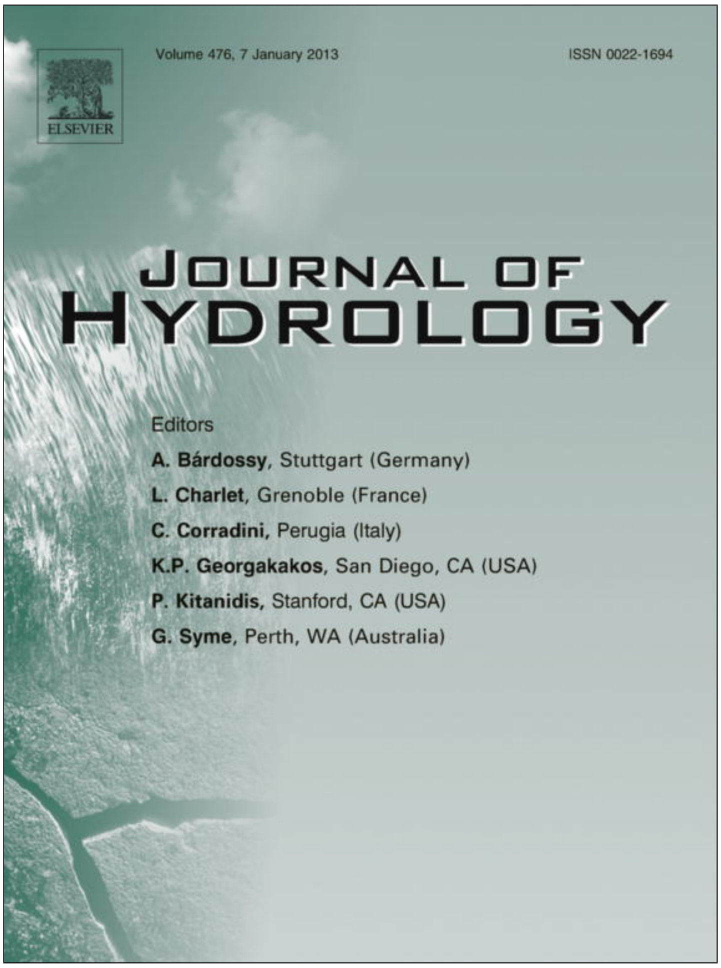
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A memory model of sedimentation in water reservoirs

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SUMMARY

We consider a one-dimensional model of water reservoir, where the sediment is diffusing according to the Fourier law modified with the introduction of a derivative of fractional distributed orders as memory formalism. The fractional order is equivalent to a time-varying diffusivity and the distributed orders represent a variety of memory mechanisms to model a sediment with a varied distribution of grain sizes. Using the Laplace transform (LT), we find the solution in the case when the flux is constant at the source and is arbitrarily given at the output. Then, the time-domain solution is obtained by means of a numerical Fourier transform. We apply a one-dimensional simplified model, with the diffusion governed by two parameters, to the Quarto Nuovo (Italy) reservoir, where the flux of sediment at the output is obtained from observed data. It is found that the flux increases when one of the parameters defining the diffusion model, the pseudo-diffusivity, is increasing or when the other parameter defining the diffusion, the order of fractional differentiation, is decreasing. When the latter parameter is nil, one obtains the classic diffusion with maximum flux.

1. Introduction

1.1. Foreword on water reservoirs and sediment diffusion

Due to the shortage of fresh water the artificial water reservoirs are of ever increasing importance. They are mostly contained by dams made of earth or concrete. Both types of reservoirs, however, are doomed to inefficiency because of the phenomenon of the diffusion of sediments filling the reservoirs. For instance, in the Italian Alps, the lifetime of the water reservoirs lies between 50 and 100 years depending on the sediments contained in their tributary streams, on their geometry and on the use which set serious constraints on their economic convenience.

Monitoring the deposits of sediment in the water reservoir in the first few years after the construction is generally sufficient to have an approximate estimate of its future efficiency which concern its investors and managers.

The phenomenon of sediment diffusion and deposit in the water reservoirs is extremely complex. Most important is the varied density distribution function of the sediment relative to its size and weight, but are also important the seasonal variation in river flow, the reservoir operation scheduling and the irregular and varying shape and volume of the reservoirs occurring at the same time of the deposit of the sediment, whose fate is influenced by the force of gravity and by the velocity of the water.

