ENERGY BALANCE AND INHOMOGENEOUS PLANE-WAVE ANALYSIS OF A CLASS OF ANISOTROPIC VISCOELASTIC CONSTITUTIVE LAWS *

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We prove a theorem of power and energy for the solutions of the linear equations of a viscoelastic material, whose rheology may be described in terms of lumped elements having the behaviour of either an elastic solid or a viscous fluid. The assumed anisotropy ensures that this class of constitutive laws is wide enough for describing most of geophysical media; yet, its a priori physical interpretation permits to avoid the mathematical ambiguities arising, in the definition of potential energy, with constitutive laws of abstract hereditary type. Moreover, sharper results for time-averaged energies are obtained by assuming a time-harmonic displacement. Finally, fundamental relations for phase-, energy- and dissipation-velocity are derived in the framework of plane inhomogeneous waves. As case studies, the Kelvin-Voigt, Maxwell and standard linear solid rheologies are worked out in detail. The use of coordinate-free notation permits to perform computations in a clean and rigorous way.

1. Introduction

The theory of mechanical waves in solid dissipative media is a classical topic: for background information on the physical and mathematical aspects, we refer to the books by Auld [1] and Caviglia and Morro [6], respectively. Fundamental papers on the energy balance for these waves are those by Buchen [3] and Borcherdt [2]. However, most of the results that can be found in the literature were proven assuming isotropy, which is a too restrictive assumption for geophysical purposes [12]. Hence Carcione and Cavallini [5] reviewed the subject in a fully anisotropic framework, using a component notation. The ideas in [5] are developed here with applications to specific case-studies, using a component-free notation [8]; indeed, the latter is more

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suitable to theoretical investigations, whereas a component notation is unavoidable in numerical modelling.

2. Waves with Arbitrary Time-Dependence

2.1. Hereditary-Elastic Media

2.1.1. Equations of Motion. Firstly, the *dynamic equation* is the continuum version of Newton's second law:

$$Div[\mathbf{S}] + \mathbf{b} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{2.1}$$

Secondly, the *kinematic equation* expresses the strain-displacement relationship as $\mathbf{E} = (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$. Here, as in the book by Gurtin [8]: $\mathbf{S}$ is stress, $\mathbf{b}$ is body force, ρ is density, t is time, $\mathbf{u}$ is displacement and $\mathbf{E}$ is strain.

Finally, the *constitutive equation* is $\mathbf{S} = \mathbf{R}*\mathbf{E}$, where $\mathbf{R}[t]$ is a linear symmetric operator $Sym \longrightarrow Sym$, with Sym the space of linear symmetric operators over the 3-D euclidean space; moreover, the asterisk indicates time convolution: $(\mathbf{R}*\mathbf{E})[t] = \int \mathbf{R}[t-\tau][\mathbf{E}[\tau]]d\tau$, where the integral is meant in the sense of naive distribution theory.

2.1.2. Kinetic Energy Theorem. Taking the scalar product of the particle velocity with the dynamic equation (2.1) and rearranging, we get the kinetic energy theorem

$$\frac{\partial \mathcal{E}_K}{\partial t} = -Div[\mathbf{J}] + \Pi_b - \Pi_S, \tag{2.2}$$

where $\mathbf{v} = \partial \mathbf{u}/\partial t$ is particle velocity, $\mathcal{E}_K = (1/2)\rho \mathbf{v}.\mathbf{v}$ is kynetic energy density, $\mathbf{J} = -\mathbf{S}[\mathbf{v}]$ is energy flux density, $\Pi_b = \mathbf{v}.\mathbf{b}$ is the power density expended by body forces, and $\Pi_S = \mathbf{S}.\nabla\mathbf{v} = \mathbf{S}.(\partial \mathbf{E}/\partial t)$ is stress power density ([8], p.111).

2.2. Visco-Elastic Media

In this paper, we call "visco-elastic" those hereditary-elastic media whose constitutive law may be modelled in terms of m elastic springs and n viscous dashpots, connected in series or in parallel [11]. The elastic elements are described by the generalized Hooke's law $\mathbf{S}_i = \mathbf{C}_i[\mathbf{E}_i]$, where $\mathbf{C}_i : Sym \longrightarrow Sym$ for $i = 1, \ldots, m$. Analogously, the viscous elements are described by the generalized Newton's dissipation law $\mathbf{S}_j = \mathbf{D}_j[\partial \mathbf{E}_j/\partial t]$, where $\mathbf{D}_j : Sym \longrightarrow Sym$ for $j = 1, \ldots, n$.

