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Energy balance and fundamental relations in anisotropic-viscoelastic media

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The first part of this work analyses the energy balance equation for inhomogeneous time-harmonic waves propagating in a linear anisotropic-viscoelastic medium whose constitutive equation is described by a general time-dependent relaxation matrix of 21 independent components. This matrix includes most linear anisotropic-viscoelastic rheologies and the generalized Hooke's law. The balance of energy allows the identification of the potential and loss energy densities, which are related to the real and imaginary parts of the complex stiffness matrix.

The second part establishes some fundamental relations valid for inhomogeneous viscoelastic plane waves. The scalar product between the complex wavenumber and the complex power flow vector is a real quantity proportional to the time-average kinetic energy density. As in the anisotropic-elastic case, it is confirmed that the phase velocity is the projection of the energy velocity vector onto the propagation direction. A similar equation is obtained by replacing the energy velocity with a velocity related to the dissipated energy. Finally, as in isotropic-viscoelastic media, the time-average energy density can be obtained from the projection of the average power flow vector onto the propagation direction, and the time-average dissipated energy density from the projection of the average power flow vector onto the attenuation direction.

1. Introduction

The study of energy propagation in dissipative and anisotropic media is a challenging topic in wave theory. In the first place, it is not clear from experimental data what the form of the constitutive relation should be. In isotropic media, two independent relaxation functions are enough since the wave modes decouple each other and the attenuation does not depend on the propagation direction. In anisotropic-viscoelastic media the problem is more complex since one has to decide the time dependence of 21 relaxation components. Most applications use the Kelvin–Voigt relaxation matrix (e.g. [1–3]) which does not consider the time dependence of the real stiffness and viscosity tensors. Recently, Carcione [4] introduced a class of anisotropic-viscoelastic constitutive relations that generalizes the Kelvin–Voigt approach by using two relaxation functions describing the anelastic properties of the quasi-dilatational and quasi-shear modes. This new constitutive equation is based on the standard linear solid model which yields the correct wave propagation properties. Carcione [5] studied the wave characteristics of viscoelastic finely layered media in the long-wavelength approximation, where after application of the correspondence principle, the averaged elasticities become complex and depend non-linearly on the frequency. Therefore, this rheology cannot be modeled by a Kelvin–Voigt stress-strain relation.

In the second place, the behaviour of anisotropic-viscoelastic waves departs substantially from the behaviour of isotropic-elastic waves. Anisotropy implies that in general the wavefield is not pure longitudinal or pure transverse, and therefore there is not a simple relation between the propagation direction and the direction of particle displacement. As a consequence, wavefronts are not spherical and the direction of energy flux (ray) does not coincide with the wavenumber direction. On the other hand, in viscoelastic media the existence of the so called ([6] and [7])

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inhomogeneous waves (not the interface waves of elastic media) is necessary to satisfy the boundary conditions at interfaces. For these waves the propagation direction does not coincide with the attenuation direction, and particle motions are in general elliptical [7].

This paper investigates the energy balance of the wave propagation process which takes place in an isothermal and linear anisotropic-viscoelastic medium. The analysis is carried out from a general point of view for time-harmonic inhomogeneous viscoelastic waves, i.e., waves for which the directions of propagation and attenuation do not necessarily coincide. The energy balance is analysed for a complex stiffness matrix which includes the Kelvin–Voigt constitutive equation studied by Auld [2]. From the energy balance equation several fundamental relations for inhomogeneous plane waves are deduced which are generalizations of more simple rheologies, like the anisotropic-elastic (e.g. [2]), and the isotropic-viscoelastic ([8] and [6]).

The first two sections establish the stress-strain relation and the equation of motion for time-harmonic fields. Then, the energy balance equation or Umov–Poynting theorem is developed, allowing the identification of the potential and dissipated energies as well as the power flow vector. Finally, the last section provides some fundamental relations involving the complex wavenumber vector, the real wavenumber and attenuation vectors, the various energy quantities, the power flow vector and the energy and phase velocities.

