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A rheological model for anelastic anisotropic media with applications to seismic wave propagation

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SUMMARY

This work presents a new constitutive law for linear viscoelastic and anisotropic media, to model rock behaviour and its effects on wave propagation. In areas with high dissipation properties (e.g. hydrocarbon reservoirs), the interpretation of seismic data based on the isotropic and purely elastic assumption might lead to misinterpretations or, even worse, to overlooking useful information. Thus, a proper description of wave propagation requires a rheology which accounts for the anisotropic and anelastic behaviour of rocks. The present model is based on the following mechanical interpretation; each eigenvector (eigenstrain) of the stiffness tensor of an anisotropic solid defines a fundamental deformation state of the medium. The six eigenvalues (eigenstiffnesses) represent the genuine elastic parameters. Since they are independent of the reference system, they have an intrinsic physical content. From this fact and the correspondence principle we infer that in a real medium the rheological properties depend essentially on six relaxation functions, which are the generalization of the eigenstiffnesses to the viscoelastic case. The existence of six or less complex moduli depends on the symmetry class of the medium. We probe the new stress–strain relation with homogeneous viscoelastic plane waves, and give expressions for the slowness, attenuation, phase velocity, energy velocity (wavefront) and quality factor of the different wave modes.

Key words: anisotropy, constitutive law, viscoelasticity, wave propagation.

INTRODUCTION

Modelling rock rheology and its effects on wave propagation has many important applications, mainly in mining and petroleum engineering, exploration geophysics, earthquake seismology, etc. For instance, in exploration geophysics, enhanced reservoir characterization requires a suitable constitutive equation that accounts for the effects of anisotropy and anelasticity on seismic wave propagation. The relevance of viscoelastic effects in seismic wave propagation is well documented (e.g. Bourbié, Coussy & Zinszner 1987) and the corresponding dissipation mechanisms are mostly known. These effects can be indicators of possible hydrocarbon accumulations: energy dissipation is enhanced in fluid-filled cracked limestones and porous sandstones; moreover, fractured formations and fine layering may show effective anisotropy.

The model introduced here is based on an idea that dates back to Lord Kelvin. As he wrote in his early papers on elasticity (Thomson 1856, 1878): '*a single system of six mutually orthogonal types* (strains) *may be determined for any homogeneous elastic solid, so that its potential energy when homogeneously strained in any way is expressed by the sum of the products of the squares of the components of the strain, according to those types, respectively multiplied by six determinate coefficients. The six strain-types thus determined are called the Six Principal Strain-types of the body*'. A few paragraphs later he refers to the coefficients as the '*six Principal Elasticities of the body*'. The equations of equilibrium imply that: '*If a body be strained to any of its six Principal Types, the stress required to hold it so is directly concurrent with* (proportional to) *the strain*'. These concepts were reinterpreted by Pipkin (1976), Walpole (1984) and recently by Mehrabadi & Cowin (1990) by using fourth-rank and second-rank tensor algebra, respectively. The *Six Principal Strains* in which any arbitrary strain can be decomposed are the eigenvectors of the elasticity tensor in 6-D space, or the *eigenstrains* when working in 3-D space. The *Six Principal Elasticities* are the eigenvalues of the second-rank elasticity tensor; they are referred to here as the *eigenstiffnesses* after Helbig (1993), who recently investigated the relation between the eigensystems and the material symmetry, and identified the wave-compatible isochoric (deformation without change of volume) eigenstrains.

The additive decomposition of the total strain energy into a sum of six or fewer terms represents energy modes which are not interactive with each other. These modes, together with their eigenstiffnesses, determine the complete set of fundamental deformations of a material body, including those compatible with wave propagation. The effective stiffness of an arbitrary strain compatible with wave propagation can be expressed as a linear combination of the eigenstiffnesses, an expansion that seems to take a simple form along longitudinal (pure mode) directions. This decomposition implies that of the 21 parameters of the elasticity tensor, six are genuine stiffnesses describing the properties of the medium, and the other 15 are geometrical parameters required to define the shape and orientation of the eigenstrains in 6-D space.

The correspondence principle allows the application of this approach in the framework of viscoelastic media. Generalizing the real eigenstiffnesses to complex and frequency-dependent moduli, we obtain a constitutive model able to describe viscoelastic behaviour. There is freedom in the choice of the frequency dependence of the eigenstiffnesses, so one can accommodate several dissipation peaks in the quality factor. Moreover, the particular symmetry determines the directional anelastic properties of the medium.

The conventions in the next sections are that 'tr' takes the trace of a 3×3 matrix, $\mathcal{R}$ and $\mathcal{I}$ take the real and imaginary parts, respectively, 'diag' denotes a diagonal matrix, and the superscript '*' indicates complex conjugate.

HOOKE'S LAW IN TENSORIAL FORM

By the generalized Hooke's law, it is assumed that stress σ and strain ϵ are linearly related by a symmetric stiffness operator $\mathbf{c}$. In other words, there exists a symmetric linear operator

$$\mathbf{c}: L_s(\mathbf{R}^3)\rightarrow L_s(\mathbf{R}^3): \varepsilon \rightarrow \sigma = \mathbf{c}[\varepsilon], \tag{1}$$

where $L_s(V)$ is the subspace of symmetric linear maps over V.

The second-rank Cartesian tensor formulation of Hooke's law in six dimensions is introduced by Mehrabadi & Cowin (1990). If the Cartesian base vectors in three dimensions are denoted by $\boldsymbol{e}_i$ ($i = 1, 2, 3$) and those in six dimensions by $\hat{\boldsymbol{e}}_I$ ($I = 1, \ldots, 6$), the canonical basis in $L_s(\mathbf{R}^3)$ is given by the following set of tensors:

$$\begin{aligned}
&\hat{\boldsymbol{e}}_1 = \boldsymbol{e}_1\otimes\boldsymbol{e}_1, \qquad \hat{\boldsymbol{e}}_4 = \alpha(\boldsymbol{e}_2\otimes\boldsymbol{e}_3+\boldsymbol{e}_3\otimes\boldsymbol{e}_2),\\
&\hat{\boldsymbol{e}}_2 = \boldsymbol{e}_2\otimes\boldsymbol{e}_2, \qquad \hat{\boldsymbol{e}}_5 = \alpha(\boldsymbol{e}_1\otimes\boldsymbol{e}_3+\boldsymbol{e}_3\otimes\boldsymbol{e}_1),\\
&\hat{\boldsymbol{e}}_3 = \boldsymbol{e}_3\otimes\boldsymbol{e}_3, \qquad \hat{\boldsymbol{e}}_6 = \alpha(\boldsymbol{e}_1\otimes\boldsymbol{e}_2+\boldsymbol{e}_2\otimes\boldsymbol{e}_1),
\end{aligned} \tag{2}$$

where $\otimes$ denotes the tensor product (Gurtin 1981) and $\alpha = 1/\sqrt{2}$. This is an orthonormal basis namely $\hat{\boldsymbol{e}}_I^{\mathrm{T}}\cdot\hat{\boldsymbol{e}}_J = \delta_{IJ}$, where the dot denotes ordinary matrix multiplication, and T indicates transpose. Hence, the symmetric stiffness operator may be expanded as

$$\mathbf{c} = \sum_{I,J=1}^{6} \hat{c}_{IJ}\hat{\boldsymbol{e}}_I\otimes\hat{\boldsymbol{e}}_J, \quad \text{where } \hat{c}_{IJ} = \hat{\boldsymbol{e}}_I^{\mathrm{T}}\cdot\mathbf{c}[\hat{\boldsymbol{e}}_J]. \tag{3}$$

Explicitly, Hooke's law in the second-rank tensor notation reads

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sqrt{2}\sigma_{yz}\\ \sqrt{2}\sigma_{xz}\\ \sqrt{2}\sigma_{xy}\end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \sqrt{2}c_{14} & \sqrt{2}c_{15} & \sqrt{2}c_{16}\\ c_{12} & c_{22} & c_{23} & \sqrt{2}c_{24} & \sqrt{2}c_{25} & \sqrt{2}c_{26}\\ c_{13} & c_{23} & c_{33} & \sqrt{2}c_{34} & \sqrt{2}c_{35} & \sqrt{2}c_{36}\\ \sqrt{2}c_{14} & \sqrt{2}c_{24} & \sqrt{2}c_{34} & 2c_{44} & 2c_{45} & 2c_{46}\\ \sqrt{2}c_{15} & \sqrt{2}c_{25} & \sqrt{2}c_{35} & 2c_{45} & 2c_{55} & 2c_{56}\\ \sqrt{2}c_{16} & \sqrt{2}c_{26} & \sqrt{2}c_{36} & 2c_{46} & 2c_{56} & 2c_{66}\end{bmatrix} \times \begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \sqrt{2}\varepsilon_{yz}\\ \sqrt{2}\varepsilon_{xz}\\ \sqrt{2}\varepsilon_{xy}\end{bmatrix}, \tag{4}$$

where c_{IJ} are the elasticities in the Voigt bases (Auld 1990). These are defined as follows: the Voigt stress basis has the form of eq. (2) with $\alpha = 1$, and the Voigt strain basis has the same form but with $\alpha = 1/2$. However, the convention will be to use the symbol ^ over the elasticity matrix, and stress and strain vectors when these quantities are expressed in the tensorial basis. It is convenient to express eq. (4) in compact notation as

$$\hat{\mathbf{T}} = \hat{c}\cdot\hat{\mathbf{S}}. \tag{5}$$

Actually, the interpretation of stress and strain as vectors is not physically essential but simplifies the mathematical treatment of the problem. Indeed, in this way the elasticity tensor $\hat{c}$ has order two instead of four and hence may be considered as a matrix: its eigenvalues and eigenvectors are then well defined.