In recent times, we have been gratified by several studies concerning the estimate of the flux of sediment versus time and at different locations along the water reservoirs. To quote the works which may seem more pertinent, we begin with that of Zyryanov (1973) reporting on the silting of the Uch-Kurgansk hydroelectric station and on the silt control. Bodulski and Górski (2007) studied the silting of the Cedzyna water reservoir in Poland in the period 1972–2003, finding that the volume had decreased by 113,000 m^3.

Rădoane and Rădoane (2005) analyzed the data of 138 reservoirs with relatively large volume affected by the phenomenon of silting and found that it is very serious for 11% of them and serious for 22%. Sharma and Dubey (2001) discussed the remote monitoring of the silting in water reservoirs for estimating the silting delivery rate. Cogollo and Villela (1988) provided means of estimating the sediment distribution in time and space inside a reservoir.

Chen et al. (1978) produced a model for the prediction of the deposit of sediment in reservoirs. The river is modeled by a single channel assuming that the one-dimensional flow phenomena are dominant, whereas a compound stream model approach is used to simulate the main river and the flood plains of the reservoir. Their jet theory is incorporated in a mathematical model by Lopez (1978) and the resulting flow field, computed with the scheme of finite differences, is used to route the sediment through the

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reservoir. The simulated bed profiles generated by the mathematical model compare well with measured data.

Some of the difficulties in modeling the diffusion could be tentatively overcome by introducing appropriate linear phenomenological equations at the price to loose some of the intuitive properties of the classic equations. On the other hand, the phenomenological equations may allow to find different points of view on the evolution of the phenomenon of diffusion and, possibly, new concepts.

There are many generalizations of the original diffusion equation for use in various fields of science. A relevant case for anisotropic media is the substitution of the scalar parameter of diffusion with a tensor. Another relevant case is the Fokker–Planck equation describing the time evolution of the probability density function of the position of the diffusing particles.

The diffusion equation has been generalized with the introduction of memory formalisms represented by fractional order derivatives (Wyss, 1986; Mainardi, 1993). In this work, concerning the diffusion of sediment in water reservoirs, the diffusion equation is generalized by introducing a distributed order fractional derivative to represent the effect of the various density distribution functions of the sizes and weights of the particles forming the sediment. The scope is then to present a mathematical model based on a memory formalism for the diffusion of sediment in a one-dimensional water reservoir. The more general model of sedimentation in the water reservoirs introduced here, when adequate data is available, would give the flux of sediment along the reservoirs and may possibly estimate the evolution of their efficiency and improve the capability of forecast of their lifetime. It may also be of help in selecting the sites of future reservoirs in connection with the estimate of sediments in the tributary streams.

1.2. The use of the mathematical memory formalism

The basic notion of memory functions is widely recognized in science in general and, in particular, in the fields of mathematical physics, engineering and biology. Numerous applications of mathematical memory formalisms to the description of physical phenomena have been published. We try here to recall some contributions being sure that some work will be unintentionally omitted.

Using fractional derivatives as memory formalisms Baleanu and Agraval (2006) studied the Hamilton formalism. Baleanu and Trujillo (2010) studied the Euler–Lagrange equations and Baleanu et al. (2009) studied the Newtonian law with memory. Körnig and Müller (1989) used a rheological model based on fractional calculus to estimate the anelastic properties of the crust of the Earth. Iaffaldano et al. (2006) and Di Giuseppe et al. (2010) modeled the flux of water through different types of sand using diffusion equations modified with the introduction of fractional derivatives and Schumer et al. (2009) modeled transport on the Earth's surface with a fractional advection diffusion equation.

Zhang et al. (2007) studied the impact of boundaries on the fractional advection–dispersion equation for solute transport in soil defining the fractional dispersive flux with the fractional derivatives. Murio and Mejia (2008) studied the generalized time inverse heat convection problems with fractional derivatives. Bagley and Torvik (2000a,b) discussed the problem of the existence of the order domain and the solution of distributed-order differential equations. Mainardi et al. (2008) generalized the partial differential equation of Gaussian diffusion by using the time-fractional derivative of distributed order between 0 and 1, in both the Riemann–Liouville and the Caputo sense.

The fractional derivative was also used in medicine: El-Shahed (2003) made a fractional calculus model of heart valve vibrations, Magin and Ovadia (2008) modeled the cardiac tissue electrode interface using fractional calculus and Freed and Diethelm (2008) applied the fractional derivatives in viscoelasticity for a non-linear finite-deformation theory of tissue.

The derivatives of fractional order are often used to model biological phenomena, as for instance the diffusion of fluids in organic and inorganic substances. For instance, Cesarone et al. (2005) and Caputo and Cametti (2008, 2009) introduced a fractional derivative in the diffusion equation to model the profile concentration of diffusing solutes inside cell membranes. The latter authors compared their model predictions with experimental results concerning the permeation of piroxicam, an anti-inflammatory drug, and of 4-cyanophenol through human skin in vivo, obtaining a good fit.

