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Editors
Salvatore Rionero
University of Napoli
Tommaso Ruggeri
University of Bologna

Quaderno CNR – Gruppo Nazionale per la Fisica Matematica

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ANISOTROPIC-VISCOELASTIC RHEOLOGIES VIA EIGENSTRAINS

J. M. CARCIONE and F. CAVALLINI
Osservatorio Geofisico Sperimentale
P. O. Box 2011, 34016 Opicina, Trieste, Italy

Each eigenvalue (eigenstiffness) and eigenvector (eigenstrain) of the stiffness tensor of an anisotropic solid defines a fundamental deformation state of the medium. The six eigenvalues represent the genuine elastic parameters. From this fact and the correspondence principle we infer that in a real medium there exist no more than six relaxation functions, which are the generalization of the eigenstiffnesses to the viscoelastic case. The existence of six or less complex moduli depends on the symmetry class of the medium. Then, we obtain a new stress-strain relation and probe it with homogeneous viscoelastic plane waves, giving 3-D representations of the quality factor and energy velocity (wavefront) of the different wave modes.

1. Introduction

As Lord Kelvin wrote in his Encyclopaedia Britannica article on Elasticity [15]: *a single system of six mutually orthogonal* (strain) *types may be determined for any homogeneous elastic solid, so that its potential energy when homogeneously strained in any way is expressed by the sum of the products of the squares of the components of the strain, according to those types, respectively multiplied by six determinate coefficients. The six strain-types thus determined are called the Six Principal Strain-types of the body.* Few paragraphs later he refers to the coefficients as the *six Principal Elasticities of the body.* The equations of equilibrium imply that: *If a body be strained to any of its six Principal Types, the stress required to hold it so is directly concurrent with* (proportional to) *the strain.* These concepts were reinterpreted by Pipkin [13], Rychlewski [14], Walpole [16] and recently by Mehrabadi and Cowin [12] by using fourth-rank and second-rank tensor algebra, respectively. The *Six Principal Strains* in which any arbitrary strain can be decomposed are the eigenvectors of the elasticity tensor in 6-dimensional space, or the eigenstrains when working in 3-dimensional space. The *Six Principal Elasticities* are the eigenvalues of the second-rank elasticity tensor referred here as the eigenstiffnesses after Helbig [10], who recently investigated the relation between the eigensystems and the material symmetry, and identified the wave-compatible isochoric (deformation without change of volume) eigenstrains.

The additive decomposition of the total strain energy into a sum of six, or fewer, terms represents energy modes which are not interactive each other. These modes, together with their eigenstiffnesses, determine the complete set of fundamental deformations of a material body including those compatible with wave propagation. The effective stiffness of an arbitrary strain compatible with wave propagation can be expressed as a linear combination of the eigenstiffnesses, expansion that seems to take a simple form along longitudinal (pure mode) directions. This decomposition implies that of the 21 parameters of the elasticity tensor, six are genuine stiffnesses describing the mechanical properties of the medium, and the other fifteen are geometrical parameters required to define the shape and orientation of the eigenstrains in 6-dimensional space.

The conventions in the next sections are that 'tr' takes the trace of a 3×3 matrix; $\Re$ and $\Im$ take real and imaginary parts, respectively; 'diag' denotes a diagonal matrix; and the superscript '$*$' indicates complex conjugation.

2. Hooke's law in tensorial form

By the generalized Hooke's law, it is assumed that stress σ and strain ϵ are linearly related by a symmetric stiffness operator $\mathbf{c}$. In other words, there exists a symmetric linear operator

$$\mathbf{c} : L_s(\mathbf{R}^3) \rightarrow L_s(\mathbf{R}^3) : \epsilon \rightarrow \sigma = \mathbf{c}[\epsilon] \tag{2.1}$$

where $L_s(V)$ is the subspace of symmetric linear maps over V.