2.2.1. Potential Energy. Here, by analogy with the elastic case, the potential energy is defined as:

$$\mathcal{E}_P = \frac{1}{2}\sum_{k=1}^{m} \mathbf{E}_k.\mathbf{C}_k[\mathbf{E}_k]. \tag{2.3}$$

Thus, the dissipated power may be defined as

$$\Pi_D = \Pi_S - \frac{\partial \mathcal{E}_P}{\partial t} \tag{2.4}$$

and hence the kinetic energy theorem (2.2) yields the following energy balance equation:

$$\frac{\partial}{\partial t}(\mathcal{E}_K + \mathcal{E}_P) = -Div[\mathbf{J}] + \Pi_b - \Pi_D.$$

2.2.2. Example: Kelvin-Voigt Rheology. The constitutive equation is $\mathbf{S} = \mathbf{C}[\mathbf{E}] + \mathbf{D}[\partial \mathbf{E}/\partial t]$, where $\mathbf{C}$ and $\mathbf{D}$ are linear symmetric operators $Sym \longrightarrow Sym$. Then

$$\mathcal{E}_P = \frac{1}{2}\mathbf{E}.\mathbf{C}[\mathbf{E}] \quad \text{and} \quad \Pi_D = \frac{\partial \mathbf{E}}{\partial t}.\mathbf{D}[\frac{\partial \mathbf{E}}{\partial t}] \tag{2.5}$$

are the potential energy density and the power density dissipated because of viscosity, respectively.

2.2.3. Example: Maxwell Rheology. The constitutive equation is given by $\partial \mathbf{E}/\partial t = \mathbf{C}^{-1}[\partial \mathbf{S}/\partial t] + \mathbf{D}^{-1}[\mathbf{S}]$, where $\mathbf{C}$ and $\mathbf{D}$ are (linear, symmetric and invertible) operators $Sym \longrightarrow Sym$. Then

$$\mathcal{E}_P = \frac{1}{2}\mathbf{S}.\mathbf{C}^{-1}[\mathbf{S}] \quad \text{and} \quad \Pi_D = \frac{1}{2}\mathbf{S}.\mathbf{D}^{-1}[\mathbf{S}] \tag{2.6}$$

are the potential energy density and the power density dissipated because of viscosity, respectively.

3. Time-Harmonic Waves

By definition, a *time-harmonic wave* has the form:

$$\mathbf{u}[\mathbf{x}, t] = \Re[\tilde{\mathbf{u}}[\mathbf{x}] \exp[-i\omega t]] = \tilde{\mathbf{u}}_1[\mathbf{x}] \cos[\omega t] + \tilde{\mathbf{u}}_2[\mathbf{x}] \sin[\omega t],$$

where $\tilde{\mathbf{u}}_1 = \Re[\tilde{\mathbf{u}}]$ and $\tilde{\mathbf{u}}_2 = \Im[\tilde{\mathbf{u}}]$.

3.1. Hereditary-Elastic Media

3.1.1. Equations of Motion. The kinematic equation implies that the strain is a time-harmonic field $\mathbf{E}[\mathbf{x}, t] = \Re[\tilde{\mathbf{E}}[\mathbf{x}] \exp[-i\omega t]]$ related to the complex velocity $\tilde{\mathbf{v}}[\mathbf{x}]$ by $\tilde{\mathbf{E}}[\mathbf{x}] = (i/2\omega)(\nabla\tilde{\mathbf{v}}[\mathbf{x}] + \nabla\tilde{\mathbf{v}}[\mathbf{x}]^T)$. The constitutive equation implies that the stress is a time-harmonic field $\mathbf{S}[\mathbf{x}, t] = \Re[\tilde{\mathbf{S}}[\mathbf{x}] \exp[-i\omega t]]$ related to the complex strain $\tilde{\mathbf{E}}[\mathbf{x}]$ by $\tilde{\mathbf{S}}[\mathbf{x}] = \mathbf{R}_0[\tilde{\mathbf{E}}[\mathbf{x}]]$, where $\mathbf{R}_0 = \int \mathbf{R}[\tau] \exp[i\omega\tau] d\tau$. The dynamic equation implies that the body force is a time-harmonic field $\mathbf{b}[\mathbf{x}, t] = \Re[\tilde{\mathbf{b}}[\mathbf{x}] \exp[-i\omega t]]$ whose complex amplitude is related to the velocity and stress by

$$\tilde{\mathbf{b}}[\mathbf{x}] = -i\rho\omega\tilde{\mathbf{v}}[\mathbf{x}] - Div[\tilde{\mathbf{S}}[\mathbf{x}]]. \tag{3.1}$$