Caputo and Carcione (2011a,b) used fractional derivatives of distributed order to model fatigue criteria and wave simulation, respectively, while Caputo et al. (2011) applied fractional derivatives to the propagation of waves in biological dissipative media.

In seismology, Carcione et al. (2002) and Carcione (2009) described the anelastic behavior of general materials over wide frequency ranges by using fractional derivatives, in particular considering propagation with constant-Q characteristics.

In finance, the fractional derivative represents the effect of memory on the economic operators concerning their action in the markets. Scalas et al. (2000) developed a theory which fully takes into account the non-Markovian and non-local character of financial time series and Mainardi et al. (2000) pointed out the consistency of the results of Scalas et al. (2000).

In physics, Laskin (2000, 2002) applied the fractional derivative in quantum mechanics, particularly to the equation of Schrödinger discussing the difference with the original equation. Závada (2002) studied relativistic wave equations involving fractional derivatives, Raspini (2000) studied the Dirac equation with a fractional derivative of order 2/3, Magin et al. (2009) solved the Bloch equation, which relates a macroscopic model of magnetization to applied radiofrequency, in gradient and static magnetic fields, in order to detect and characterize neurodegenerative, malignant and ischemic diseases.

In information theory, Frederico and Torres (2008) studied the optimal control in the sense of the fractional Noether theorem. Introducing this derivative in the stress strain relation of elasticity is possible to model the phenomenon of dissipation of the elastic energy; that dissipation which renders harmless an earthquake at sufficiently large distances from the epicenter.

All the equations generalizing the Fourier equation, and used in the works previously mentioned, are called phenomenological, since they are not obtained from first principles only. The reputation of this type of equations has been confirmed for their important contribution given in various forms to the rapid developments of the superconductive materials. These phenomenological equations, when adequately verified with experimental data, represent a step forward with respect to the usual empirical equations which are still very useful in many branches of applied science and technology.

The fractional calculus is used here to describe a one-dimensional model of water reservoir, where the sediment is diffusing according to the Fourier law. The model is applied to the Quarto Nuovo reservoir. A list of symbols is given in Appendix A.

2. The Quarto Nuovo reservoir

The ITCOLD is the Italian research group supervising the management of the water reservoirs and of the operations of removal of their sediment according to an Italian law issued in 2006. This group studied the Quarto Nuovo reservoir which was built between 1923 and 1925 in the Italian province of Forli, along the State Road 71 from Cesena to Bagno di Romagna, at the elevation of 320 m above the sea level. The reservoir has a surface of about 85,000 m^2 with an elongated form (it is about 1000 m long) and collects the waters of the rivers Para and Savio.

Soon after the construction, it was noted the formation of a relevant deposit of sediment on the bottom in the reservoir and, because of this phenomenon, the variation of the actual volume of water of the reservoir was monitored almost regularly and numerically (Piro et al., 2007). These data inspired us to make a simple analytic model of the diffusion of sediment in the reservoir, assuming that the yearly introduction of the sediment at the source is constant and that the fraction of the sediment eliminated at the output is similar to that implied by the data supplied in the report of Piro et al. (2007).

One is forced to severe simplifications, due to the lack of data on the geometry of the reservoir, to produce a simple mathematical model of the sediment diffusion. We will assume here a one-dimensional model of the reservoir with length h displayed along the x axis and with the water entering the reservoir at the origin $x = 0$. We also assume that the input of sediment has a constant yearly rate A and that, at the output, $x = h$, the flux of sediment is given. In short, the reservoir is a simple one-dimensional input–output system.

From the data available on the reservoir (Piro et al., 2007), one obtains the rate of sediment filling it. We assume that the flux of sediment at the input and output of the reservoir are

$$q(0,t) = A,$$
$$q(h,t) = k(t) = \frac{B\gamma}{\gamma - \epsilon}[\exp(-\epsilon t) - \exp(-\gamma t)], \quad (1)$$

respectively, where t is the time, A, B and γ are constants to be determined from the data and $\epsilon t \ll 1$ and $\epsilon \ll \gamma$, where ϵ is a non-physical parameter to avoid the singularity in the inverse Fourier transform (see below). Asymptotically ($t \to \infty$), the flux at the mouth will be the same as at the source if $B = A$.