EIGENSTIFFNESSES AND EIGENSTRAINS IN ELASTIC MEDIA

The *Six Orthogonal Strain Types* and the *Six Principal Elasticities* referred to by Lord Kelvin can be found by seeking those strain states σ for which ε and σ are parallel in 6-D Cartesian space, i.e.

$$\sigma = \mathbf{c}[\varepsilon] = \Lambda\varepsilon, \tag{6}$$

where Λ is a scalar quantity. This is mathematically equivalent to diagonalizing the stiffness matrix $\hat{\mathbf{c}}$:

$$(\hat{\mathbf{c}} - \Lambda\mathbf{1})\cdot\hat{\mathbf{S}} = 0, \tag{7}$$

where $\mathbf{1}$ is the 6×6 identity matrix. Hence, the eigenstiffnesses and eigenstrains are the eigenvalues and eigenvectors of $\hat{\mathbf{c}}$, respectively.

The diagonal matrix of the eigenstiffnesses, taken with their multiplicity, can be expressed as

$$\mathbf{\Lambda} = \mathbf{A}\cdot\hat{\mathbf{c}}\cdot\mathbf{A}^{\mathrm{T}}, \tag{8}$$

where $\mathbf{A}$ is the matrix formed with the eigenstrains, or more precisely, with the columns of the right (orthonormal) eigenvectors of $\hat{\mathbf{c}}$ (note that the symmetry of $\hat{\mathbf{c}}$ implies that $\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}}$). Then,

$$\hat{\mathbf{c}} = \mathbf{A}^{\mathrm{T}} \cdot \mathbf{\Lambda} \cdot \mathbf{A}. \tag{9}$$

The fact that the eigenvalues of the elasticity matrix are invariant with respect to any base and coordinate system confers to the eigenstiffnesses an intrinsic character. To illustrate the utility of the decomposition eq. (9) we consider briefly the isotropic and transversely isotropic cases.

Isotropic media

An isotropic medium is characterized by a stiffness operator $\mathbf{c}$ defined by

$$\mathbf{c}[\varepsilon] = 2\mu\varepsilon + \lambda(\mathrm{tr}\,\varepsilon)\mathbf{I}, \quad \text{i.e. } \mathbf{c} = 2\mu\mathbf{1} + \lambda\mathbf{I}\otimes\mathbf{I}, \tag{10}$$

where λ and μ are the Lamé constants, and $\mathbf{I}$ is the identity map in $\mathbf{R}^3$. The characteristic equation for the stiffness operator is then

$$2\mu\varepsilon + (\mathrm{tr}\,\varepsilon)\mathbf{I} = \Lambda\varepsilon. \tag{11}$$

Taking the trace of this equation, we see that a strain with non-zero trace is an eigenstrain if and only if it is proportional to $\mathbf{I}$, and the corresponding eigenvalue is then $\Lambda_1 = 2\mu + 3\lambda$, with multiplicity 1. Moreover, all non-zero strains with zero trace are eigenstrains corresponding to the eigenstiffness $\Lambda = 2\mu$, with multiplicity 5. No other eigenstiffnesses or eigenstrains are possible. It is clear that eigenstrains and eigenstresses are related by

$$\begin{aligned} &\mathrm{tr}\,\sigma = \Lambda_1(\mathrm{tr}\,\varepsilon), \\ &\text{and} \\ &\tilde{\sigma} = \Lambda_I\tilde{\varepsilon}, \qquad I = 2, \ldots, 6, \end{aligned} \tag{12}$$

where the tilde denotes the deviatoric tensors. Then, in unbounded and homogeneous isotropic media, the total stress can be decomposed into pure dilatational and shear stresses, and they produce pure deformations which are not interactive with each other.

Transversely isotropic media

In this case the eigenstiffnesses are the eigenvalues of the matrix

$$\hat{\mathbf{c}} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{11} - c_{12} \end{bmatrix} \tag{13}$$

Moreover, if $(\hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_6)$ is an eigenvector of the $\hat{\mathbf{c}}$ matrix, then $\sum \hat{\varepsilon}_I \hat{\mathbf{e}}_I$ is an eigenstrain, and conversely. The eigenvalues of the $\hat{\mathbf{c}}$ matrix are the following:

$$\begin{aligned} &\Lambda_1 = 2c_{44} && \text{with multiplicity 2} \\ &\Lambda_2 = c_{11} - c_{12} && \text{with multiplicity 2} \\ &\Lambda_3 = \tfrac{1}{2}(c_{11} + c_{12} + c_{33} - \sqrt{E}) && \text{with multiplicity 1} \\ &\Lambda_4 = \tfrac{1}{2}(c_{11} + c_{12} + c_{33} + \sqrt{E}) && \text{with multiplicity 1} \end{aligned} \tag{14}$$

where

$$E = c_{11}^2 + c_{12}^2 + c_{33}^2 + 8c_{13}^2 + 2c_{11}c_{12} - 2c_{11}c_{33} - 2c_{12}c_{33}. \tag{15}$$

The eigenspace associated with the first eigenvalue is spanned by

$$(0, 0, 0, 1, 0, 0)^{\mathrm{T}} \quad \text{and} \quad (0, 0, 0, 0, 1, 0)^{\mathrm{T}}, \tag{16}$$

which represent isochoric modes. Likewise, the eigenspace associated with the second eigenvalue is spanned by

$$(0, 0, 0, 0, 0, 1)^{\mathrm{T}} \quad \text{and} \quad (-1, 1, 0, 0, 0, 0)^{\mathrm{T}}, \tag{17}$$

and also these eigenvectors represent isochoric modes. The eigenspace associated with the third eigenvalue is spanned by

$$\left(1, 1, \frac{2c_{13}}{\Lambda_3 - c_{33}}, 0, 0, 0\right)^{\mathrm{T}}, \tag{18}$$

which can be interpreted as a quasi-isochoric mode since in the isotropic limit it corresponds to an isochoric eigenstrain. The eigenspace associated with the fourth eigenvalue is spanned by

$$\left(1, 1, \frac{2c_{13}}{\Lambda_4 - c_{33}}, 0, 0, 0\right)^{\mathrm{T}}, \tag{19}$$

which can be interpreted as a quasi-dilatational mode since in the isotropic limit it corresponds to the dilatation eigenstrain.

The eigenstiffnesses and eigenstrains of materials of lower symmetry are given by Mehrabadi & Cowin (1990). The eigentensors may be represented as 3×3 symmetric matrices in 3-D space; in that case their eigenvalues are invariant under rotations and describe the magnitude of the deformation. On the other hand, their eigenvectors describe the orientation of the eigentensor in a given coordinate system. For instance, pure volume dilatations correspond to eigenstrains with three equal eigenvalues (e.g. eq. (19) in the isotropic limit), and the trace of an isochoric eigenstrain is zero (e.g. eq. (18) in the isotropic limit). Isochoric strains with two equal eigenvalues but opposite sign and a third eigenvalue zero are plane shear tensors (e.g. in the second of eqs (17)). To summarize, the eigentensors identify preferred modes of deformation associated with the particular symmetry of the material. An illustrative pictorial representation of these modes or eigenstrains was designed by Helbig (1993).

THE VISCOELASTIC CONSTITUTIVE LAW

The above discussion of the elastic case leads us to create a model in which *six relaxation functions* together with the eigenstrains describe the deformation and anelastic properties of an anisotropic and viscoelastic medium. These six or less relaxation functions (*complex moduli* in the frequency domain) are the generalization of the eigenstiffnesses, by using the correspondence principle (Ben-Menahem & Singh 1981), to appropriate complex moduli satisfying the Kramers–Krönig dispersion relations (causality principle). The existence of six or less complex eigenstiffnesses depends on the *symmetry* class of the medium.

