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Attenuation and quality factor surfaces in anisotropic-viscoelastic media

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Abstract

We obtain expressions of the attenuation vector and quality factor of the three possible wave modes propagating in a linear anisotropic medium. The theory assumes, in principle, a general stiffness matrix. Probing the medium with a time-harmonic homogeneous plane wave gives the attenuations and quality factors as simple forms of the propagation direction, complex stiffnesses and mass density. As an application, we introduce a new constitutive relation, based on four complex moduli, for which the values of the quality factor along three preferred directions can be matched with experimentally pre-determined values. The rheology is causal and allows an arbitrary frequency-dependence of the stiffnesses based on the generalized standard linear solid model. Two examples are explicitly worked out. The first is clay shale, a material of hexagonal symmetry. Since, by Neumann's principle, the attenuation symmetries are determined by the crystal class, the medium presents isotropic attenuation in a plane normal to the symmetry axis. For instance, in materials with $c_{11} > c_{33}$, it is found that the quasi-compressional wave attenuates more along the symmetry axis direction than in the plane of isotropy. The second medium is tellurium dioxide, a strongly anisotropic material of tetragonal symmetry. In this case, the diagrams show that strong attenuation is associated with high slowness values, as at around 45° in the horizontal plane. Both case studies show that the features of the attenuation surfaces strongly depend on the values of the elasticities.

1. Introduction

Wave attenuation is caused by a variety of dissipation mechanisms whose effects depend on the frequency, as well as on the type of elastic symmetry of the medium. In macroscopically isotropic media, the attenuation properties do not depend on the direction of propagation. In anisotropic media, any kind of symmetry possessed by the attenuation follows the symmetry of the crystallographic form of the material. These statements are derived from an empirical law known as Neumann's principle (Neumann, 1885; Love, 1927). This implies that the symmetries of any property must be higher or equal to the symmetries of the solid themselves.

The constitutive relation is Boltzmann's after-effect principle combined with Hooke's law in its general form, where the attenuation properties are modeled by a viscoelastic constitutive law, and anisotropy includes all the symmetry classes. As in the elastic case, there are at most 21 independent components defining the relaxation matrix relating the stress and strain vectors. Moreover, the degree of symmetry of this matrix depends upon the crystal class in the same way as

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the elasticity matrix. This automatically implies that the attenuation properties have the same symmetries as the elasticities.

Dissipation is quantified by the quality factor and the attenuation factor which can be measured experimentally by various techniques. In this work, the probing wave is a time-harmonic homogeneous viscoelastic plane wave, and the attenuation constants obtained in this way should be compared with results from wave propagation experiments. The theory of propagation of viscoelastic waves in isotropic media has been investigated by several researchers, notably Buchen (1971), Borcherdt (1977), Krebes (1984) and Caviglia and Morro (1992). However, research in the framework of anisotropic media is relatively recent. Carcione (1990) obtained the expressions of the phase, group and energy velocities, and quality factors for homogeneous viscoelastic plane waves in a transversely-isotropic medium.

For homogeneous waves, the wave number is assumed to be complex and the attenuation and propagation vectors are collinear. In general, there are three possible modes for plane wave propagation along a specified direction: they are solutions of the complex Christoffel equation whose roots can be expressed in terms of the complex velocities. The attenuation factor is simply the imaginary part of the reciprocal of the complex velocity. Although the quality factor involves the calculation of the potential and loss energy densities, it results in a simple function of the complex velocity.

Most applications use the Kelvin–Voigt stress–strain relation as constitutive law, but this rheology models a particular type of frequency-dependent relaxation matrix (Auld, 1991; Lamb and Richter, 1966). A more general model was introduced by Carcione (1990), which is based on standard linear solid relaxation functions. In its general form, this model contains several standard linear elements, thus describing any arbitrary frequency dependence of the attenuation and quality factors.