3. The one-dimensional analytical model of diffusion

The mathematical formalism used to model the diffusion of sediments in a reservoir consists of the Fourier equation modified with the application of a memory formalism, in the form of a fractional derivative of distributed order of the concentration gradient. This derivative is used to model the complexity of the time evolution of the local distribution of sediment due to the previous transit of fluid. Then, the flux of sediment can be expressed as

$$q(x,t) = -cp_{,x}(x,t) - d\, D^{[a,b]}p_{,x}(x,t), \quad (2)$$

where $p(x,t)$ represents the concentration of sediment in a section at distance x from the input, c and d are constants (whose dimensions are given in Appendix A), and the subindex "$,x$" denotes a partial derivative with respect to the variable x. In Eq. (2), $D^{[a,b]}$ is the operator of fractional differentiation of distributed order introduced by Caputo (1967, 1995), used in Caputo (2001) and Caputo and Carcione (2011b) and extensively reported in Jiao et al. (2012):

$$D^{[a,b]}p(t) = \int_a^b D^{\nu}p(t)d\nu, \quad (3)$$

where

$$D^{\nu}p(x,t) = \frac{1}{\Gamma(1-\nu)}\int_{-\infty}^{t}\frac{p_{,\tau}(x,\tau)}{(t-\tau)^{\nu}}d\tau = \frac{1}{\Gamma(1-\nu)}\; p_{,t} * \frac{1}{t^{\nu}} \quad (4)$$

is the so-called Caputo fractional derivative of order $\nu \in [0,1]$ (Podlubny, 1999; Carcione et al., 2002; Diethelm, 2010), where "$*$" denotes time convolution.

The operator D^{ν} describes the perturbation of the local concentration of homogeneous sediment due to the previous transit of water. It represents the perturbation due to inhomogeneous sediment formed with particles of varied shape, weight and size. The mathematical formalism defined by Eq. (4) is constructed with a weighted mean of the first-order derivative $p_{,\tau}(x,\tau)$ in the time interval [0,t], which is a sort of feedback system, i.e., the values of $p_{,\tau}(x,\tau)$ at time τ far apart from t are given smaller weight than those at times τ closer to t. Hence, the weights are increasingly smaller with increasing time separation from the time t to imply that the effect of the past is fading with increasing time. Importantly, the weights multiplying the first-order derivative of $p(x,\tau)$ inside the integral appearing in Eq. (4) can be chosen in many ways. The definition adopted in Eq. (4) is appropriate because is algebraically simple, allows easy solutions, and has been commonly applied in the previously cited scientific studies (Podlubny, 1999; Mainardi, 2010; Diethelm, 2010).

A simple example of diffusion with memory experimentally verified is the flux of water through sand which causes a rotation of its grains and thus generates a matrix with porosity variable in time (Iaffaldano et al., 2006; Di Giuseppe et al., 2010). The variation of the porosity is then function of the quantity of water which went through the matrix which in turn has a memory of this quantity. In our case, the memory is materialized in the exposure of the particles forming the sediment to the gravity force, namely, the duration of this exposure combined with the velocity of the water. The distributed-order fractional equation expressed by Eqs. (2) and (3) aims to model the difference of the memory of the sediments with varied shape, weight and size.

Eq. (2) has to be considered with the continuity equation

$$q_{,x} + p_{,t} = 0. \quad (5)$$

The LT domain of Eqs. (2) and (5) are

$$Q = -\left(c + \frac{s^b - s^a}{\ln s}d\right)P_{,x},$$
$$Q_{,x} + sP = 0, \quad (6)$$

where s is the LT variable, and the LT pair $t^{-\nu} \leftrightarrow \Gamma(1-\nu)s^{\nu-1}$ and $\int s^{\nu}\, d\nu = s^{\nu}/\ln s + C$, where C is a constant, have been used. Eliminating $P(x,s)$ in Eq. (6), one obtains

$$Q_{,xx}(x,s) = \left(c + \frac{s^b - s^a}{\ln s}d\right)^{-1} sQ \quad (7)$$

(Caputo, 1995). The boundary conditions are given in Eq. (1). The Laplace-domain solution of Eq. (7) is given in Appendix B,

$$Q(x,s) = K(s)F(x,s) - (A/s)F(x-h,s), \quad (8)$$

where

$$F(x,s) = \frac{\sinh(\alpha x)}{\sinh(\alpha h)} \quad (9)$$

and

$$\alpha = \sqrt{s}\left(c + \frac{s^b - s^a}{\ln s}d\right)^{-1/2}. \quad (10)$$

The time-domain solution is calculated by recasting this solution in the Fourier domain and performing an inverse transform using the fast-Fourier transform (FFT) (see Appendix B).