2. Constitutive relation

The most general relation between the components of the stress tensor σ_{ij} and the components of the strain tensor ϵ_{ij} for an anisotropic linear viscoelastic medium is given by [9],

$$\sigma_{ij}(\boldsymbol{x}, t) = \psi_{ijkl}(\boldsymbol{x}, t) * \dot{\epsilon}_{kl}(\boldsymbol{x}, t), \quad i, j, k, l = 1, \ldots, 3 \tag{1}$$

where t is the time variable, $\boldsymbol{x}$ is the position vector, ψ_{ijkl} are the components of the fourth-order tensorial relaxation function, and the symbol $*$ indicates time convolution. A dot above a variable denotes time differentiation, and the Einstein convention for repeated indices is used.

Equation (1) is the formulation of the isothermal anisotropic-viscoelastic stress-strain constitutive relation, also called the Boltzmann superposition principle. The fourth-rank tensor contains all the information about the behaviour of the medium under infinitesimal deformations. In the most general case, the number of components is 81. Following the same arguments as Auld [2] to obtain his Kelvin–Voigt anisotropic-viscoelastic rheology, it is assumed here that the stress tensor is symmetric and that the strain and loss energy densities are positive and real. Hence, since the strain tensor is symmetric, the number of independent components reduces to 21.

The stress-strain relation is simplified by introducing the well-known shortened matrix notation where pairs of subscripts are replaced by a single number according to the following correspondences:

$$(11) \to 1, \quad (22) \to 2, \quad (33) \to 3, \quad (23) = (32) \to 4,$$
$$(13) = (31) \to 5, \quad (12) = (21) \to 6.$$

Upper case letters will denote abbreviated subscripts and lower case letters full subscripts. In the new notation, Boltzmann's superposition principle (1) can be expressed as:

$$T_I = \psi_{IJ} * \dot{S}_J, \quad I, J = 1, \ldots, 6 \tag{2}$$

where

$$\boldsymbol{T}^{\mathrm{T}} = [T_1, T_2, T_3, T_4, T_5, T_6] = [\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{xz}, \sigma_{xy}] \tag{3}$$

is the stress vector,

$$S^{\mathrm{T}} = [S_1, S_2, S_3, S_4, S_5, S_6] = [\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, 2\epsilon_{yz}, 2\epsilon_{xz}, 2\epsilon_{xy}] \tag{4}$$

is the strain vector, and ψ_{IJ} are the components of the relaxation matrix $\Psi(x, t)$, such that $\psi_{IJ} = \psi_{JI}$. Vectors are written as columns with the superscript 'T' denoting the transpose. By using the properties of convolution, eq. (2) gives

$$T_I = \dot{\psi}_{IJ} * S_J . \tag{5}$$

Time-harmonic fields are represented by the real part of

$$[\cdot]\, \mathrm{e}^{\mathrm{i}\omega t}, \tag{6}$$

where $[\cdot]$ represents a complex vector that depends solely on the spatial coordinates, and ω is the angular frequency. Substituting the time dependence (6) into the stress-strain relation (5) yields

$$T_I = p_{IJ} S_J , \tag{7}$$

where

$$p_{IJ} = \int_{-\infty}^{\infty} \dot{\psi}_{IJ}(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t \tag{8}$$

are the components of the stiffness matrix $p(x, \omega)$. In matrix notation, eq. (7) reads

$$T = p \cdot S , \tag{9}$$

where the dot indicates ordinary matrix multiplication. For anelastic media, the components of p are complex and frequency dependent. Note that the anelastic rheology discussed by Auld ([2], p. 87) is a particular case of (9). In fact, Auld introduces a viscosity tensor η such that $p(\omega) = c + \mathrm{i}\omega\eta$, with c the low-frequency limit elasticity matrix. This equation corresponds to a Kelvin–Voigt constitutive equation. Equation (9) includes a class of anisotropic-viscoelastic constitutive equations studied by Carcione [4], and based on two relaxation functions of the standard linear solid type.

