kosloff, "Wave propagation simulation in a linear viscoelastic medium," *Geophys. J. Roy. Astr. Soc.* **95**, 597 (1988).
6. J. M. Carcione, D. Kosloff, and R. Kosloff, "Wave propagation simulation in an elastic (transversely isotropic) solid," *Q. Jl. Mech. Appl. Math.* **41**, 320 (1988).
7. J. M. Carcione, "Viscoelastic effective rheologies for modelling wave propagation in porous media," *Geophysical Prospecting* **46**, 249 (1998).
8. J. G. Broida and S. G. Williamson, *A Comprehensive Introduction to Linear Algebra* (Addison-Wesley, Wokingham, 1989).
9. F. R. Gantmacher, The theory of matrices, Vol. 1. (Chelsea, New York, 1977).
10. Q. Qi and T. L. Geers, "Evaluation of the perfectly matched layer (PML) for computational acoustics," *J. Comput. Phys.* **139**, 166 (1998).