The second-rank Cartesian tensor formulation of Hooke's law in six dimensions is introduced by Mehrabadi and Cowin [12]. If the Cartesian basis vectors in three dimensions are denoted by $\mathbf{e}_i$ $(i = 1,2,3)$ and those in six dimensions by $\hat{\mathbf{e}}_I$ $(I = 1,\ldots,6)$, the canonical basis in $L_s(\mathbf{R}^3)$ is given by the following set of tensors:

$$\begin{aligned} \hat{\mathbf{e}}_1 &= \mathbf{e}_1 \otimes \mathbf{e}_1, & \hat{\mathbf{e}}_4 &= \alpha(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2) \\ \hat{\mathbf{e}}_2 &= \mathbf{e}_2 \otimes \mathbf{e}_2, & \hat{\mathbf{e}}_5 &= \alpha(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1) \\ \hat{\mathbf{e}}_3 &= \mathbf{e}_3 \otimes \mathbf{e}_3, & \hat{\mathbf{e}}_6 &= \alpha(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) \end{aligned} \tag{2.2}$$

where $\otimes$ denotes the tensor product and $\alpha = 1/\sqrt{2}$. This is an orthonormal basis, namely: $\hat{\mathbf{e}}_I \cdot \hat{\mathbf{e}}_J = \delta_{IJ}$, where the point denotes scalar product in $L_s(\mathbf{R}^3)$. Explicitly, Hooke's law in the second-rank tensor notation reads

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sqrt{2}\sigma_{yz} \\ \sqrt{2}\sigma_{xz} \\ \sqrt{2}\sigma_{xy} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \sqrt{2}c_{14} & \sqrt{2}c_{15} & \sqrt{2}c_{16} \\ c_{12} & c_{22} & c_{23} & \sqrt{2}c_{24} & \sqrt{2}c_{25} & \sqrt{2}c_{26} \\ c_{13} & c_{23} & c_{33} & \sqrt{2}c_{34} & \sqrt{2}c_{35} & \sqrt{2}c_{36} \\ \sqrt{2}c_{14} & \sqrt{2}c_{24} & \sqrt{2}c_{34} & 2c_{44} & 2c_{45} & 2c_{46} \\ \sqrt{2}c_{15} & \sqrt{2}c_{25} & \sqrt{2}c_{35} & 2c_{45} & 2c_{55} & 2c_{56} \\ \sqrt{2}c_{16} & \sqrt{2}c_{26} & \sqrt{2}c_{36} & 2c_{46} & 2c_{56} & 2c_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \sqrt{2}\epsilon_{yz} \\ \sqrt{2}\epsilon_{xz} \\ \sqrt{2}\epsilon_{xy} \end{bmatrix} \tag{2.3}$$

and $\hat{c}$ coincide, indicating that both the elastic and the viscoelastic rheologies possess the same eigenstrains. Each complex eigenstiffness $\Lambda_I{}^{(\nu)}$ defines a fundamental deformation state of the solid, associated with a set of dissipation mechanisms.

A given wave mode is characterized by its proper complex effective stiffness that can be expressed in terms of the complex eigenstiffnesses. For example, let us consider an isotropic viscoelastic solid. We have seen that the total strain can be decomposed into the dilatational and deviatoric eigenstrains, whose eigenstiffnesses are related to the compressibility and the shear moduli, respectively, the last with multiplicity five. Therefore, there are only two relaxation functions (or two complex eigenstiffnesses) in an isotropic medium, one describing pure dilatational anelastic behaviour, and the other describing pure shear anelastic behaviour. Every eigenstress is directly proportional to its eigenstrain of identical form, the proportionality constant being the complex eigenstiffness. As is well known, the properties of the shear waves are described by the shear relaxation function, and the properties of the compressional wave by a linear combination of the dilatational and shear relaxation functions.