3. Equation of motion

The equation of momentum for a three-dimensional anisotropic linear anelastic medium is

$$\nabla \cdot T = \rho \dot{v} - F , \tag{10}$$

where $v(x, t)$ is the particle velocity vector, $F(x, t)$ is the body force vector, $\rho(x)$ is the density, and ∇ is a differential operator defined by

$$\nabla = \begin{bmatrix} \partial/\partial x & 0 & 0 & 0 & \partial/\partial z & \partial/\partial y \\ 0 & \partial/\partial y & 0 & \partial/\partial z & 0 & \partial/\partial x \\ 0 & 0 & \partial/\partial z & \partial/\partial y & \partial/\partial x & 0 \end{bmatrix} . \tag{11}$$

For time-harmonic fields of the form (6), equation (10) reads

$$\nabla \cdot T = \mathrm{i}\omega\rho v - F . \tag{12}$$

On the other hand, the strain-velocity relation is

$$\nabla^{T}\cdot v = i\omega S . \quad (13)$$

4. Energy balance equation for time-harmonic fields

The derivation of the energy balance equation or Umov–Poynting theorem is straightforward when using complex notation. The basic equations for time-average and peak values of the different quantities involved in the energy balance equation are given in the Appendix. To derive this equation the dot product of the equation of motion (12) is first taken with $-v^{*T}$ to give

$$-v^{*T}\cdot\nabla\cdot T = -i\omega\rho v^{*T}\cdot v + v^{*T}\cdot F . \quad (14)$$

On the other hand, the dot product of $-T^{T}$ with the complex conjugate of (13) is

$$-T^{T}\cdot\nabla^{T}\cdot v^{*} = i\omega T^{T}\cdot S^{*} . \quad (15)$$

Adding eqs. (14) and (15) gives

$$-v^{*T}\cdot\nabla\cdot T - T^{T}\cdot\nabla^{T}\cdot v^{*} = -i\omega\rho v^{*T}\cdot v + i\omega T^{T}\cdot S^{*} + v^{*T}\cdot F . \quad (16)$$

The left-hand side of (16) is simply

$$-v^{*T}\cdot\nabla\cdot T - T^{T}\cdot\nabla^{T}\cdot v^{*} = -\mathrm{div}(\Sigma\cdot v^{*}) , \quad (17)$$

where

$$\Sigma = \begin{bmatrix} T_1 & T_6 & T_5 \\ T_6 & T_2 & T_4 \\ T_5 & T_4 & T_3 \end{bmatrix} , \quad (18)$$

is the stress tensor, and div is the well-known divergence operator. Then, (16) can be expressed as

$$-\tfrac{1}{2}\mathrm{div}(\Sigma\cdot v^{*}) = -i\omega\tfrac{1}{2}\rho v^{*T}\cdot v + i\omega\tfrac{1}{2}T^{T}\cdot S^{*} + \tfrac{1}{2}v^{*T}\cdot F . \quad (19)$$

After substitution of the stress-strain relation (9), eq. (19) gives

$$-\tfrac{1}{2}\mathrm{div}(\Sigma\cdot v^{*}) = 2i\omega\left(-\frac{1}{4}\rho v^{*T}\cdot v + \frac{1}{4}\mathrm{Re}[S^{T}\cdot p\cdot S^{*}]\right) - \frac{\omega}{2}\mathrm{Im}[S^{T}\cdot p\cdot S^{*}] + \tfrac{1}{2}v^{*T}\cdot F , \quad (20)$$

where Re[·] and Im[·] take real and imaginary parts, respectively. The significance of this equation becomes clear when one recognizes that each of its terms has a precise physical meaning on a time-average basis. For instance, from eq. (A.1),

$$\tfrac{1}{4}\rho v^{*T}\cdot v = \tfrac{1}{2}\rho\langle \mathrm{Re}[v^{T}]\cdot\mathrm{Re}[v]\rangle = \langle\epsilon_v\rangle \quad (21)$$

is the time-average stored kinetic energy density; from (A.5)

$$\tfrac{1}{4}\mathrm{Re}[S^{T}\cdot p\cdot S^{*}] = \tfrac{1}{2}\langle \mathrm{Re}[S^{T}]\cdot\mathrm{Re}[p]\cdot\mathrm{Re}[S]\rangle = \langle\epsilon_s\rangle \quad (22)$$