3.1.2. Time-Average Kinetic Energy Theorem. The time average, over one period, of the scalar product of two time-harmonic waves $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\langle \Re[\tilde{\mathbf{a}} \exp[-i\omega t]].\Re[\tilde{\mathbf{b}} \exp[-i\omega t]] \rangle = \frac{1}{2}\Re[\tilde{\mathbf{a}}.\tilde{\mathbf{b}}^*].$$

Then, it follows that

$$\langle\mathcal{E}_K\rangle = \frac{1}{4}\rho\tilde{\mathrm{v}}^*.\tilde{\mathrm{v}}, \quad \langle\mathbf{J}\rangle = \Re[\mathbf{P}], \quad \langle\Pi_b\rangle = \frac{1}{2}\Re[\tilde{\mathrm{v}}.\tilde{\mathrm{b}}^*], \quad \langle\Pi_S\rangle = -\frac{1}{2}\omega\Im[\tilde{\mathrm{S}}.\tilde{\mathrm{E}}^*], \tag{3.2}$$

where $\mathbf{P} = -(1/2)\tilde{\mathrm{S}}[\tilde{\mathrm{v}}^*]$ is the complex Poynting vector, by analogy with the electromagnetic case [7]. Now, taking the scalar product of the dynamic equation (3.1) with $\tilde{\mathrm{v}}^*$, rearranging, and separating the real and the imaginary parts, we get

$$Div[\langle\mathbf{J}\rangle] = \langle\Pi_b\rangle - \langle\Pi_S\rangle \quad \text{and} \quad -2\omega\langle\mathcal{E}_K\rangle = -Div[\Im[\mathbf{P}]] + \frac{1}{2}\Im[\tilde{\mathrm{b}}.\tilde{\mathrm{v}}^*] - \omega\frac{1}{2}\Re[\tilde{\mathrm{S}}.\tilde{\mathrm{E}}^*]. \tag{3.3}$$

The stress power density may be expanded, in the time-harmonic regime, as $\Pi_S = \langle\Pi_S\rangle + \frac{1}{2}\omega\Im[\tilde{\mathrm{S}}.\tilde{\mathrm{E}}e^{-2i\omega t}]$. The second term of the r.h.s. of this equation is a periodic function of time with zero mean: it corresponds to the completely reversible work done by the elastic forces. Hence we are led to infer that a good definition of dissipated power implies that its average value must coincide with the average stored power. This argument is a 3D anisotropic generalization of the 1D treatment in Sec. 17 of the book by Rabotnov [11]; equation (3.4) below shows that this view is correct, at least in the case of viscoelastic media.

3.2. Visco-Elastic Media: Time-Average Energy Balance

The average time rates of kinetic and potential energies result to be zero: $\langle\partial\mathcal{E}_K/\partial t\rangle = \langle\partial\mathcal{E}_P/\partial t\rangle = 0$; hence equation (2.4) yields

$$\langle\Pi_S\rangle = \langle\Pi_D\rangle. \tag{3.4}$$

Therefore, using the first equation in (3.3), we get the following mean energy flux equation:

$$Div[\langle\mathbf{J}\rangle] = \langle\Pi_b\rangle - \langle\Pi_D\rangle.$$

3.2.1. Example: Kelvin-Voigt Rheology. In this case, equations (2.5) yield $\langle\mathcal{E}_P\rangle = (1/4)\tilde{\mathrm{E}}.\mathbf{C}[\tilde{\mathrm{E}}^*]$ and $\langle\Pi_D\rangle = \langle\Pi_S\rangle = (1/2)\omega^2\tilde{\mathrm{E}}.\mathbf{D}[\tilde{\mathrm{E}}^*]$. Therefore $\Re[\tilde{\mathrm{S}}.\tilde{\mathrm{E}}^*] = 4\langle\mathcal{E}_P\rangle$ and hence, substituting into the second equation in (3.3), we get the energy balance equation

$$2\omega(\langle\mathcal{E}_P\rangle - \langle\mathcal{E}_K\rangle) = -Div[\Im[\mathbf{P}]] + \frac{1}{2}\Im[\tilde{\mathrm{b}}.\tilde{\mathrm{v}}^*]. \tag{3.5}$$