Hence, in virtue of the correspondence principle and its application to eq. (9), we introduce the viscoelastic stiffness tensor

$$\hat{\mathbf{p}}(\omega) = \mathbf{A}^{\mathrm{T}} \cdot \mathbf{\Lambda}^{(v)}(\omega) \cdot \mathbf{A}, \tag{20}$$

where ω is the angular frequency, and $\mathbf{\Lambda}^{(v)}$ is a diagonal matrix with entries

$$\Lambda_I^{(v)} = \Lambda_I M_I(\omega), \qquad I = 1, \ldots, 6. \tag{21}$$

The quantities M_I are complex and frequency-dependent dimensionless moduli. Alternatively, the viscoelastic stiffness tensor may be expressed as

$$\hat{\mathbf{p}}(\omega) = \hat{\mathbf{c}} \cdot \mathbf{A}^{\mathrm{T}} \cdot \mathrm{diag}\,[M_1(\omega), \ldots, M_6(\omega)] \cdot \mathbf{A}. \tag{22}$$

It can be easily shown that the viscoelastic stiffness tensor is symmetric, in agreement with the result obtained by Gurtin & Hrusa (1991). Moreover, from definition (20) it follows that the eigenvectors of $\hat{\mathbf{p}}$ and $\hat{\mathbf{c}}$ coincide, indicating that both the elastic and the viscoelastic rheologies possess the same eigenstrains. Each complex eigenstiffness $\Lambda_I^{(v)}$ defines a fundamental deformation state of the solid, associated with a set of dissipation mechanisms.

The six relaxation functions are the inverse time Fourier transform of the complex eigenstiffnesses divided by $i\omega$

$$\psi_I = \Lambda_I \mathscr{F}^{-1}\left[\frac{M_I}{i\omega}\right]. \tag{23}$$

Therefore, the viscoelastic stress–strain relation is given by

$$\hat{\mathbf{T}} = \hat{\mathbf{\Psi}} * \frac{\partial \hat{\mathbf{S}}}{\partial t}, \tag{24}$$

where $*$ denotes time convolution, and from eq. (20) the relaxation tensor is

$$\hat{\mathbf{\Psi}}(t) = \mathbf{A}^{\mathrm{T}} \cdot \mathrm{diag}\,[\psi_1(t), \ldots, \psi_6(t)] \cdot \mathbf{A}. \tag{25}$$

We note that the behaviour of the material is elastic at both the low- and high-frequency limits. However, for dynamics problems (wave propagation) the elasticity matrix $\hat{\mathbf{c}}$ corresponds to the unrelaxed viscoelastic matrix (Herrera & Gurtin 1965), i.e.

$$\hat{\mathbf{\Psi}}(t = 0^+) = \hat{\mathbf{p}}(\omega = \infty) = \hat{\mathbf{c}}. \tag{26}$$

This implies that the complex moduli and relaxation functions must satisfy

$$M_I(\omega = \infty) = 1 \quad \text{and} \quad \psi_I(0^+) = \Lambda_I, \tag{27}$$

by eqs (20) and (25), respectively. In the example below, we consider relaxation functions, represented by simple mechanical models, which satisfy eq. (27).

A given wave mode is characterized by its proper complex effective stiffness that can be expressed, and hence defined, in terms of the complex eigenstiffnesses. For example, let us consider an isotropic viscoelastic solid. We have seen that the total strain can be decomposed into the dilatational and deviatoric eigenstrains, whose eigenstiffnesses are related to the compressibility and the shear moduli, respectively, the last with multiplicity five. Therefore, there are only two relaxation functions (or two complex eigenstiffnesses) in an isotropic medium, one describing pure dilatational anelastic behaviour, and the other describing pure shear anelastic behaviour. Every eigenstress is directly proportional to its eigenstrain of identical form, the proportionality constant being the complex eigenstiffness. As is well known (e.g. Carcione, Kosloff & Kosloff, 1988), the properties of the shear waves are described by the shear relaxation function, and the properties of the compressional wave by a linear combination of the dilatational and shear relaxation functions.

APPLICATIONS TO WAVE PROPAGATION

The theory of propagation of viscoelastic waves in isotropic media has been investigated by several researchers, notably Buchen (1971), Borcherdt (1977), Krebes (1984) and Caviglia, Morro & Pagani (1990). However, research into anisotropic media is relatively recent. Carcione (1990) obtained the expressions of the phase, group and energy velocities, and quality factors for homogeneous viscoelastic plane waves in a transversely isotropic medium. In the following, $\mathbf{T}$, $\mathbf{S}$, and $\mathbf{p}$ will denote the stress and strain vectors, and the viscoelastic stiffness matrix in the Voigt basis. This is because the equation of motion and the strain-displacement relations involve awkward factors of $\sqrt{2}$ in the tensorial basis. A homogeneous viscoelastic plane wave, solution of the wave equation, is of the form

$$\mathbf{u} = \mathbf{U}_0 \exp\,[i(\omega t - \mathbf{k} \cdot \mathbf{x})], \tag{28}$$

where $\mathbf{U}_0$ represents a constant complex vector, and

$$\mathbf{k} = (\kappa - i\alpha)\hat{\kappa} \equiv \mathrm{k}\hat{\kappa} \tag{29}$$

is the complex wavenumber vector, with κ and α the magnitudes of the real wavenumber and attenuation, respectively, and

$$\hat{\kappa} = l_x \hat{\mathbf{e}}_x + l_y \hat{\mathbf{e}}_y + l_z \hat{\mathbf{e}}_z \tag{30}$$

defines the propagation direction through the direction cosines l_x, l_y and l_z.

For homogeneous waves the Christoffel equation takes the following simple form (Carcione 1990):

$$(\mathbf{L} \cdot \mathbf{p} \cdot \mathbf{L}^{\mathrm{T}} - \rho V^2 \mathbf{I}) \cdot \mathbf{u} = \mathbf{0}, \tag{31}$$

where

$$\mathbf{L} = \begin{bmatrix} l_x & 0 & 0 & 0 & l_z & l_y \\ 0 & l_y & 0 & l_z & 0 & l_x \\ 0 & 0 & l_z & l_y & l_x & 0 \end{bmatrix} \tag{32}$$

is the direction cosine matrix and ρ is the material density. The complex velocity

$$V = \frac{\omega}{\mathrm{k}} \tag{33}$$

is a fundamental quantity since it determines uniquely the slowness, the attenuation, the phase velocity and the quality factor. The complex velocities of the three wave modes are obtained from the dispersion relation

$$\det[\mathbf{L}\cdot\mathbf{p}\cdot\mathbf{L}^{\mathrm{T}}-\rho V^2\mathbf{I}]=0, \tag{34}$$

which is the characteristic equation of (31). Using eq. (33), the slowness and attenuation vectors can be expressed in terms of the complex velocity as

$$\mathbf{s}=\mathcal{R}\left[\frac{1}{V}\right]\hat{\kappa} \quad \text{and} \quad \alpha=-\omega\mathcal{I}\left[\frac{1}{V}\right]\hat{\kappa}, \tag{35}$$

and the phase velocity is the reciprocal of the slowness. In vector form it is given by

$$\mathbf{V}_{\mathrm{p}}=\left(\mathcal{R}\left[\frac{1}{V}\right]\right)^{-1}\hat{\kappa}. \tag{36}$$

The quality factor is defined as the ratio of the peak strain energy density to the average loss energy density. The peak strain energy for homogeneous plane waves is twice the average value, and is given by Carcione & Cavallini (1993)

$$(\varepsilon_s)_{\mathrm{peak}}=\tfrac{1}{2}\mathcal{R}[\mathbf{S}^{\mathrm{T}}\cdot\mathbf{p}\cdot\mathbf{S}^*]. \tag{37}$$

The average loss energy density is

$$\langle\varepsilon_d\rangle=\tfrac{1}{2}\mathcal{I}[\mathbf{S}^{\mathrm{T}}\cdot\mathbf{p}\cdot\mathbf{S}^*]. \tag{38}$$

From the definition, the quality factor is then

$$Q=\frac{(\varepsilon_s)_{\mathrm{peak}}}{\langle\varepsilon_d\rangle}=\frac{\mathcal{R}[\mathbf{S}^{\mathrm{T}}\cdot\mathbf{p}\cdot\mathbf{S}^*]}{\mathcal{I}[\mathbf{S}^{\mathrm{T}}\cdot\mathbf{p}\cdot\mathbf{S}^*]}. \tag{39}$$