is the time-average stored strain energy density, and from (A.6),

$$\tfrac{1}{2}\mathrm{Im}[\boldsymbol{S}^{\mathrm{T}}\cdot\boldsymbol{p}\cdot\boldsymbol{S}^*] = \langle \mathrm{Re}[\boldsymbol{S}^{\mathrm{T}}]\cdot\mathrm{Im}[\boldsymbol{p}]\cdot\mathrm{Re}[\boldsymbol{S}]\rangle = \langle\epsilon_{\mathrm{d}}\rangle \tag{23}$$

is defined as the time-average dissipated energy density, where the fact that $\boldsymbol{p}$ is a symmetric matrix has been used. The complex power flow vector or Umov–Poynting vector is defined as

$$\boldsymbol{P} = -\tfrac{1}{2}\boldsymbol{\Sigma}\cdot\boldsymbol{v}^*\,, \tag{24}$$

and

$$P_{\mathrm{s}} = \tfrac{1}{2}\boldsymbol{v}^{*\mathrm{T}}\cdot\boldsymbol{F} \tag{25}$$

is the complex power per unit volume supplied by the body forces. Substituting the preceding expressions into eq. (20) gives the energy balance equation for linear-anisotropic viscoelastic media,

$$\mathrm{div}\,\boldsymbol{P} - 2\mathrm{i}\omega[\langle\epsilon_{\mathrm{s}}\rangle - \langle\epsilon_{\mathrm{v}}\rangle] + \omega\langle\epsilon_{\mathrm{d}}\rangle = P_{\mathrm{s}}\,. \tag{26}$$

The time-average stored energy density is

$$\langle\epsilon\rangle = \langle\epsilon_{\mathrm{v}}\rangle + \langle\epsilon_{\mathrm{s}}\rangle = \tfrac{1}{4}\{\rho\boldsymbol{v}^{\mathrm{T}}\cdot\boldsymbol{v}^* + \mathrm{Re}[\boldsymbol{S}^{\mathrm{T}}\cdot\boldsymbol{p}\cdot\boldsymbol{S}^*]\}\,. \tag{27}$$

In elastic media, $\langle\epsilon_{\mathrm{d}}\rangle = 0$, and since in the absence of sources the net energy flow into, or out of a given closed surface S must vanish, $\mathrm{div}\,\boldsymbol{P} = 0$. Thus, the average kinetic energy equals the average potential energy. As a consequence, the average stored energy is twice the average potential energy.

On separating the real and imaginary parts of eq. (26), two independent and separately meaningful physical relations are obtained:

$$-\mathrm{Re}[\mathrm{div}\,\boldsymbol{P}] + \mathrm{Re}[P_{\mathrm{s}}] = \omega\langle\epsilon_{\mathrm{d}}\rangle \quad \text{and} \quad -\mathrm{Im}[\mathrm{div}\,\boldsymbol{P}] + \mathrm{Im}[P_{\mathrm{s}}] = 2\omega[\langle\epsilon_{\mathrm{v}}\rangle - \langle\epsilon_{\mathrm{s}}\rangle]\,. \tag{28a, b}$$

The real part states that the total time-average power supplied (per unit volume) to a point, both from internal sources $\mathrm{Re}[P_{\mathrm{S}}]$ and from inwardly-directed radiation (via the negative divergence of $\boldsymbol{P}$), must be equal to the time-average dissipated energy density at that point multiplied by ω. The imaginary part equates the total reactive power supplied at any point to the difference in the time-average values of the kinetic and potential energy densities stored at that point multiplied by 2ω.

The complex Umov–Poynting theorem as given by eq. (26) is analogous to a similar relation for complex power in sinusoidal steady-state circuit theory given by (e.g., [10]),

$$\tfrac{1}{2}VI^* - 2\mathrm{i}\omega[\langle E_{\mathrm{C}}\rangle - \langle E_{\mathrm{L}}\rangle] + \langle P_{\mathrm{d}}\rangle = 0\,,$$

where $1/2(VI^*)$ is the complex power, V is the voltage phasor, I is the current phasor, $\langle E_{\mathrm{C}}\rangle$ and $\langle E_{\mathrm{L}}\rangle$ are the time-average stored energies in the capacitors and inductors, respectively, and $\langle P_{\mathrm{d}}\rangle$ is the average power dissipated in the resistors.