4. Application to the Quarto-Nuovo water reservoir

We now apply the model to the flux of sediment in the Quarto-Nuovo water reservoir. The data of Piro et al. (2007) on the evolution of the capacity of the reservoir allow only six values of the yearly rate of sediment distributed in the time window from the year 1924 through the year 2006. They are shown in Fig. 1.

We assume that the memory is represented by a single mathematical memory formalism, re-writing Eq. (2) as

$$q = -cp_{,x} - dD^{\nu}p_{,x}. \quad (11)$$

In this case, combining Eqs. (5) and (11) yields

$$q_{,t} = (c + dD^{\nu})q_{,xx}. \quad (12)$$

In the LT domain,

$$Q_{,xx} = \alpha^2 Q, \quad \text{with } \alpha = \sqrt{\frac{s}{c + ds^{\nu}}}. \quad (13)$$

We also assume that the flux of sediment at $x = 0$ be A = constant and, as in the case of the Quarto-Nuovo reservoir, and that the flux at $x = h$ be approximated by Eq. (1), whose LT is

$$K(s) = \frac{B\gamma}{(s+\epsilon)(s+\gamma)}, \quad (14)$$

where ϵ is small and is required to perform the numerical inverse transform to the time domain.

It is reasonable to assume that the phenomenon of diffusion with memory affects all the sediment. In this case, we may assume that $c = 0$ in Eq. (11). The flux is defined by the parameter d, with dimension $T^{\nu-1}L^2$, which we call pseudo-diffusivity, which together with the order ν of fractional differentiation, are new parameters defining the diffusion. We note then that the diffusion is characterized by two parameters and not by a single one as in the classic case.

Moreover, it seems rational to assume that, asymptotically, the flux at $x = h$ be the same as that at the source, that is, Eq. (1) with $B = A$

$$q(h,t) = k(t) = \frac{A\gamma}{\gamma - \epsilon}[\exp(-\epsilon t) - \exp(-\gamma t)]. \quad (15)$$

In this case, the solution (8) becomes

$$Q(x,s) = \frac{A}{s+\epsilon}\left[\frac{\gamma}{s+\gamma}F(x,s) + F(h-x,s)\right], \quad (16)$$

where

$$\alpha = \sqrt{\frac{s^{1-\nu}}{d}}. \quad (17)$$

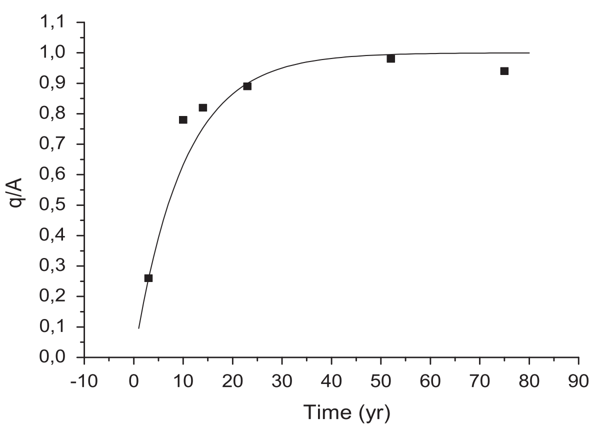

Fig. 1. Normalized yearly flux of sediment at the output of the Quarto Nuovo water reservoir. The squares are the data and the solid line is the fit.

5. Results

The fitting of Eq. (15) to the data of the Quarto-Nuovo reservoir gives $\gamma = 0.1$/yr. The normalized data and the fitting are shown in Fig. 1.

Bridge (2003) reports values of d between 1000 m^2/yr and 10^8 m^2/yr. Firstly, as an example, we chose to present the flux at $x = 0.9h = 2.7$ km, with $h = 3$ km, $d = 10^4$ m^2 yr$^{\nu-1}$ (Fig. 2a) and $d = 10^5$ m^2 yr$^{\nu-1}$ (Figure 2b) in the cases $\nu = 0$, $\nu = 0.3$ and $\nu = 0.6$, with $\gamma = 0.1$/yr. The algorithm uses $\epsilon = 0.5 \times 10^{-4}$/yr, while the FFT length is 2^{21} with a time step of 0.5 yr. The flux is normalized to the final value observed and the time is normalized to the time of last observation when the reservoir was practically full of sediment. We note that the flux is increasing when d increases and decreases when ν increase. Obviously the case $\nu = 0$ is the classic one which, in absence of memory, gives the largest flux for all values of t.