The paper is organized as follows. Section 2 introduces the general time-domain constitutive relation and the corresponding equation for homogeneous plane waves. In a similar way, the equation of motion gives the complex Christoffel equation whose roots define the complex velocities. This is done in Section 3 together with the calculation of the attenuation vector. Section 4 computes the quality factor from the potential and loss energy densities. In Section 5 the rheological model is introduced and, finally, the examples of Section 6 illustrate the dissipation properties of hexagonal and tetragonal media by computing the three-dimensional attenuation and quality factor surfaces.

2. Constitutive relation

The most general relation (Christensen, 1982) between the components of the stress tensor σ_{ij} and the components of the strain tensor ϵ_{ij} for an anisotropic linear viscoelastic medium is given by

$$\sigma_{ij}(\boldsymbol{x}, t) = \psi_{ijkl}(t) * \dot{\epsilon}_{kl}(\boldsymbol{x}, t), \quad i, j, k, l = 1, 2, 3, \tag{2.1}$$

where t is the time variable, $\boldsymbol{x}$ is the position vector, ψ_{ijkl} are the components of a fourth-order tensorial relaxation function, and the asterisk ($*$) indicates time convolution. A dot above a variable denotes differentiation with respect to time, and the Einstein convention for repeated indices is used.

Eq. (2.1) is the formulation of the isothermal anisotropic-viscoelastic stress–strain constitutive relation, also called the Boltzmann superposition principle. The fourth-rank tensor contains all the information about the behavior of the medium under infinitesimal deformations. In the most general case, the number of components is 81, but since the stress and strain tensors are symmetrical, and from the real nature of the strain and loss in energy densities (Auld, 1991), it follows that the number of independent real components reduces to 21.

By using properties of the convolution, Eq. (2.1) gives, in condensed subscript notation (Auld, 1991),

$$T_I(\boldsymbol{x}, t) = \dot{\psi}_{IJ}(t)^* S_J(\boldsymbol{x}, t), \quad I, J = 1, \ldots, 6, \tag{2.2}$$

where

$$T = [T_1, T_2, T_3, T_4, T_5, T_6]^T = [\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{xz}, \sigma_{xy}]^T \tag{2.3}$$

and

$$S = [S_1, S_2, S_3, S_4, S_5, S_6]^T = [\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}]^T \tag{2.4}$$

are the stress and strain vectors, respectively. A general solution for the displacement field representing viscoelastic plane waves is of the form

$$u = U_0 \, e^{i(\omega t - k \cdot x)}, \tag{2.5}$$

where U_0 represents a constant complex vector and ω is the angular frequency. The wavenumber vector is, in general, complex and can be written as

$$k = \kappa - i\alpha, \tag{2.6}$$

where the real vectors κ and α indicate the directions and magnitudes of propagation and attenuation, respectively. In general, these directions are different and the plane wave is termed inhomogeneous, with $\kappa^T \cdot \alpha$ strictly different from zero, unlike the interface waves in elastic media (the dot denotes ordinary matrix multiplication); otherwise, the wave is called homogeneous and in this case

$$k = (\kappa - i\alpha)\hat{\kappa} \equiv k\hat{\kappa}, \tag{2.7}$$

where

$$\hat{\kappa} = l_x \hat{e}_x + l_y \hat{e}_y + l_z \hat{e}_z \tag{2.8}$$

defines the propagation direction through the direction cosines l_x, l_y and l_z. For this kind of wave, planes of constant phase (planes normal to the propagation vector κ) are parallel to planes of constant amplitude (defined by $\alpha^T \cdot x = \text{const}$). Substituting the plane wave (2.5) into the stress–strain relation (2.2) yields

$$T(\omega) = p(\omega) \cdot S, \tag{2.9}$$

where $T(\omega)$ and S are related to stress and strain by formulas analogous to (2.5), while the components of the complex stiffness matrix are

$$p_{IJ}(\omega) = \int_{-\infty}^{\infty} \dot{\psi}_{IJ}(t) \, e^{-i\omega t} \, dt. \tag{2.10}$$