3.2.2. Example: Maxwell Rheology. In this case, equations (2.6) yield $\langle\mathcal{E}_P\rangle = (1/4)\tilde{\mathrm{S}}.\mathbf{C}^{-1}[\tilde{\mathrm{S}}^*]$ and $\langle\Pi_D\rangle = \langle\Pi_S\rangle = (1/2)\tilde{\mathrm{S}}.\mathbf{D}^{-1}[\tilde{\mathrm{S}}^*]$. Therefore, as in the case of the just seen Kelvin-Voigt rheology, $\Re[\tilde{\mathrm{S}}.\tilde{\mathrm{E}}^*] = 4\langle\mathcal{E}_P\rangle$ and hence, substituting into the second equation in (3.3), we obtain again the energy balance equation (3.5).

3.2.3. Example: Standard Linear Solid Rheology. The rheological model called standard linear solid results from connecting in series an elastic element, $\mathbf{C}_1$, with a Kelvin-Voigt element, formed by a spring $\mathbf{C}$ and a dashpot $\mathbf{D}$ connected in parallel. The corresponding constitutive law is

$$\mathbf{A}[\mathrm{S}] + \mathbf{B}[\frac{\partial \mathrm{S}}{\partial t}] = \mathbf{C}[\mathrm{E}] + \mathbf{D}[\frac{\partial \mathrm{E}}{\partial t}], \tag{3.6}$$

where, for simplicity, we have put

$$\mathbf{A} = \mathbf{I} + \mathbf{C} \circ \mathbf{C}_1^{-1} \quad \text{and} \quad \mathbf{B} = \mathbf{D} \circ \mathbf{C}_1^{-1}, \tag{3.7}$$

the symbol $\circ$ denoting map composition. If, instead of equation (3.7), we take $\mathbf{A} = \mathbf{I}$ and $\mathbf{B} = 0$, then we get from (3.6) the Kelvin-Voigt model. Analogously, for $\mathbf{C} = 0$, equations (3.6) and (3.7) yield Maxwell's rheology.

In the case of a time-harmonic wave, equations (2.3) and (2.4) yield $\langle \mathcal{E}_P \rangle = (\tilde{\mathrm{E}}_1.\mathbf{C}[\tilde{\mathrm{E}}_1^*] + \tilde{\mathrm{E}}_2.\mathbf{C}[\tilde{\mathrm{E}}_2^*])/4$ and $\langle \Pi_D \rangle = \langle \Pi_S \rangle = (1/2)\omega^2 \tilde{\mathrm{E}}.\mathbf{D}[\tilde{\mathrm{E}}^*]$, where $\mathrm{E}_1 = \mathbf{C}_1^{-1}[\mathrm{S}]$ and $\mathrm{E}_2 = \mathrm{E} - \mathrm{E}_1$. Therefore, as in the preceding examples, $\Re[\tilde{\mathrm{S}}.\tilde{\mathrm{E}}^*] = 4\langle \mathcal{E}_P \rangle$ and hence, substituting into the second equation in (3.3), we obtain again the energy balance equation (3.5).

4. Inhomogeneous Plane Waves

By definition, an *inhomogeneous plane wave* has the form

$$\mathrm{u}[\mathrm{x}, t] = \Re[\mathrm{u}_0 \exp[i(\mathrm{k}.\mathrm{x} - \omega t)]] = \exp[-\mathrm{k}_2.\mathrm{x}](\mathrm{u}_{01}\cos[\phi] - \mathrm{u}_{02}\sin[\phi]),$$

where $\mathrm{k}_1 = \Re[\mathrm{k}]$ is the propagation vector, $\phi = \mathrm{k}_1.\mathrm{x} - \omega t$ and $\mathrm{k}_2 = \Im[\mathrm{k}]$ is the attenuation vector. In other words, an inhomogeneous plane wave is a time-harmonic wave whose complex amplitude is given by $\tilde{\mathrm{u}}[\mathrm{x}] = \mathrm{u}_0 \exp[i\mathrm{k}.\mathrm{x}]$.