Equation (26) gives the energy balance for general viscoelastic time-harmonic fields. For linearly polarized fields, the components of the particle velocity vector $\boldsymbol{v}$ are in phase, and the average kinetic energy is half the peak kinetic energy by virtue of eq. (A.4). The same property holds for the potential energy if the components of the strain vector $\boldsymbol{S}$ are in phase (see eq. (A.7)). In this case, the energy balance equation reads

$$\mathrm{div}\,\boldsymbol{P} - \mathrm{i}\omega[(\epsilon_{\mathrm{s}})_{\mathrm{peak}} - (\epsilon_{\mathrm{v}})_{\mathrm{peak}}] + \omega\langle\epsilon_{\mathrm{d}}\rangle = P_{\mathrm{s}}\,, \tag{29}$$

in agreement with [2] and [4]. Equation (29) is found to be valid only for homogeneous viscoelastic plane waves, i.e. when the propagation direction coincides with the attenuation direction, although Auld ([2], eq. 5.76) seems to attribute a general validity to it. Instead, it is to be pointed out that for inhomogeneous viscoelastic plane waves the peak value is not twice the average value. The same remark applies to Ben-Menahem and Singh ([6], p. 883).

5. Fundamental relations for plane waves

A general solution representing viscoelastic plane waves is of the form

$$[\cdot]e^{i(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})}, \tag{30}$$

where $[\cdot]$ is a constant complex vector. The wavenumber vector is in general complex and can be written as

$$\boldsymbol{k}=\boldsymbol{\kappa}-i\boldsymbol{\alpha}, \tag{31}$$

where the real vectors $\boldsymbol{\kappa}$ and $\boldsymbol{\alpha}$ are the real wavenumber and attenuation, respectively. They indicate the directions and magnitudes of propagation and attenuation. In general, these directions are different and the plane wave is termed inhomogeneous, with $\boldsymbol{\kappa}^{\mathrm{T}}\cdot\boldsymbol{\alpha}$ strictly different from zero unlike the interface waves in elastic media.

For inhomogeneous viscoelastic plane waves, the operator (11) takes the form

$$\nabla\rightarrow -i\boldsymbol{K}, \tag{32}$$

where

$$\boldsymbol{K}=\begin{bmatrix} k_x & 0 & 0 & 0 & k_z & k_y \\ 0 & k_y & 0 & k_z & 0 & k_x \\ 0 & 0 & k_z & k_y & k_x & 0 \end{bmatrix}, \tag{33}$$

with k_x, k_y and k_z the components of the complex wavenumber $\boldsymbol{k}$. Note that for the corresponding conjugated fields, the operator should be replaced by $i\boldsymbol{K}^*$.

Substituting the differential operator ∇ into eqs. (14) and (15) and assuming zero body forces yields

$$\boldsymbol{v}^{*\mathrm{T}}\cdot\boldsymbol{K}\cdot\boldsymbol{T}=-\omega\rho\boldsymbol{v}^{*\mathrm{T}}\cdot\boldsymbol{v} \tag{34}$$

and

$$\boldsymbol{T}^{\mathrm{T}}\cdot\boldsymbol{K}^{*\mathrm{T}}\cdot\boldsymbol{v}^*=-\omega\boldsymbol{T}^{\mathrm{T}}\cdot\boldsymbol{S}^*, \tag{35}$$

respectively. Replacing the stresses on the right-hand side of (35) gives

$$-\omega\boldsymbol{T}^{\mathrm{T}}\cdot\boldsymbol{S}^*=-\omega\boldsymbol{S}^{\mathrm{T}}\cdot\boldsymbol{p}\cdot\boldsymbol{S}^*,$$

since $\boldsymbol{p}$ is symmetric. The left-hand sides of (34) and (35) contain the Umov–Poynting vector (24) because $\boldsymbol{K}\cdot\boldsymbol{T}=\boldsymbol{\Sigma}\cdot\boldsymbol{k}$ and $\boldsymbol{K}^*\cdot\boldsymbol{T}=\boldsymbol{\Sigma}\cdot\boldsymbol{k}^*$, with $\boldsymbol{\Sigma}$ the stress tensor (18); thus

$$2\boldsymbol{k}^{\mathrm{T}}\cdot\boldsymbol{P}=\omega\rho\boldsymbol{v}^{*\mathrm{T}}\cdot\boldsymbol{v}, \tag{36}$$

and

$$2\boldsymbol{k}^{*\mathrm{T}}\cdot\boldsymbol{P}=\omega\boldsymbol{S}^{\mathrm{T}}\cdot\boldsymbol{p}\cdot\boldsymbol{S}^*. \tag{37}$$