3. Complex velocity and attenuation factor

The equation of momentum for a three-dimensional anisotropic linear anelastic medium in absence of body forces is (Auld, 1991)

$$\nabla \cdot T - \rho \ddot{u} = 0, \tag{3.1}$$

where $u(x, t)$ is the displacement vector, $\rho(x)$ is the density, and ∇ is a differential operator defined by

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}. \tag{3.2}$$

For the plane wave (2.5), Eq. (3.1) reads

$$(\Gamma - \rho V^2 I) \cdot u = 0, \tag{3.3}$$

where I is the 3×3 identity matrix,

$$L = \begin{bmatrix} l_x & 0 & 0 & 0 & l_z & l_y \\ 0 & l_y & 0 & l_z & 0 & l_x \\ 0 & 0 & l_z & l_y & l_x & 0 \end{bmatrix} \tag{3.4}$$

is the direction cosine matrix, and

$$\Gamma = L \cdot p \cdot L^T \tag{3.5}$$

is the Christoffel matrix. The complex velocity

$$V = \frac{\omega}{k} \tag{3.6}$$

is a fundamental quantity since it determines uniquely both the attenuation vector (3.8) and the quality factor (4.8). The complex velocities of the three wave modes are obtained from the dispersion relation

$$\det[L \cdot p \cdot L^T - \rho V^2 I] = 0, \tag{3.7}$$

which is the characteristic equation of (3.3). Using (3.6), the attenuation vector can be expressed in terms of the complex velocity as

$$\alpha = -\omega \, \mathrm{Im}\left[\frac{1}{V}\right]\hat{\kappa} \tag{3.8}$$

and its magnitude is the attenuation factor.

4. Quality factor

Roughly speaking, the quality factor expresses the amount of the dissipated energy relative to the stored elastic energy. To make this concept precise, several definitions have been proposed (O'Connell and Budiansky, 1978). Owing to its widespread use in the geophysical community (see, e.g., Borcherdt, 1973), we choose the definition of the quality factor as the ratio of the peak strain energy density to the average loss energy density. The peak strain energy for homogeneous plane waves is twice the average value, and is given by (Carcione and Cavallini, 1993)

$$(\epsilon_s)_{peak} = \tfrac{1}{2}\,\mathrm{Re}\left[\boldsymbol{S}^{\mathrm{T}} \cdot \boldsymbol{p} \cdot \boldsymbol{S}^*\right], \tag{4.1}$$

where Re denotes the real part. The average loss energy density is

$$\langle \epsilon_d \rangle = \tfrac{1}{2}\,\mathrm{Im}\left[\boldsymbol{S}^{\mathrm{T}} \cdot \boldsymbol{p} \cdot \boldsymbol{S}^*\right]. \tag{4.2}$$

From the definition, the quality factor is then

$$Q = \frac{(\epsilon_s)_{\mathrm{peak}}}{\langle \epsilon_{\mathrm{d}} \rangle} = \frac{\mathrm{Re}\left[\boldsymbol{S}^{\mathrm{T}} \cdot \boldsymbol{p} \cdot \boldsymbol{S}^*\right]}{\mathrm{Im}\left[\boldsymbol{S}^{\mathrm{T}} \cdot \boldsymbol{p} \cdot \boldsymbol{S}^*\right]}. \tag{4.3}$$