4.1. Equations of Motion

The kinematic equation implies that the strain is an inhomogeneous plane wave $\mathrm{E}[\mathrm{x}, t] = \Re[\mathrm{E}_0 \exp[i(\mathrm{k}.\mathrm{x} - \omega t)]]$ whose complex amplitude is given by

$$\mathrm{E}_0 = \frac{-1}{2\omega}(\mathrm{v}_0 \otimes \mathrm{k} + \mathrm{k} \otimes \mathrm{v}_0). \tag{4.1}$$

The constitutive equation implies that the stress is an inhomogeneous plane wave $\mathrm{S}[\mathrm{x}, t] = \Re[\mathrm{S}_0 \exp[i(\mathrm{k}.\mathrm{x} - \omega t)]]$ whose complex amplitude is given by $\mathrm{S}_0 = \mathbf{R}_0[\mathrm{E}_0]$, where $\mathbf{R}_0 = \int \mathbf{R}[\tau] \exp[i\omega\tau] d\tau$. The dynamic equation implies that the body force is an inhomogeneous plane wave $\mathrm{b}[\mathrm{x}, t] = \Re[\mathrm{b}_0 \exp[i(\mathrm{k}.\mathrm{x} - \omega t)]]$ whose complex amplitude is given by

$$\mathrm{b}_0 = -i\rho\omega \mathrm{v}_0 - i\mathrm{S}_0[\mathrm{k}]. \tag{4.2}$$

4.2. Energies via Poynting Vector

We first note that, for inhomogeneous plane waves, the Poynting vector and the average kinetic energy are given by

$$\mathbf{P} = -\frac{1}{2}\mathrm{S}_0[\mathrm{v}_0^*]e^{-2\mathrm{k}_2.\mathrm{x}} \quad \text{and} \quad \langle \mathcal{E}_K \rangle = \frac{1}{4}\rho \mathrm{v}_0^*.\mathrm{v}_0 e^{-2\mathrm{k}_2.\mathrm{x}}, \tag{4.3}$$

respectively.

Taking the scalar product of $\mathbf{v}_0^*$ with the dynamic equation (4.2) yields, assuming zero body forces and using the second of equations (4.3), that the average kinetic energy can be obtained from the Poynting vector through

$$\langle \mathcal{E}_K \rangle = \frac{1}{2\omega} \mathbf{k}.\mathbf{P}. \tag{4.4}$$

Likewise, the scalar product of the stress $\mathbf{S}_0$ with the complex conjugate of the kinematic equation (4.1) gives

$$\mathbf{S}_0.\mathbf{E}_0^* = \frac{2}{\omega} \mathbf{k}^*.\mathbf{P} e^{2\mathbf{k}_2.\mathbf{x}}, \tag{4.5}$$

where the symmetry of $\mathbf{S}_0$ and the first equation in (4.3) have been used. Let's assume that, as in the examples of Section 3.2, $\Re[\tilde{\mathbf{S}}.\tilde{\mathbf{E}}^*] = 4\langle \mathcal{E}_P \rangle$ holds; then, by equation (3.4), (4.5) and the last of equations (3.2), we get

$$\mathbf{k}^*.\mathbf{P} = 2\omega \langle \mathcal{E}_P \rangle - i \langle \Pi_D \rangle. \tag{4.6}$$

Moreover, equations (4.4) and (4.6) yield

$$\langle \mathcal{E}_P \rangle + \langle \mathcal{E}_K \rangle = \frac{1}{\omega} \mathbf{k}_1.\Re[\mathbf{P}] \quad \text{and} \quad \langle \Pi_D \rangle = -2\mathbf{k}_1.\Im[\mathbf{P}]. \tag{4.7}$$

These equations show that the knowledge of the Poynting vector permits to compute the total average stored energy as well as the average dissipated power.

Finally, we note that, by equation (4.4), the scalar product between the propagation vector and the Poynting vector is real; therefore $\mathbf{k}_1.\Im[\mathbf{P}] = -\mathbf{k}_2.\Re[\mathbf{P}]$ and this implies, by the second of equations (4.7),

$$\langle \Pi_D \rangle = 2\mathbf{k}_2.\Re[\mathbf{P}],$$

which is an alternative way of expressing the dissipated power in terms of the Poynting vector.