It is shown in the Appendix that the quality factor in anisotropic-viscoelastic media takes the following simple form:

$$Q=\frac{\mathcal{R}[V^2]}{\mathcal{I}[V^2]}. \tag{40}$$

The energy velocity vector is defined as the ratio of the average power flow density to the mean energy density. The average power flow density is the real part of the complex Poynting vector

$$\mathbf{P}=-\frac{1}{2}\begin{bmatrix} T_1 & T_6 & T_5\\ T_6 & T_2 & T_4\\ T_5 & T_4 & T_3\end{bmatrix}\cdot\mathbf{v}^*, \tag{41}$$

where $\mathbf{v}=\partial\mathbf{u}/\partial t$. On the other hand, the mean energy density is the sum of the kinetic and strain energy density densities, where the kinetic energy is simply

$$(\varepsilon_v)_{\mathrm{peak}}=\tfrac{1}{2}\rho\mathbf{v}^{\mathrm{T}}\cdot\mathbf{v}^*. \tag{42}$$

Then, the energy velocity vector is

$$\mathbf{V}_{\mathrm{e}}=\frac{2\mathcal{R}[\mathbf{P}]}{(\varepsilon_v+\varepsilon_s)_{\mathrm{peak}}}. \tag{43}$$

We consider that the wavefront is the locus of the end of the energy velocity vector multiplied by one unit of propagation time.

EXAMPLE

We consider two transversely isotropic materials: Mesaverde clay shale whose material properties are

$$\begin{aligned}&c_{11}=66.6\,\mathrm{GPa},\quad c_{12}=19.7\,\mathrm{GPa},\quad c_{13}=39.4\,\mathrm{GPa},\\ &c_{33}=39.9\,\mathrm{GPa},\quad c_{44}=10.9\,\mathrm{GPa},\quad \rho=2590\,\mathrm{kg\,m^{-3}},\end{aligned} \tag{44}$$

and Taylor sandstone, for which

$$\begin{aligned}&c_{11}=34.6\,\mathrm{GPa},\quad c_{12}=9.4\,\mathrm{GPa},\quad c_{13}=10.6\,\mathrm{GPa},\\ &c_{33}=28.3\,\mathrm{GPa},\quad c_{44}=8.4\,\mathrm{GPa},\quad \rho=2500\,\mathrm{kg\,m^{-3}}.\end{aligned} \tag{45}$$

Clay shales and sandstones are characteristic of a reservoir environment: sandstones as recipient rocks, and shales as seal rocks. The numerical values in tables (44) and (45) are taken from the article by Thomsen (1986) who collected experimental data for a variety of anisotropic materials. The elastic constants (44) are untypical of normal shales (e.g. Sayers 1994) but have been chosen to illustrate the unusual but interesting case when the cusps are along the symmetry axis. Both media possess four distinct eigenstiffnesses, and therefore four complex moduli, one quasi-dilatational, one quasi-isochoric and two isochoric of multiplicity two. As we have seen previously, these eigenstiffnesses relax to one pure dilatational and five isochoric eigenstiffnesses, respectively, in the isotropic limit. We choose the following relaxation functions

$$\psi_l(t)=\Lambda_l\frac{\tau_\sigma^{(l)}}{\tau_\varepsilon^{(l)}}\left\{1-\left[1-\frac{\tau_\varepsilon^{(l)}}{\tau_\sigma^{(l)}}\right]\exp\left[-t/\tau_\sigma^{(l)}\right]\right\}H(t),$$
$$l=1,\ldots,4, \tag{46}$$

where $H(t)$ is the Heaviside function. The material relaxation times $\tau_\sigma^{(l)}$ and $\tau_\varepsilon^{(l)}$, characterizing the dissipation mechanism, satisfy $\tau_\sigma^{(1)}<\tau_\varepsilon^{(l)}$. A mechanical model corresponding to $\psi_l(t)$ is represented in Fig. 1. The instantaneous response of the material depends solely on the series spring. Several dissipation mechanisms can be modelled by a series or parallel connection of such single elements. The dimensionless complex moduli associated with the relaxation functions (46) are

$$M_l=\frac{\tau_\sigma^{(l)}}{\tau_\varepsilon^{(l)}}\left[\frac{1+i\omega\tau_\varepsilon^{(l)}}{1+i\omega\tau_\sigma^{(l)}}\right],\qquad l=1,\ldots,4. \tag{47}$$

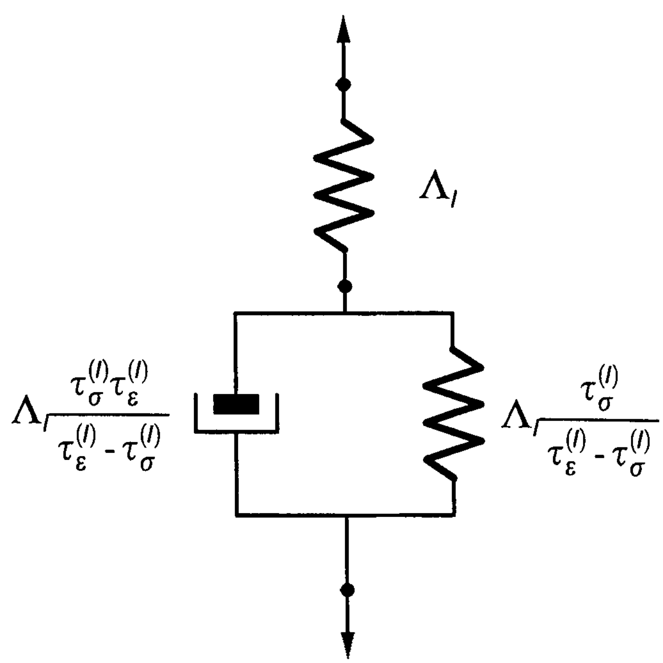


Figure 1. Mechanical model of the relaxation functions. Note that the series spring corresponds to the elastic stiffnesses (Λ_l, $l=1,\ldots,6$).

Note that the relaxation (46) and complex moduli (47) satisfy eqs (27), respectively, and moreover

$$\psi_l(0^+) = \Lambda_l > \psi_l(\infty) = \Lambda_l \frac{\tau_\sigma^{(l)}}{\tau_\varepsilon^{(l)}}, \tag{48}$$

in agreement with realistic behaviour of relaxation functions (Coleman 1964).

The quality factors associated with the four relaxation functions are (Ben-Menahem & Singh 1981),

$$Q^{(l)}(\omega) = \frac{\mathcal{R}[M_l]}{\mathcal{I}[M_l]} = Q_0^{(l)} \frac{1+\omega^2\tau_0^2}{2\omega\tau_0}, \tag{49}$$

where $Q_0^{(l)} = 2\tau_0/(\tau_\varepsilon^{(l)} - \tau_\sigma^{(l)})$, and $\tau_0 = \sqrt{\tau_\varepsilon^{(l)}\tau_\sigma^{(l)}}$. The curve $Q^{(l)}(\omega)$ has its peak at $\omega_0 = 1/\tau_0$, and the value of $Q^{(l)}$ at the peak is $Q_0^{(l)}$.

The complex velocities are the key to obtaining the attenuation and propagation properties. In the natural coordinate system, the eigenvalues of the characteristic eq. (34) in the (x, z)-plane of symmetry are as follows:

$$\rho V_{1(2)}^2 = \tfrac{1}{2}(p_{44} + p_{11}l_x^2 + p_{33}l_z^2 \pm \sqrt{D}), \tag{50}$$

and

$$\rho V_3^2 = p_{66}l_x^2 + p_{44}l_z^2, \tag{51}$$

where

$$D = [(p_{33} - p_{44})l_z^2 - (p_{11} - p_{44})l_x^2]^2 + 4(p_{13} + p_{44})^2 l_x^2 l_z^2. \tag{52}$$

In principle, V_1 (+ sign) is the velocity of the qP wave, while V_2 (− sign) and V_3 correspond to the shear waves, with V_3 a pure mode. In complex materials this identification does not apply, since along the same wavefront the wave may change from quasi-compressional to quasi-shear or vice versa. Note that, in a weak transversely isotropic medium the quasi-shear wave is defined by the velocity V_2. For completeness, the expressions of the energy velocities are given below (Carcione 1992):

$$\begin{aligned}\mathbf{V}_{e(m)} = V_{p(m)} D_m^{-1} \mathcal{R}[V_m^{-1}\{[l_x(p_{11} + p_{44}|B_m|^2) \\ + l_z(p_{13}B_m + p_{44}B_m^*)]\hat{e}_x + [l_x(p_{44}B_m + p_{13}B_m^*) \\ + l_z(p_{44} + p_{33}|B_m|^2)]\hat{e}_z\}], \qquad m = 1, 2\end{aligned} \tag{53}$$

where

$$D_m = \rho(1 + |B_m|^2 \mathcal{R}[V_m]), \tag{54}$$

and

$$B_m = -\frac{p_{11}l_x^2 + p_{44}l_z^2 - \rho V_m^2}{(p_{13} + p_{44})l_x l_z}. \tag{55}$$