In terms of the energy densities (21), (22) and (23),

$$\boldsymbol{k}^{\mathrm{T}}\cdot\boldsymbol{P}=2\omega\langle\epsilon_v\rangle \tag{38}$$

and

$$k^{*T} \cdot P = \omega[2\langle \epsilon_s \rangle + i\langle \epsilon_d \rangle] . \quad (39)$$

Since the right-hand side of (38) is real, the product $k^T \cdot P$ is also real. For elastic media, k and the Umov–Poynting vector are both real quantities.

Adding eqs. (38) and (39) and using $k^* + k = 2\kappa$, yields

$$\kappa^T \cdot P = \omega\left[\langle \epsilon \rangle + \frac{i}{2}\langle \epsilon_d \rangle\right], \quad (40)$$

where the time-average stored energy density (27) has been substituted. Splitting equation (40) into real and imaginary parts gives

$$\kappa^T \cdot \langle P \rangle = \omega\langle \epsilon \rangle , \quad (41)$$

and

$$\kappa^T \cdot \mathrm{Im}[P] = \frac{\omega}{2}\langle \epsilon_d \rangle , \quad (42)$$

where

$$\langle P \rangle = \mathrm{Re}[P] \quad (43)$$

is the average power flow density. An important concept derived from these equations is the energy velocity,

$$V_e = \frac{\langle P \rangle}{\langle \epsilon \rangle} , \quad (44)$$

which defines the orientation of the wavefront. Since the phase velocity is

$$V_p = \frac{\omega}{\kappa}\hat{\kappa} , \quad (45)$$

where $\hat{\kappa}$ defines the propagation direction, the following important relation is obtained from (41):

$$\hat{\kappa}^T \cdot V_e = V_p . \quad (46)$$

This relation, as in the elastic case [2], means that the phase velocity is simply the projection of the energy velocity onto the propagation direction. Note also that eq. (42) can be written as

$$\hat{\kappa}^T \cdot V_d = V_p , \quad (47)$$

where V_d is a velocity defined by

$$V_d = \frac{2\,\mathrm{Im}[P]}{\langle \epsilon_d \rangle} \quad (48)$$

and associated with the dissipated energy. Equations (46) and (47) are illustrated in Fig. 1.

Another important relation is obtained from eq. (41):

$$\langle \epsilon \rangle = \frac{1}{\omega}\kappa^T \cdot \langle P \rangle , \quad (49)$$

which is the extension to anisotropic media of the first equation (10.183) for isotropic media given in [6]. It means

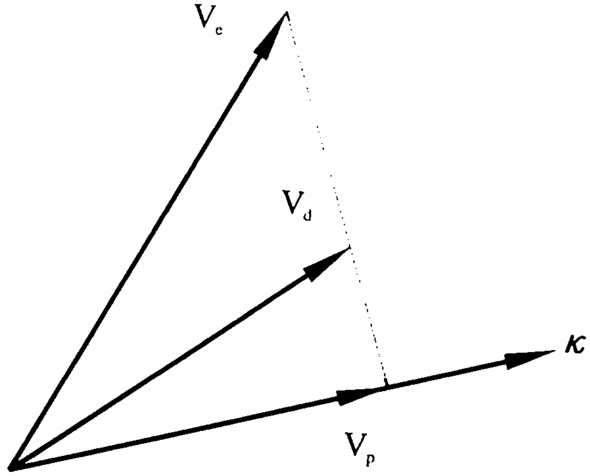

Fig. 1. Graphical representation of eqs. (46) and (47). The projection of the energy velocity over the propagation direction gives the phase velocity. The same result is obtained by projecting the velocity related to the dissipated energy.