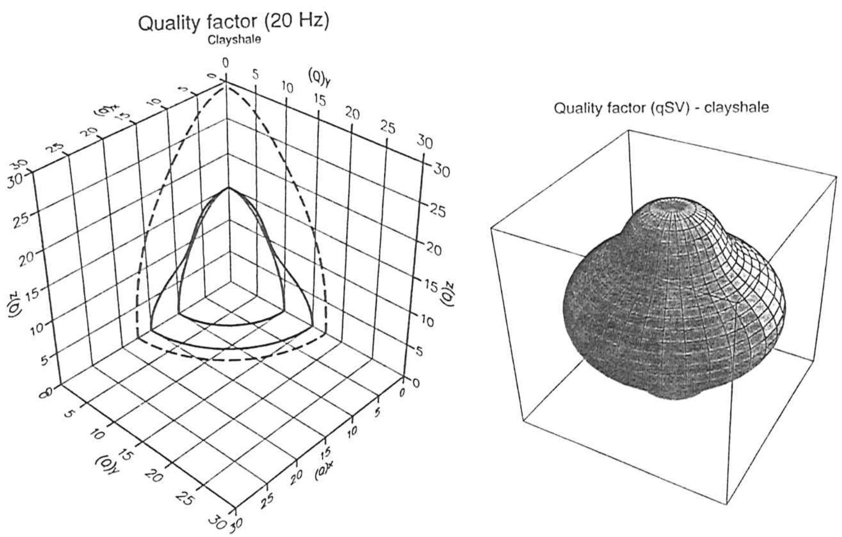

Figure 1. Sections of the quality factor surfaces (left) and quality factor surface of the qSV wave (right).

5. Example

The theory of propagation of viscoelastic waves in isotropic media has been investigated by several researchers, notably Buchen [3], Borcherdt [2], Krebes [11] and Caviglia et al. [8]. Modeling results can be found, for instance, in Carcione et al., [7]. However, research in the framework of anisotropic media is relatively recent. Carcione [4] and Carcione and Cavallini [6] obtained the expressions of the phase, group and energy velocities, and quality factors for homogeneous viscoelastic plane waves in a transversely-isotropic medium. Results for inhomogeneous viscoelastic waves in general anisotropic media were recently published by Carcione and Cavallini [5].

In this work, we assume that the wavefront is the locus of the end of the energy velocity vector multiplied by one unit of propagation time. We consider Mesaverde clayshale, a material of hexagonal symmetry. The medium possesses four distinct eigenstiffnesses [12], and therefore four complex moduli: one quasi-dilatational, one quasi-isochoric and two isochoric of multiplicity two. In the isotropic limit, these eigenstiffnesses relax to a pure dilatational and five isochoric complex moduli, respectively. Each modulus represents a dissipation mechanism modeled by the standard linear solid with minimum quality factors of 30 and 20 for the quasi-dilatational and quasi-isochoric eigenstrains, respectively, and 15 and 10 for the isochoric eigenstrains, at a frequency of 20 Hz.

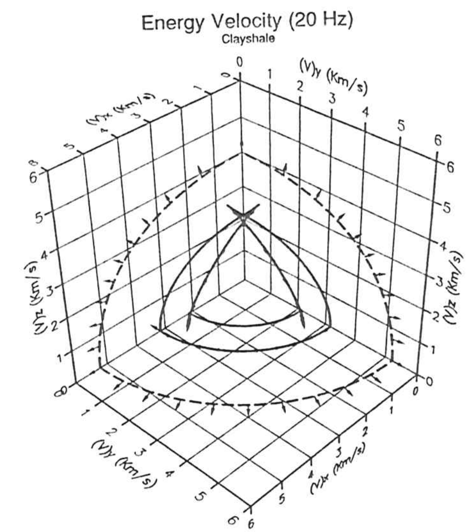

Figure 2. Sections of the energy velocity surfaces.

where c_{IJ} are the elasticities in the Voigt bases [1]. The convention will be to use the symbol '^' over the elasticity matrix, and stress and strain vectors, when these quantities are expressed in the tensorial basis. It is convenient to express (2.3) in compact notation as

$$\hat{\mathrm{T}} = \hat{\mathrm{c}} \bullet \hat{\mathrm{S}}. \tag{2.4}$$

where the dot indicates ordinary matrix multiplication. Actually, the interpretation of stress and strain as vectors is not physically essential but simplifies the mathematical treatment of the problem. Indeed, in this way the elasticity tensor $\hat{\mathrm{c}}$ has order two instead of four and hence may be considered as a matrix: its eigenvalues and eigenvectors are then well defined.