The case $m = 1$ corresponds to the qP wave, and the qS wave is given by $m = 2$. A careful numerical evaluation of eq. (53) should consider the limits when either l_x or $l_z \to 0$. For instance, when $l_x \to 0$ and $l_z \to 1$, $B_1 \to \infty$ and $B_2 \to 0$. Taking these limits gives the appropriate energy velocities. The energy velocity for the pure shear mode is given by (Carcione 1992),

$$\mathbf{V}_{e(3)} = \rho^{-1}\{V_{p(3)}/\mathcal{R}[V_3]\}\mathcal{R}\{[l_x p_{66}\hat{e}_x + l_z p_{44}\hat{e}_z]/V_3\}. \tag{56}$$

The peak quality factors corresponding to each complex modulus are chosen as follows: $Q_0^{(1)} = 15$ and $Q_0^{(2)} = 10$ for the isochoric eigenstrains; $Q_0^{(3)} = 20$ and $Q_0^{(4)} = 30$ for the quasi-isochoric and quasi-dilatational eigenstrains. Each modulus gives a Debye peak in the quality factor at a frequency of $f_0 = \omega_0/2\pi = 20$ Hz, a typical value for seismic waves in geophysical exploration. The preceding values control the attenuation along the principal axes of the medium, as can be observed in Fig. 2, where the quality factor 40 of each wave mode is represented. The figure illustrates a polar diagram of quality factor curves in an (x, z)-plane of the medium. Only one quadrant of the plane is displayed from symmetry considerations. It can be shown that the complex stiffness matrix (20), when the elasticity matrix has the form (13), is the following, in the Voigt basis:

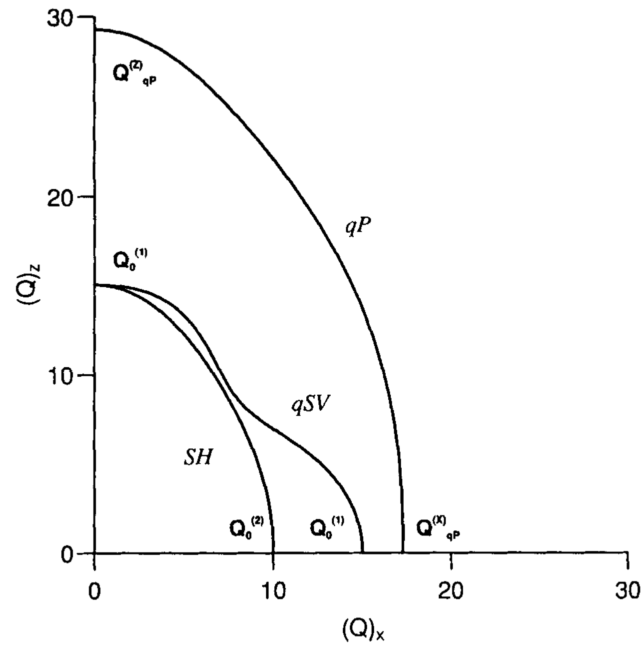


Figure 2. Polar diagram of clay shale quality factor in the first quadrant of the (x, z)-plane.

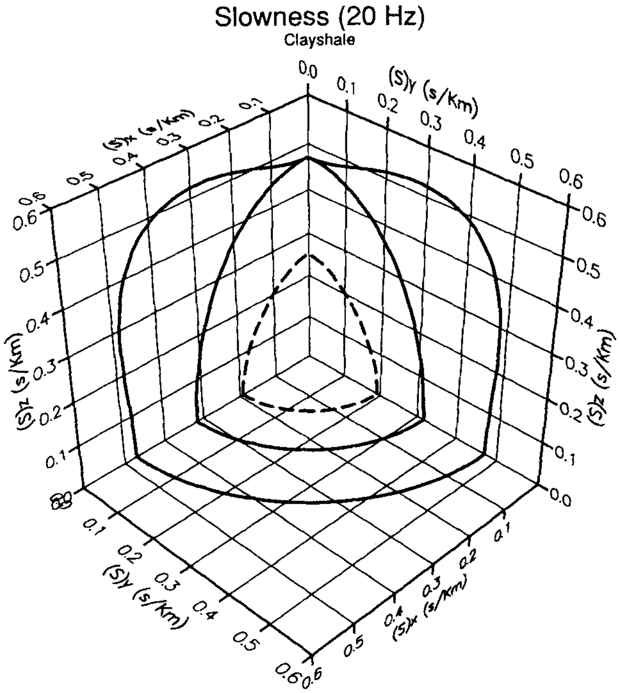
$$\mathbf{p} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & 0 & 0 & 0 \\ p_{12} & p_{22} & p_{13} & 0 & 0 & 0 \\ p_{13} & p_{13} & p_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & p_{66} \end{bmatrix}, \tag{57}$$

where, in particular, $p_{44} = c_{44}M_1$ and $p_{66} = c_{66}M_2$, with $c_{66} = (c_{11} - c_{12})/2$. As shown in the figure, the values of the quality factor of the shear modes along the principal axes are uniquely determined by the peak quality factors $Q^{(l)}(\omega_0) = Q_0^{(l)}$, $l = 1, 2$. On the other hand, the quality factors of the qP wave along the principal axes are mainly dependent on $Q_0^{(3)}$ and $Q_0^{(4)}$, and are given by

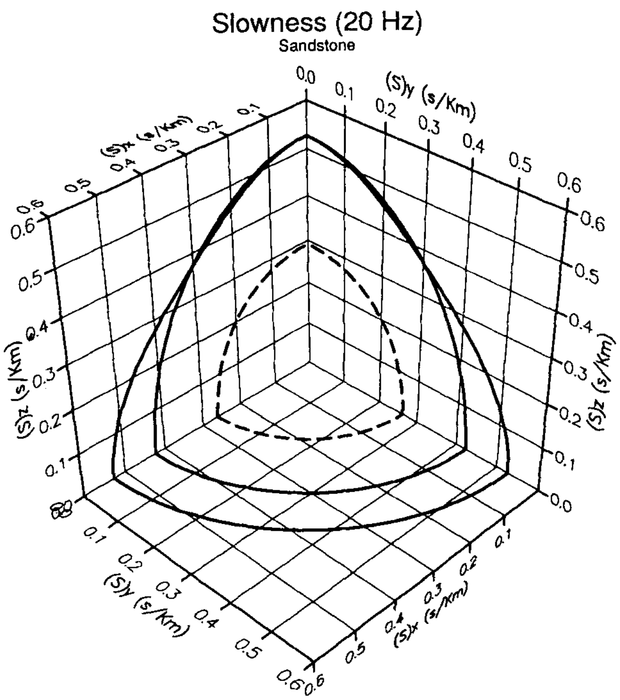
$$Q_{qP}^{(X)} = \frac{\mathcal{R}[p_{11}]}{\mathcal{I}[p_{11}]} \quad \text{and} \quad Q_{qP}^{(Z)} = \frac{\mathcal{R}[p_{33}]}{\mathcal{I}[p_{33}]}. \tag{58}$$

Figs 3, 5, 7 and 9 display sections of surfaces, representing physical quantities at 20 Hz, across the three mutually perpendicular coordinate planes where the symmetry axis coincides with the vertical axis. Only one octant of the sections is displayed from symmetry considerations.