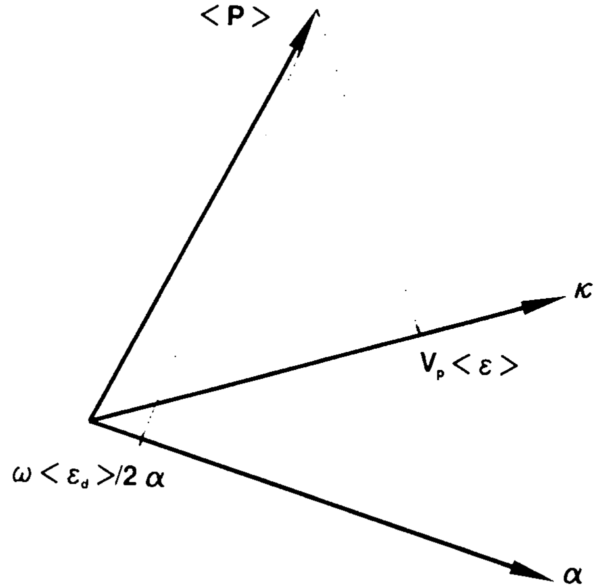

Fig. 2. Graphical representation of eqs. (49) and (51). The time-average energy density can be calculated as the component of the average power flow vector along the propagation direction, while the time-average dissipated energy density depends on the projection of the average power flow vector in the attenuation direction.

that the time-average energy density can be computed from the component of the average power flow density along the propagation direction.

On the other hand, subtracting (39) from (38) yields

$$-2\boldsymbol{\alpha}^{\mathrm{T}}\cdot\boldsymbol{P}=2\mathrm{i}\omega[\langle\epsilon_{\mathrm{s}}\rangle-\langle\epsilon_{\mathrm{v}}\rangle]-\omega\langle\epsilon_{\mathrm{d}}\rangle\,, \tag{50}$$

which can be deduced also from the energy balance equation (26), since for plane waves div $\boldsymbol{P}=-2\boldsymbol{\alpha}^{\mathrm{T}}\cdot\boldsymbol{P}$. Taking the real part of (50) gives the following relation:

$$\langle\epsilon_{\mathrm{d}}\rangle=\frac{2}{\omega}\,\boldsymbol{\alpha}^{\mathrm{T}}\cdot\langle\boldsymbol{P}\rangle\,, \tag{51}$$

which is equivalent to the second equation (10.183) in [6], stating that the time-average dissipated energy depends on the projection of the average power flow density onto the attenuation direction. Relations (49) and (51) are illustrated in Fig. 2.

6. Conclusions

The energy balance equation found in this work holds for inhomogeneous plane waves propagating in three-dimensional anisotropic-viscoelastic media described by Boltzmann's linear rheological law. For these waves the propagation and attenuation directions can be arbitrary. This result can be interpreted as generalization of the balance equation obtained by Auld [2] for homogeneous plane waves (for which the propagation and attenuation vectors are coincident) assuming a Kelvin–Voigt constitutive relation. Moreover, from the present theory it is clear that the much used equivalence between the peak value of an energy quantity and twice the corresponding time-average value does not hold for inhomogeneous plane waves, but only for homogeneous waves.

The interpretation of the phase velocity as the projection of the energy velocity vector onto the propagation direction, eq. (46), is shown to be valid for inhomogeneous waves, generalizing the results obtained by Auld [2] for the anisotropic-elastic case, and by Borcherdt [8] for the isotropic-viscoelastic case. Similar generalizations are obtained for the time-average stored and dissipated energies, eqs. (49) and (51), which are obtained from the projection of the average power flow vector onto the propagation and attenuation directions, respectively.

Acknowledgements

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Appendix. Averages and peak values for time-harmonic fields