Next, we consider two different values of γ defining the flux at the output of the reservoir ($x = h$). Fig. 3 shows the normalized yearly flux at $x = 2.7$ km, where $\nu = 0.3$ and $d = 10^4$ m^2 yr$^{\nu-1}$. The increase of γ implies an increase of the flux rate of sediment at the output of the reservoir.

The flux at different locations is displayed in Fig. 4, corresponding to $\gamma = 0.1$/yr, $\nu = 0.3$ and $d = 10^4$ m^2 yr$^{\nu-1}$. The flux decreases with decreasing x and at a given location x_0 increases and approaches the value at $x = 0$, i.e., $q/A = 1$. Fig. 5 shows the flux as a function of x at 20 yr, where $x_0 \approx 1592$ m.

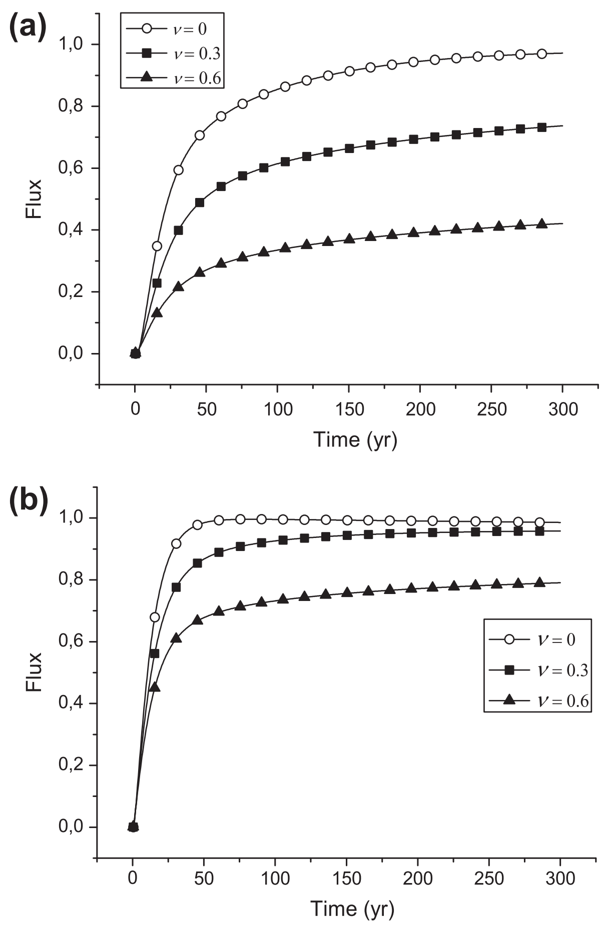

Fig. 2. Normalized yearly flux of sediment at $x = 2.7$ km for $\gamma = 0.1$/yr, various values of ν and $d = 10^4$ m^2 yr$^{\nu-1}$ (a) and $d = 10^5$ m^2 yr$^{\nu-1}$ (b).

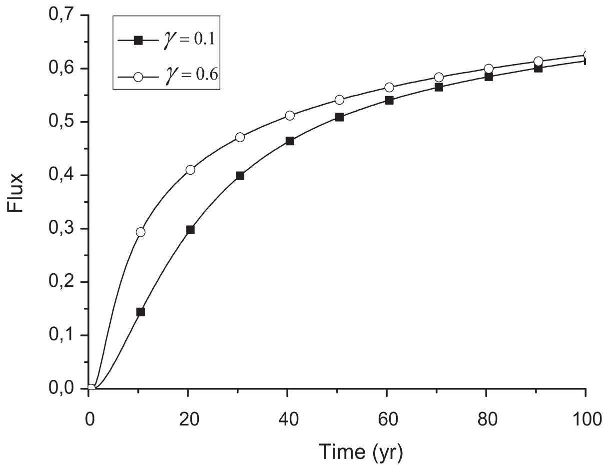

Fig. 3. Normalized yearly flux of sediment at $x = 2.7$ km for $\gamma = 0.1$/yr and $\gamma = 0.6$/yr, $\nu = 0.3$ and $d = 10^4\ \mathrm{m}^2\ \mathrm{yr}^{\nu-1}$.