Figure 3 represents sections of the clay shale and sandstone slowness surfaces. The inner curve (broken line)

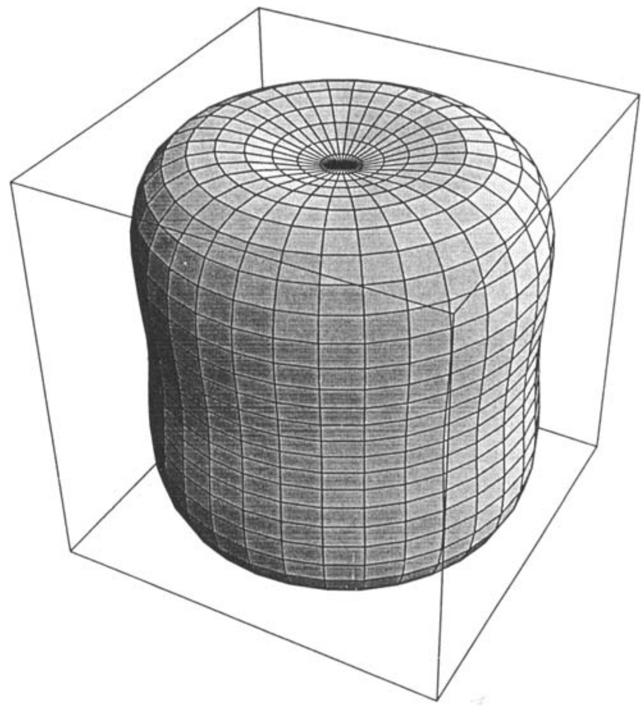


(a)

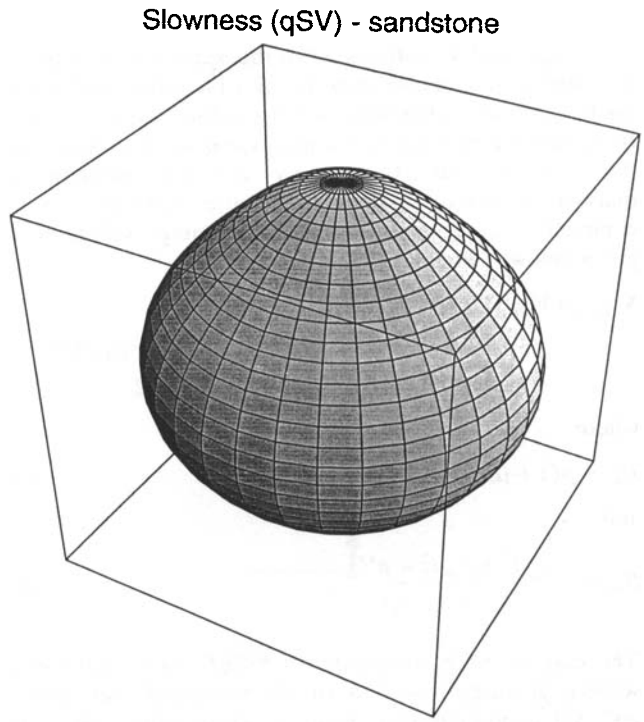


(b)

Figure 3. Sections of the clay shale and sandstone slowness surfaces. The inner curve (broken line) corresponds to the quasi-compressional wave (qP); then follows the pure shear (SH) and quasi-shear (qSV) waves.



(a)

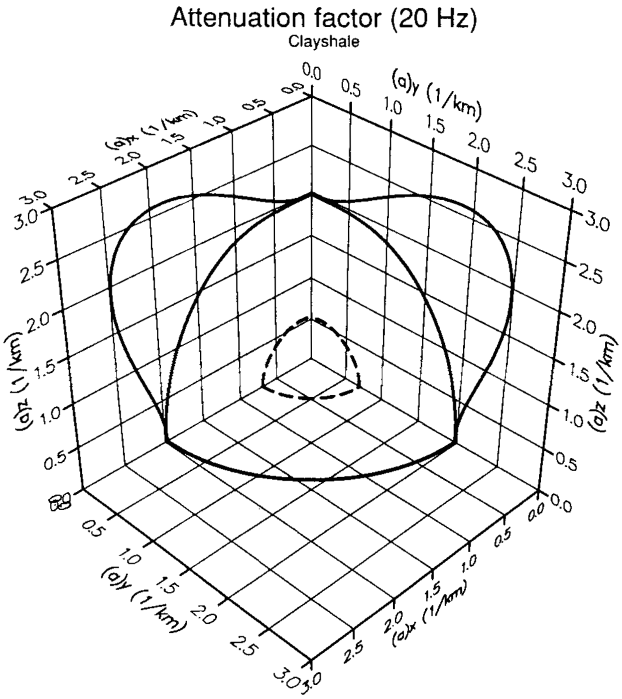


(b)

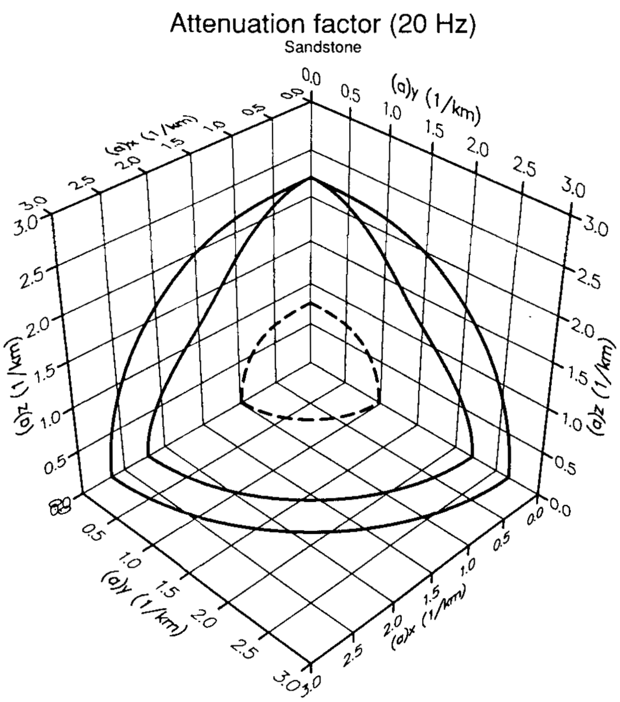
Figure 4. 3-D surfaces of the qSV wave slowness surface for clay shale and sandstone.

corresponds to the quasi-compressional wave (qP); then follow the pure shear (SH) and quasi-shear (qSV) waves, which have a kiss singularity at the symmetry axis. The corresponding three-dimensional surfaces for the qSV waves are illustrated in Fig. 4. The 3-D surfaces of the other waves are not displayed since they are very similar and can be deduced from Fig. 3: indeed, the qSV mode shows, usually, the highest degree of anisotropy.

Figure 5 represents sections of the clay shale and sandstone attenuation surfaces. The inner curve (broken line) corresponds to the quasi-compressional wave (qP).

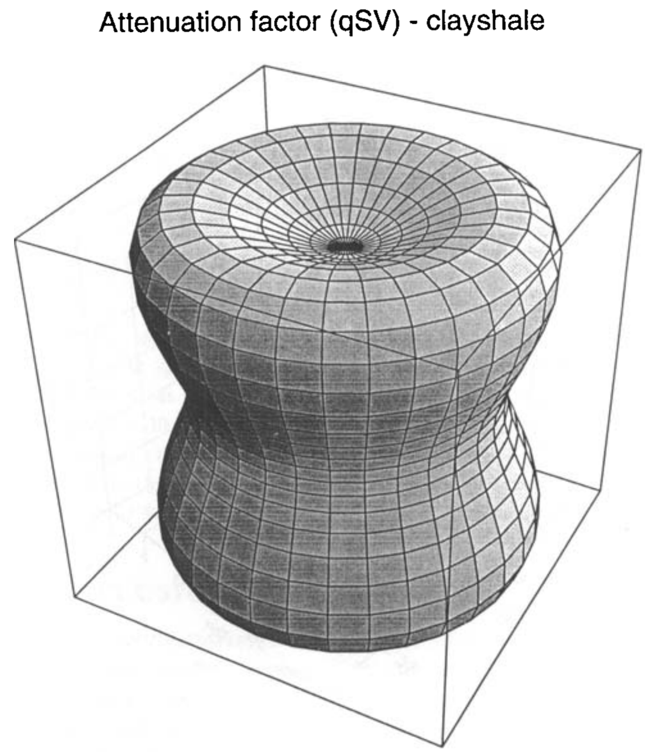


(a)

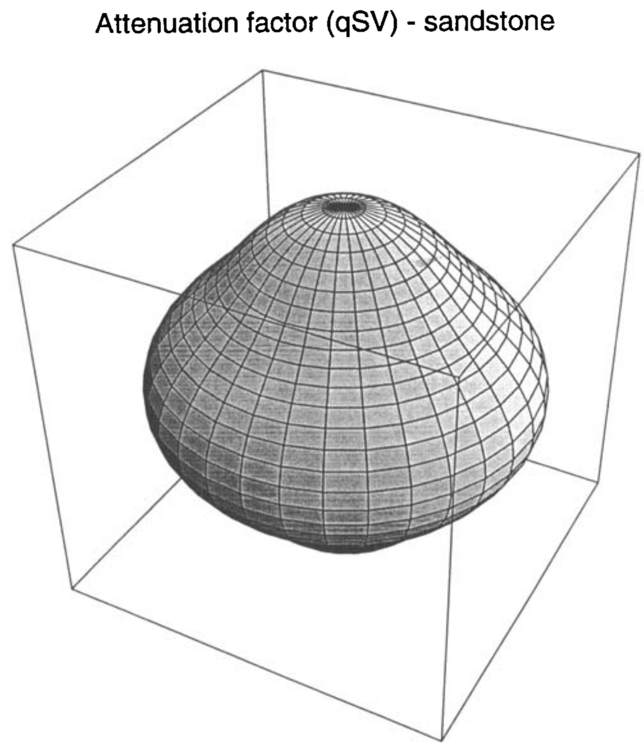


(b)

Figure 5. Sections of the clay shale and sandstone attenuation surfaces. The inner curve (broken line) corresponds to the quasi-compressional wave (qP).