This Appendix gives the basic formulae for the calculation of the different quantities of the energy balance equation which involves time-average and peak values of harmonic fields. The field variables are represented by the real part of (6). For two field variables $\boldsymbol{A}$ and $\boldsymbol{B}$ of dimension n, the time-average over a cycle of period $T = 2\pi/\omega$ is given by (e.g. [11])

$$\langle \mathrm{Re}[\boldsymbol{A}^{\mathrm{T}}] \cdot \mathrm{Re}[\boldsymbol{B}] \rangle = \tfrac{1}{2}\mathrm{Re}[\boldsymbol{A}^{*\mathrm{T}} \cdot \boldsymbol{B}] , \tag{A.1}$$

where $\mathrm{Re}[\,\cdot\,]$ takes the real part. On the other hand, through a lengthy but straightforward calculation, it can be shown that the peak or maximum value is given by

$$(\mathrm{Re}[\boldsymbol{A}^{\mathrm{T}}] \cdot \mathrm{Re}[\boldsymbol{B}])_{\mathrm{peak}} = \tfrac{1}{2}[\,|\boldsymbol{A}_k|\,|\boldsymbol{B}_k|\,\cos(\arg \boldsymbol{A}_k - \arg \boldsymbol{B}_k) + \sqrt{|\boldsymbol{A}_k|\,|\boldsymbol{B}_k|\,|\boldsymbol{A}_j|\,|\boldsymbol{B}_j|\,\cos(\arg \boldsymbol{A}_k + \arg \boldsymbol{B}_k - \arg \boldsymbol{A}_j - \arg \boldsymbol{B}_j)}\,] , \tag{A.2}$$

where $|\boldsymbol{A}_k|$ is the magnitude of the k-component of the vector $\boldsymbol{A}$, and implicit summation over repeated indices from 1 to n is assumed. When, for every k, $\arg \boldsymbol{A}_k = \phi_A$ and $\arg \boldsymbol{B}_k = \phi_B$, i.e. all the components of each vector are in phase, eq. (A.2) reduces to

$$(\mathrm{Re}[\boldsymbol{A}^{\mathrm{T}}]\cdot\mathrm{Re}[\boldsymbol{B}])_{\mathrm{peak}} = |\boldsymbol{A}_k|\,|\boldsymbol{B}_k|\,\cos^2\!\left(\frac{\phi_A-\phi_B}{2}\right), \tag{A.3}$$

and if, moreover, $\boldsymbol{A}=\boldsymbol{B}$, then

$$(\mathrm{Re}[\boldsymbol{A}^{\mathrm{T}}]\cdot\mathrm{Re}[\boldsymbol{A}])_{\mathrm{peak}} = 2\langle\mathrm{Re}[\boldsymbol{A}^{\mathrm{T}}]\cdot\mathrm{Re}[\boldsymbol{A}]\rangle\,. \tag{A.4}$$

Equations (A.1) to (A.4) are used for the calculation of the time-average Umov–Poynting vector and time-average and peak kinetic energy densities.

The calculation of the potential and loss energy densities (22) and (23) uses the following identities involving the strain $\boldsymbol{S}$ and the symmetric and complex stiffness matrix $\boldsymbol{p}$:

$$\langle\mathrm{Re}[\boldsymbol{S}^{\mathrm{T}}]\cdot\mathrm{Re}[\boldsymbol{p}]\cdot\mathrm{Re}[\boldsymbol{S}]\rangle = \tfrac{1}{2}\mathrm{Re}[\boldsymbol{S}^{\mathrm{T}}\cdot\boldsymbol{p}\cdot\boldsymbol{S}^*]\,, \tag{A.5}$$

$$\langle\mathrm{Re}[\boldsymbol{S}^{\mathrm{T}}]\cdot\mathrm{Im}[\boldsymbol{p}]\cdot\mathrm{Re}[\boldsymbol{S}]\rangle = \tfrac{1}{2}\mathrm{Im}[\boldsymbol{S}^{\mathrm{T}}\cdot\boldsymbol{p}\cdot\boldsymbol{S}^*]\,. \tag{A.6}$$

When all the components of $\boldsymbol{S}$ are in phase,

$$(\mathrm{Re}[\boldsymbol{S}^{\mathrm{T}}]\cdot\mathrm{Re}[\boldsymbol{p}]\cdot\mathrm{Re}[\boldsymbol{S}])_{\mathrm{peak}} = 2\langle\mathrm{Re}[\boldsymbol{S}^{\mathrm{T}}]\cdot\mathrm{Re}[\boldsymbol{p}]\cdot\mathrm{Re}[\boldsymbol{S}]\rangle\,, \tag{A.7}$$

where Im[·] takes the imaginary part. This identity is used to obtain the energy balance for homogeneous plane waves, eq. (29).