3. Eigenstiffnesses and eigenstrains in elastic media

The ***six orthogonal strain types*** and the ***six principal elasticities*** referred by Lord Kelvin can be found by seeking those strain states σ for which ϵ and σ are parallel in 6-dimensional Cartesian space, i.e.

$$\sigma = \mathrm{c}[\epsilon] = \Lambda\epsilon, \tag{3.1}$$

where Λ is a scalar quantity. This is mathematically equivalent to diagonalizing the stiffness matrix $\hat{\mathrm{c}}$:

$$(\hat{\mathrm{c}} - \Lambda 1) \bullet \hat{\mathrm{S}} = 0, \tag{3.2}$$

where 1 is the 6×6 identity matrix. Hence, the eigenstiffnesses and eigenstrains are the eigenvalues and eigenvectors of $\hat{\mathrm{c}}$, respectively. Matrix $\hat{\mathrm{c}}$ can be expressed as

$$\hat{\mathrm{c}} = \mathrm{A}^T \bullet \Lambda \bullet \mathrm{A}, \tag{3.3}$$

where A is the matrix formed with the eigenstrains; more precisely, with the columns of the right (orthonormal) eigenvectors of $\hat{\mathrm{c}}$ (note that the symmetry of $\hat{\mathrm{c}}$ implies that $\mathrm{A}^{-1} = \mathrm{A}^T$). The eigenvalues of the elasticity matrix are invariant with respect to any change of basis (or coordinate system): this confers the eigenstiffnesses an intrinsic character. To illustrate the usefulness of the decomposition (3.3), we consider briefly the isotropic case. An isotropic medium is characterized by a stiffness operator $\mathbf{c}$ defined by

$$\mathrm{c}[\epsilon] = 2\mu\epsilon + \lambda(tr\ \epsilon)\mathrm{I}, \quad \text{i.e.,} \quad \mathbf{c} = 2\mu 1 + \lambda \mathrm{I} \otimes \mathrm{I}, \tag{3.4}$$

where λ and μ are the Lame constants, and I is the identity map in $\mathbb{R}^3$. The characteristic equation for the stiffness operator is then

$$2\mu\epsilon + \lambda(tr\ \epsilon)\mathrm{I} = \Lambda\epsilon. \tag{3.5}$$

Taking the trace of this equation, we see that a strain with nonzero trace is an eigenstrain if and only if it is proportional to I and the corresponding eigenvalue

is then $\Lambda_1 = 2\mu + 3\lambda$, with multiplicity 1. Moreover, all nonzero strains with zero trace are eigenstrains corresponding to the eigenstiffness $\Lambda = 2\mu$, with multiplicity 5. No other eigenstiffnesses or eigenstrains are possible. It is clear that eigenstrains and eigenstresses are related by

$$tr\ \sigma = \Lambda_1(tr\ \epsilon), \quad and \quad \tilde{\sigma} = \Lambda_I\tilde{\epsilon}, \ I = 2, ..., 6, \tag{3.6}$$

where the tilde denotes the deviatoric tensors. Then, in unbounded and homogeneous isotropic media, the total stress can be decomposed in pure dilatational and shear stresses, and these produce pure deformations that are not interactive each other.

The eigenstiffness and eigenstrains of materials of lower symmetry are given by Mehrabadi and Cowin [12]. The eigentensors are 3×3 symmetric matrices in 3-D space. Their eigenvalues are invariant under rotations and describe the magnitude of the deformation. On the other hand, the eigenvectors describe the orientation of the eigentensor in a given coordinate system. For instance, pure volume dilatations correspond to three equal eigenvalues, and the sum of the eigenvalues of an isochoric eigenstrain is zero. Isochoric strains with two equal eigenvalues but opposite sign and a third eigenvalue zero, are plane shear tensors. To summarize, the eigentensors identify preferred modes of deformation associated with the particular symmetry of the material. An illustrative pictorial representation of these modes or eigenstrains was designed by Helbig [10].