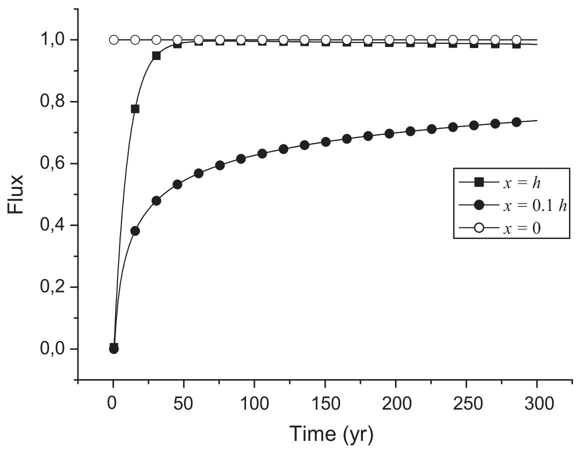

Fig. 4. Normalized yearly flux of sediment at $x = 3$ km, $x = 300$ m and $x = 0$ m, for $\gamma = 0.1$/yr, $\nu = 0.3$ and $d = 10^4\ \mathrm{m}^2\ \mathrm{yr}^{\nu-1}$.

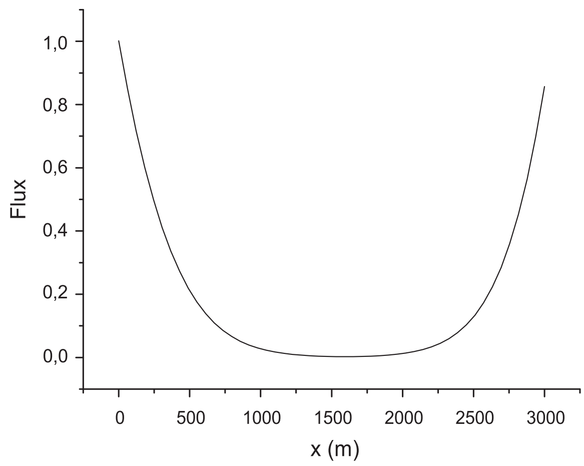

Fig. 5. Normalized flux versus x at $t = 20$ yr, for $\gamma = 0.1$/yr, $\nu = 0.3$ and $d = 10^4\ \mathrm{m}^2\ \mathrm{yr}^{\nu-1}$.

Finally, we illustrate the more general case of distributed orders of differentiation by comparing the single order $\nu = 0.3$ to the cases $[a,b] = [0.2, 0.4]$ and $[0.1, 0.5]$. The normalized flux is displayed in Fig. 6 for $c = 0$, $\gamma = 0.1$/yr and $d = 10^4\ \mathrm{m}^2\ \mathrm{yr}^{\nu-1}$ at $x = 0.9h = 2.7$ km.

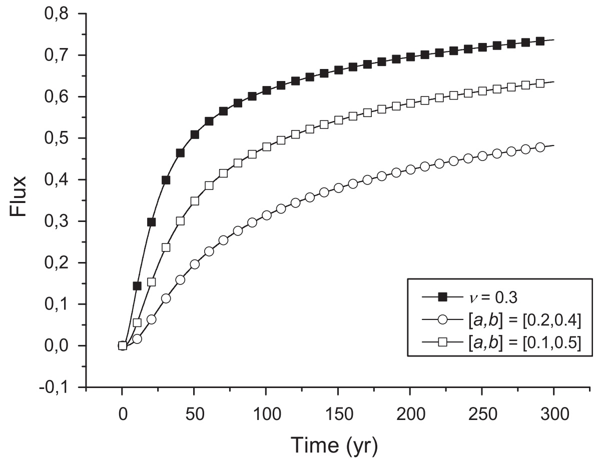

Fig. 6. Normalized yearly flux of sediment at $x = 2.7$ km for $\gamma = 0.1$/yr, $c = 0$, $d = 10^4\ \mathrm{m}^2\ \mathrm{yr}^{\nu-1}$ and $\nu = 0.3$ compared to cases of distributed order fractional derivatives.

Extending the range of the order of differentiation increases the flux.

The use of this model to estimate the diffusion of the sediment filling the water reservoir is limited by the lack of knowledge of the two parameters d and ν which characterize the flux across the reservoir. To obtain the values of these parameters, that is, to obtain a model of the flux of sediment in the reservoir, is then necessary to observe the flux in few locations of the reservoir and at different depths.

Moreover, the model with the fractional derivative of distributed order, as in Eq. (2), would represent a variety of memory mechanisms and would be more adequate to deal with a non-homogeneous sediment. It would then allow to represent the case when, as often occurs, it is present a sediment with a varied density distribution of sizes and weight of grains which in some sands may vary a few orders of magnitude i.e., in the range [0.075, 10] mm (Di Giuseppe et al., 2010) or in the range [2–2.8] mm (Iaffaldano et al., 2006) or in urban drainage in the range [0.4–500] μm (Maione and Moisello, 1993; Piro et al., 2007).