(a)



(b)

Figure 6. 3-D surfaces of the qSV wave attenuation surface for clay shale and sandstone.

Although the values of the quality factor along the principal axes have been chosen similar for both media, the behaviour of the qSV curve is notably different. Indeed, for clay shale, this wave corresponds to the outer continuous line, and for sandstone it corresponds to the inner continuous curve. This behaviour is mainly dictated by the values of the elastic constants. The corresponding 3-D surfaces for the qSV waves are illustrated in Fig. 6.

Figure 7 represents sections of the clay shale and sandstone quality factor surfaces. The outer curve (broken line) corresponds to the quasi-compressional wave (qP). We

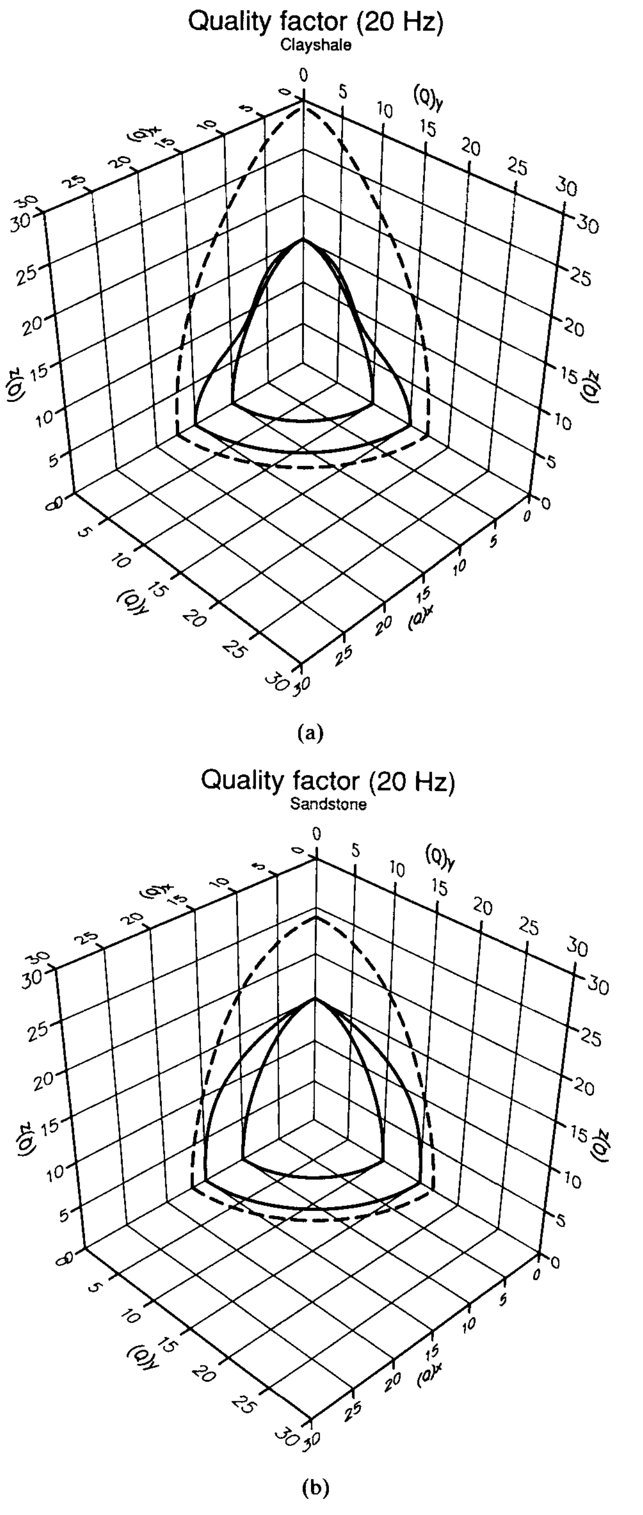


Figure 7. Sections of the clay shale and sandstone quality factor surfaces. The outer curve (broken line) corresponds to the quasi-compressional wave (qP).

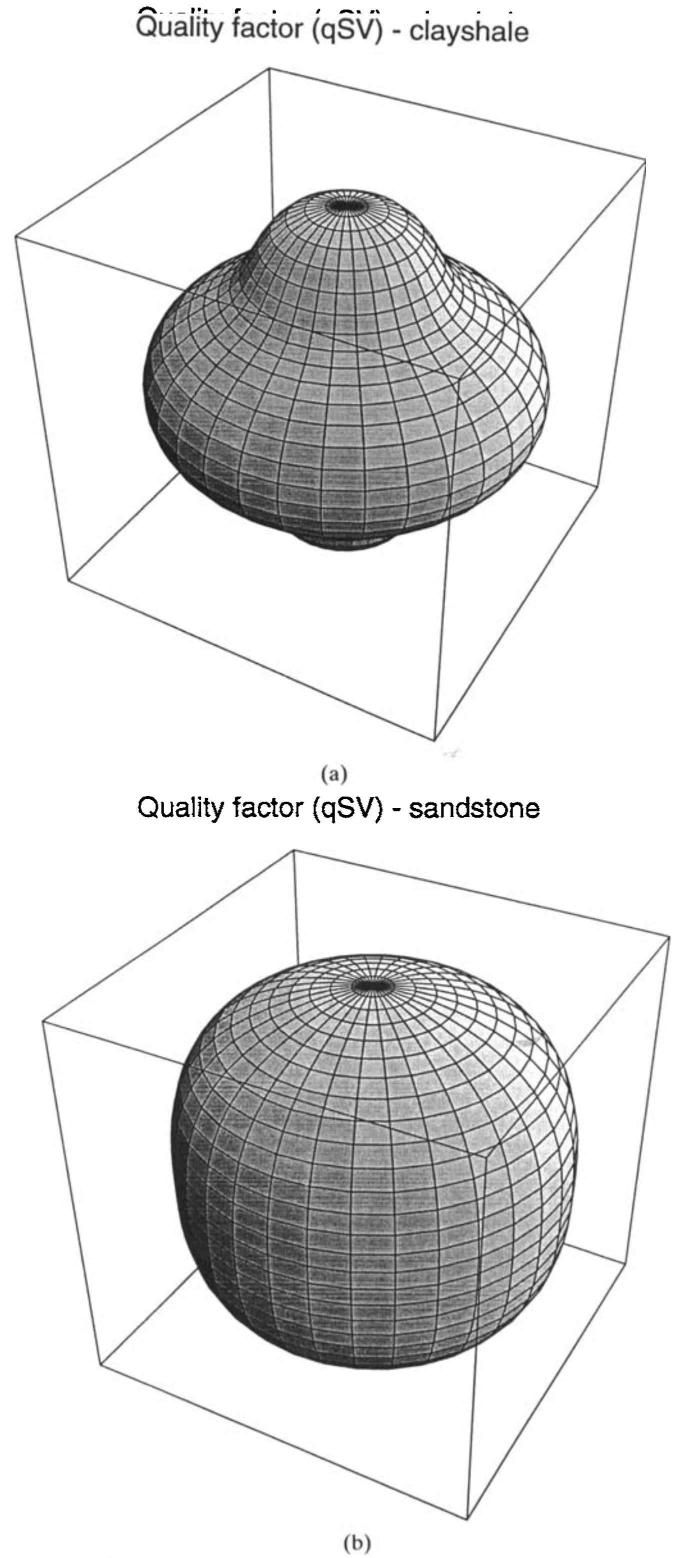


Figure 8. 3-D surfaces of the qSV wave quality factor surface for clay shale and sandstone, respectively.

note the unusual feature that, in the isotropy plane, while the quality factors of the shear waves are two distinct circles, the corresponding attenuation sections (Fig. 5) coincide. As before, the values of the elastic constants influence the dissipation: the difference between the clay shale and sandstone qP attenuation and quality factors along the symmetry axis is noticeable. The 3-D surfaces for the qSV waves are illustrated in Fig. 8.