4. The viscoelastic constitutive law

The above discussion of the elastic case lead us to infer that in a viscoelastic material there exist no more than **six relaxation functions** describing the deformation and anelastic properties of the medium. These six, or less, relaxation functions (**complex moduli** in the frequency-domain) are the generalization of the eigenstiffnesses, by using the correspondence principle, to appropriate complex moduli satisfying the Kramers-Krōnig dispersion relations (causality principle). The existence of six or less complex eigenstiffnesses depends on the symmetry class of the medium.

Hence, in virtue of the correspondence principle and its application to (3.3), we introduce the viscoelastic stiffness tensor

$$\hat{\mathrm{p}}(\omega) = \mathrm{A}^T \bullet \Lambda^{(v)}(\omega) \bullet \mathrm{A}, \tag{4.1}$$

where ω is the angular frequency, and $\Lambda^{(v)}$ is a diagonal matrix with entries

$$\Lambda_I^{(v)} = \Lambda_I M_I(\omega), \quad I = 1, ..., 6. \tag{4.2}$$

The quantities M_I are complex and frequency-dependent dimensionless moduli. It can be easily shown that the viscoelastic stiffness tensor is symmetric, in agreement with the result obtained by Gurtin and Hrusa [9]. Moreover, the eigenvectors of $\hat{\mathrm{p}}$

These values define the quality factor along the principal axes as can be observed in Figure 1, where sections across three mutually orthogonal planes are represented. The broken lines correspond to the quasi-compressional wave and the quality factor of the SH mode is the outer continuous curve. Only one octant of the space is displayed, from symmetry considerations. The right picture illustrates the 3-D surface corresponding to the qSV quality factor.

Finally, Figure 2 represents sections of the energy velocity curves. As before, the broken line is the qP wave. The polarization of the different modes are plotted in the energy velocity curves; when not plotted, polarizations are normal to the planes.

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ENERGY BALANCE AND INHOMOGENEOUS PLANE-WAVE ANALYSIS OF A CLASS OF ANISOTROPIC VISCOELASTIC CONSTITUTIVE LAWS *

F. CAVALLINI and J. M. CARCIONE

Osservatorio Geofisico Sperimentale

P. O. Box 2011, I-34016 Trieste, Italy

We prove a theorem of power and energy for the solutions of the linear equations of a viscoelastic material, whose rheology may be described in terms of lumped elements having the behaviour of either an elastic solid or a viscous fluid. The assumed anisotropy ensures that this class of constitutive laws is wide enough for describing most of geophysical media; yet, its a priori physical interpretation permits to avoid the mathematical ambiguities arising, in the definition of potential energy, with constitutive laws of abstract hereditary type. Moreover, sharper results for time-averaged energies are obtained by assuming a time-harmonic displacement. Finally, fundamental relations for phase-, energy- and dissipation-velocity are derived in the framework of plane inhomogeneous waves. As case studies, the Kelvin-Voigt, Maxwell and standard linear solid rheologies are worked out in detail. The use of coordinate-free notation permits to perform computations in a clean and rigorous way.

1. Introduction

The theory of mechanical waves in solid dissipative media is a classical topic: for background information on the physical and mathematical aspects, we refer to the books by Auld [1] and Caviglia and Morro [6], respectively. Fundamental papers on the energy balance for these waves are those by Buchen [3] and Borcherdt [2]. However, most of the results that can be found in the literature were proven assuming isotropy, which is a too restrictive assumption for geophysical purposes [12]. Hence Carcione and Cavallini [5] reviewed the subject in a fully anisotropic framework, using a component notation. The ideas in [5] are developed here with applications to specific case-studies, using a component-free notation [8]; indeed, the latter is more

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