6. Conclusions

In order to model the diffusion of sediment in a one-dimensional model of water reservoir, the Fourier diffusion equation is modified with the introduction of a memory formalism represented by a derivative of fractional order. This model describes the diffusion with two parameters instead of the classic one, i.e., the pseudo-diffusivity d and the order of fractional differentiation ν.

When adequate data on the diffusion of sediments in the reservoir is available, the model seems capable to represent the flux of sediment occurring along the reservoirs. The parameter d is increasing with increasing flux. It is also seen that ν increases with decreasing flux and, when ν is nil, one obtains the classic diffusion with maximum flux. The flux increases also with decreasing $1/\gamma$, i.e., with the decay time of the sediment accumulation.

The knowledge of the two parameters, d and ν, allows to describe the flux of sediment along the reservoir. This type of model with two parameters could help in understanding better the evolution of the deposit of the sediment and implies a more detailed description of the phenomenon of sediment deposit. It may also throw some light on the differences between the various types of water reservoirs and be of help in estimating the possible performance of perspective reservoirs.

The model with the more general operator based on the fractional derivative of distributed order, describing a variety of memory mechanisms, would be more adequate to deal with a non-homogeneous sediment, having a distribution of sizes and weight of grains which in some sands may vary a few orders of magnitude.

One practical use of the model is to reduce the cost of estimating the diffusion and accumulation of sediments by calculating them instead of monitoring them with instruments. This process can readily be made since the solutions are given with simple closed-form formulae.

Acknowledgements

The inverse Fourier transforms have been computed with Fortran. We thank Fabio Cavallini for cross-checking the calculations with the software Mathematica.

Appendix A. List of symbols and dimensions

Mass of sediment	M
t	Time, T
x	Distance from the origin, L
h	Length of the reservoir, L
s	LT variable, T^{-1}
ω	Angular frequency, T^{-1}
$p(x,t)$	Concentration of sediment in the water, $M\,L^{-3}$
$P(x,s)$	LT of $p(x,t)$
$q(x,t)$	Flux of sediment in the reservoir, $M\,L^{-2}\,T^{-1}$
$Q(x,s)$	LT of $q(x,t)$
$k(h,t)$	Flux of sediment at the output of the reservoir, $M\,L^{-2}\,T^{-1}$
$K(h,s)$	LT of $k(h,t)$
c	$L^2\,T^{-1}$
d	$L^2\,T^{\nu-1}$
ν	Fractional order of differentiation
$[a,b]$	Variation interval of ν
α	L^{-1}
A, B	$M\,L^{-2}\,T^{-1}$
γ	T^{-1}
ϵ	T^{-1}

Appendix B. Calculation of the sediment flux

The Laplace domain solution of Eq. (7) is

$$Q = Q_1 \exp(\alpha x) + Q_2 \exp(-\alpha x), \quad \alpha(s) = \sqrt{s}\left(c + \frac{s^b - s^a}{\ln s} d\right)^{-1/2}. \tag{18}$$

Introducing the boundary conditions (1) in Eq. (18) and solving for Q_1 and Q_2, we find

$$Q_1 = \frac{K(s) - (A/s)\exp(-\alpha h)}{2\sinh(\alpha h)} \quad \text{and} \quad Q_2 = \frac{(A/s)\exp(\alpha h) - K(s)}{2\sinh(\alpha h)}, \tag{19}$$

where $K(s)$ is the LT of $k(t)$ and LT $(A) = A/s$. We obtain

$$Q(x,s) = K(s)F(x,s) + (A/s)F(h-x,s), \tag{20}$$

where

$$F(x,s) = F(x,\alpha) = \frac{\sinh(\alpha x)}{\sinh(\alpha h)}. \tag{21}$$

It is readily verified in Eq. (20) that $q(h,t) = k(t)$ and that $q(0,t) = A$. Because $\alpha(s=0) = 0$, $F(x,0) = x/h$ and $F(x-h,0) = x/h - 1$. Then, invoking the final value theorem, $\lim_{t\to\infty} q(x,t) = \lim_{s\to 0} s\,Q(s)$, we obtain

$$q(x,\infty) = A\left(1 - \frac{x}{h}\right) + \frac{x k(\infty)}{h}, \tag{22}$$

which is $k(\infty)$ at $x = h$ and A at $x = 0$.

In order to compute the time-domain solution, we perform a numerical inverse Fourier transform. Hence, we take $s = i\omega$, where $i = \sqrt{-1}$ and ω is the angular frequency. Then, we use the fast Fourier transform (FFT). To avoid numerical errors, such as ringing at early times, the length of the FFT has to be chosen long enough and the time sampling small. The solution is valid for times satisfying $\epsilon t \ll 1$.

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