4.3. Fundamental Relations for Phase-, Energy- and Dissipation-Velocity

The phase velocity is defined as

$$\mathbf{V}_{ph} = \frac{\omega}{||\mathbf{k}_1||} \hat{\mathbf{k}}_1 \ , \quad \text{where} \quad \hat{\mathbf{k}}_1 = \frac{1}{||\mathbf{k}_1||} \mathbf{k}_1,$$

while the energy velocity and the dissipation velocity are defined as

$$\mathbf{V}_e = \frac{1}{\langle \mathcal{E}_K \rangle + \langle \mathcal{E}_P \rangle} \langle \mathbf{J} \rangle \quad \text{and} \quad \mathbf{V}_d = \frac{2\omega}{\langle \Pi_d \rangle} \Im[\mathbf{P}],$$

respectively. Using the second of equations (3.2) and equations (4.7), it is readily shown that

$$\hat{\mathbf{k}}_1.\mathbf{V}_e = ||\mathbf{V}_{ph}|| \quad \text{and} \quad \hat{\mathbf{k}}_1.\mathbf{V}_d = -||\mathbf{V}_{ph}||.$$

5. Conclusions

The classical theory of the energy balance for viscoelastic inhomogeneous waves has been reviewed here, and somewhat generalized by dropping any isotropy assumption. A kinetic energy theorem has been obtained for general linear-hereditary materials, whose rheology is described by Boltzmann's superposition principle.

But the concept of dissipated power and its role in the energy balance have been considered under the assumption of viscoelasticity in the strict sense, namely the constitutive relation can be described in terms of (3-D anisotropic) springs and dashpots modelled by generalized Hooke's and Newton's law, respectively. In this way, the a priori physical insight permits to overcome the ambiguities arising with a general hereditary-elastic law [9].

The results in Sections 4.2 and 4.3 have been obtained under the seemingly awkward assumption $\Re[\tilde{\mathrm{S}}.\tilde{\mathrm{E}}^*] = 4\langle \mathcal{E}_P \rangle$. However, it is straightforward to check that this condition is fulfilled by a (3-D anisotropic) generalized standard linear solid, namely by a class of constitutive laws large enough to capture the main rheological features of materials of interest in seismology and mantle composition studies [10], as well as in exploration geophysics [4].

It should be interesting to clarify the theory in the general framework of linear hereditary media: perhaps the variational principles underlying the dynamics might be helpful in the formulation of a physically justified definition of dissipated power.

References

1. A. B. Auld, *Acoustic Fields and Waves in Solids*, Vol. 1, 2nd edition, Robert E. Krieger Publishing Co., Malabar, Florida, 1990.
2. R. D. Borcherdt, *Energy and Plane Waves in Linear Viscoelastic Media*, J. of Geophysical Res. **78** (1973) 2422-2453.
3. P. W. Buchen, *Plane Waves in Linear Viscoelastic Media*, Geophys. J. R. astr. Soc., **23** (1971) 531-542.
4. J. M. Carcione, *Anisotropic Q and Velocity Dispersion of Finely Layered Media*, Geophysical Prospecting, **40** (1992) 761-783.
5. J. M. Carcione and F. Cavallini, *Energy Balance and Fundamental Relations in Viscoelastic Anisotropic Media*, Wave Motion, **18** (1993) 11-20.
6. G. Caviglia and A. Morro, *Inhomogeneous Waves in Solids and Fluids*, World Scientific, Singapore, 1992.
7. H. C. Chen, *Theory of Electromagnetic Waves*, McGraw-Hill, New York, 1983.
8. M. E. Gurtin, *Introduction to Continuum Mechanics*, Academic Press, 1981.
9. E. S. Krebes, *Discrepancies in Energy Calculations for Inhomogeneous Waves*, Geophys. J. R. astr. Soc., **75** (1983) 839-846.
10. H. P. Liu, D. L. Anderson, and H. Kanamori, *Velocity Dispersion due to Anelasticity; Implications for Seismology and Mantle Composition*, Geophys. J. R. astr. Soc. **47** (1976) 41-58.
11. Yu. N. Rabotnov, *Elements of Hereditary Solid Mechanics*, Mir, Moscow, 1980.
12. D. F. Winterstein, *Velocity Anisotropy Terminology for Geophysicists*, Geophysics, **55** (1990) 1070-1088.