Figure 9 shows sections of the clay shale and sandstone energy velocity surfaces where the polarizations of each wave mode are represented. When not plotted, the

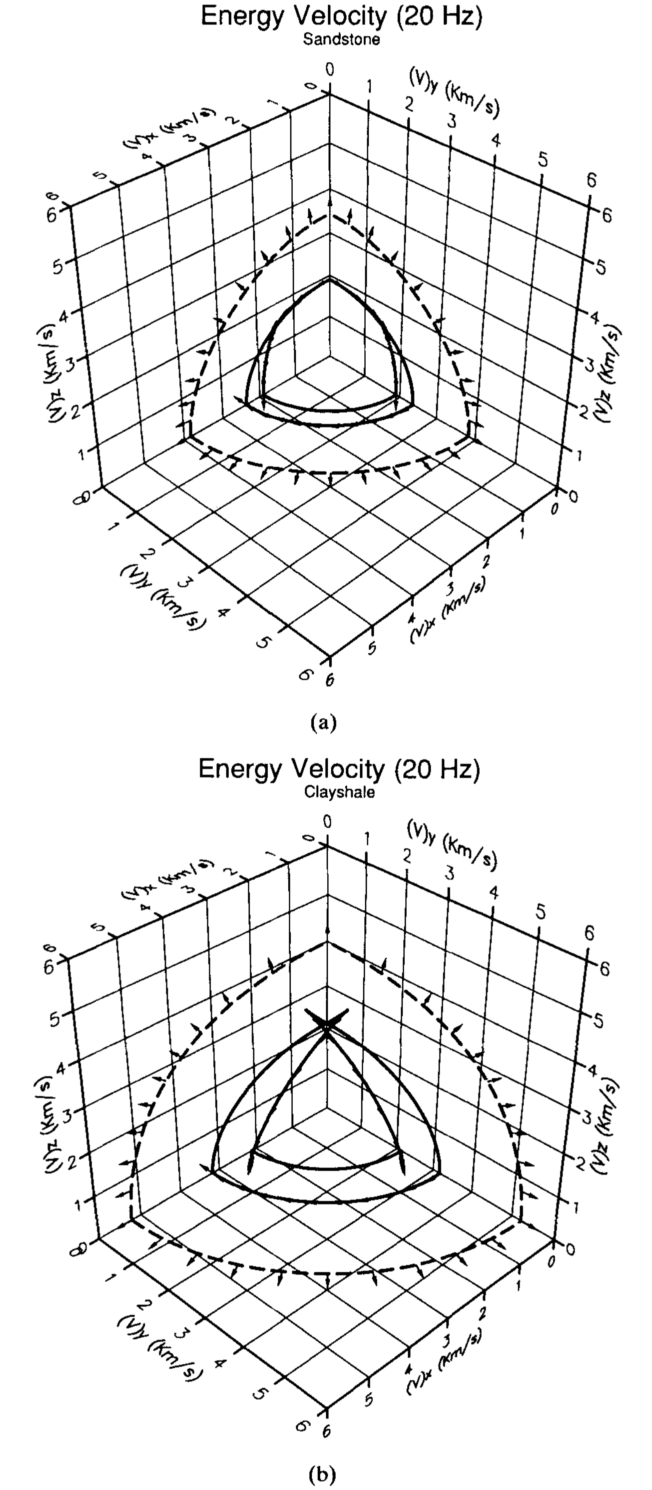


Figure 9. Sections of the clay shale and sandstone energy velocity surfaces where the polarizations of each wave mode are represented. When not plotted, the polarizations are perpendicular to the plane.

polarizations are perpendicular to the plane. The outer curve (broken line) corresponds to the quasi-compressional wave (qP). The polarizations can be calculated from the Christoffel equation (31). For instance, in the (x, z)-plane, the SH mode is polarized along the y-direction, while the coupled modes have components exclusively in the plane.

The first line of eq. (31) yields

$$(p_{11}l_x^2 + p_{44}l_z^2 - \rho V^2)u_x + (p_{13} + p_{44})l_x l_z u_z = 0. \tag{59}$$

It is clear that the complex vector $[1, 0, u_z/u_x]^{\mathrm{T}}$ is also an eigenvector of the Christoffel matrix. Then, the normalized polarization vectors of the coupled modes are

$$\frac{1}{\sqrt{1 + (\Re[B_m])^2}} \begin{bmatrix} 1 \\ 0 \\ \Re[B_m] \end{bmatrix}, \qquad m = 1, 2, \tag{60}$$

with B_m given by eq. (55). As can be appreciated in the figure, the qP polarization is almost perpendicular to the wavefront, as expected. Also, the qSV and SH waves can be identified through their polarizations, which are mostly tangential to the respective wavefronts. The plane of isotropy ((x, y)-plane) supports only pure compressional and shear waves.

CONCLUSIONS

Linear constitutive laws for general anisotropic and dissipative media, reported in the literature, are almost exclusively based on the Kelvin–Voigt stress–strain relation (e.g. Auld 1990). That is, dissipation is modelled by a viscosity matrix of 21 independent coefficients independent of frequency. In this work we introduce a new rheological relation where:

(1) based on physical grounds, the ambiguity on the time dependence of the relaxation components has been reduced by assuming that a maximum of six relaxation functions is enough to describe the anelastic properties of the material.

(2) The theory allows for an arbitrary time dependence of the relaxation components based on the relaxation kernels, and gives elastic behaviour at both the low- and high-frequency limits.

(3) The model identifies each relaxation function (or eigenstiffness, in the frequency domain) with a given state of deformation of the solid. In this way, it is possible to define the anelastic properties of the three different waves propagating in the medium: once an eigenstrain corresponding to a given propagating mode has been identified, the associated relaxation function defines its anelastic characteristics, and can be used to define the quality factor or the attenuation along preferred propagation directions.

In the example, we obtain close expressions of measurable quantities, like the attenuation and the quality factor, in terms of the complex velocities of the medium. In this way, the material properties can be determined from analysis of the propagation of plane homogeneous waves. The theory can be used either for matching experimental data for material characterization, or for predicting directional attenuation behaviour of anisotropic media.

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APPENDIX: QUALITY FACTOR FOR HOMOGENEOUS WAVES IN GENERAL ANISOTROPIC MEDIA

Equation (39) gives the quality factor for homogeneous waves:

$$Q=\frac{(\varepsilon_s)_{\text{peak}}}{\langle\varepsilon_d\rangle}=\frac{\mathscr{R}[\mathbf{S}^{\mathrm{T}}\cdot\mathbf{p}\cdot\mathbf{S}^*]}{\mathscr{I}[\mathbf{S}^{\mathrm{T}}\cdot\mathbf{p}\cdot\mathbf{S}^*]}. \tag{A1}$$

This equation requires the calculation of $\mathbf{S}^{\mathrm{T}}\cdot\mathbf{p}\cdot\mathbf{S}^*$. By definition the strain associated with the plane wave (28) is

$$\mathbf{S}=-ik\mathbf{L}^{\mathrm{T}}\cdot\mathbf{u}, \tag{A2}$$

and therefore its complex conjugate is

$$\mathbf{S}^*=ik^*\mathbf{L}^{\mathrm{T}}\cdot\mathbf{u}^*. \tag{A3}$$

Replacing these equations into $\mathbf{S}^{\mathrm{T}}\cdot\mathbf{p}\cdot\mathbf{S}^*$ gives

$$\mathbf{S}^{\mathrm{T}}\cdot\mathbf{p}\cdot\mathbf{S}^*=|k|^2\,\mathbf{u}^{\mathrm{T}}\cdot\mathbf{\Gamma}\cdot\mathbf{u}^*, \tag{A4}$$

where $\mathbf{\Gamma}=\mathbf{L}\cdot\mathbf{p}\cdot\mathbf{L}^{\mathrm{T}}$ is the Christoffel matrix. But, from the transpose of eq. (31) and the symmetry of $\mathbf{\Gamma}$,

$$\mathbf{u}^{\mathrm{T}}\cdot\mathbf{\Gamma}=\rho V^2\mathbf{u}^{\mathrm{T}}. \tag{A5}$$

Therefore, substituting these expressions into eq. (4) gives

$$\mathbf{S}^{\mathrm{T}}\cdot\mathbf{p}\cdot\mathbf{S}^*=\rho\,|k|^2\,V^2\mathbf{u}^{\mathrm{T}}\cdot\mathbf{u}^*. \tag{A6}$$

In consequence, since the matrix product on the right-hand side of eq. (6) is real, the quality factor in anisotropic-viscoelastic media takes the following simple form as a function of the complex velocity:

$$Q=\frac{\mathscr{R}[V^2]}{\mathscr{I}[V^2]}. \tag{A7}$$