This equation requires the calculation of $\boldsymbol{S}^{\mathrm{T}} \cdot \boldsymbol{p} \cdot \boldsymbol{S}^*$. Since $\boldsymbol{S} = \boldsymbol{\nabla}^{\mathrm{T}} \cdot \boldsymbol{u}$ (Auld, 1991),

$$\boldsymbol{S} = -\mathrm{i}k\boldsymbol{L}^{\mathrm{T}} \cdot \boldsymbol{u}, \tag{4.4a}$$

and

$$\boldsymbol{S}^* = \mathrm{i}k^*\boldsymbol{L}^{\mathrm{T}} \cdot \boldsymbol{u}^* \tag{4.4b}$$

for the plane wave (2.5). Replacing these equations into $\boldsymbol{S}^{\mathrm{T}} \cdot \boldsymbol{p} \cdot \boldsymbol{S}^*$ yields

$$\boldsymbol{S}^{\mathrm{T}} \cdot \boldsymbol{p} \cdot \boldsymbol{S}^* = |k|^2 \boldsymbol{u}^{\mathrm{T}} \cdot \boldsymbol{\Gamma} \cdot \boldsymbol{u}^*, \tag{4.5}$$

where $\boldsymbol{\Gamma}$ is the Christoffel matrix (3.5). But from the transpose of (2.9),

$$\boldsymbol{u}^{\mathrm{T}} \cdot \boldsymbol{\Gamma} = \rho V^2 \boldsymbol{u}^{\mathrm{T}}. \tag{4.6}$$

Therefore, substituting this expression into (4.5) gives

$$\boldsymbol{S}^{\mathrm{T}} \cdot \boldsymbol{p} \cdot \boldsymbol{S}^* = \rho |k|^2 \boldsymbol{u}^{\mathrm{T}} \cdot \boldsymbol{u}^* V^2. \tag{4.7}$$

In consequence, since the matrix product in the right-hand side of (4.7) is real, the quality factor in anisotropic-viscoelastic media takes the following simple form as a function of the complex velocity:

$$Q = \frac{\mathrm{Re}[V^2]}{\mathrm{Im}[V^2]}. \tag{4.8}$$

From the definition of complex velocity (3.6), it can be easily shown that the quality factor (4.8) can be written as

$$Q = -\frac{\mathrm{Re}[k^2]}{\mathrm{Im}[k^2]}. \tag{4.9}$$

For isotropic media, this expression is well known (see, e.g., Ben-Menahem and Singh, 1981); here it has been proved that in general anisotropic media the quality factor for homogeneous viscoelastic waves keeps the same form.

Since for a homogeneous wave $k^2 = \kappa^2 - \alpha^2 - 2\mathrm{i}\kappa\alpha$, it follows from (3.8) and (4.9) that the quality factor relates to the wavenumber and attenuation vectors as

$$\boldsymbol{\alpha} = \left(\sqrt{Q^2+1} - Q\right)\boldsymbol{\kappa}. \tag{4.10}$$

For low-loss solids, it is $Q \gg 1$, and a Taylor expansion yields

$$\boldsymbol{\alpha} = \frac{1}{2Q}\boldsymbol{\kappa}, \tag{4.11}$$

which is the well known expression for the attenuation vector.

5. The rheological model

A class of constitutive equations for anisotropic-viscoelastic media based on two complex moduli was investigated by Carcione (1990). In the present work, two additional complex moduli are introduced to model in more detail the anelastic properties of the shear modes. The stiffness matrix of this new rheology is

$$\boldsymbol{p} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & c_{14} & c_{15} & c_{16} \\ & p_{22} & p_{23} & c_{24} & c_{25} & c_{26} \\ & & p_{33} & c_{34} & c_{35} & c_{36} \\ & & & c_{44}M_2 & c_{45} & c_{46} \\ & & & & c_{55}M_3 & c_{56} \\ & & & & & c_{66}M_4 \end{bmatrix}, \tag{5.1}$$

where

$$p_{I(I)} = c_{I(I)} - D + KM_1 + \tfrac{4}{3}GM_\delta, \quad \text{for } I = 1, 2, 3, \tag{5.2a}$$

$$p_{IJ} = c_{IJ} - D + 2G + KM_1 - \tfrac{2}{3}GM_\delta \quad \text{for } I, J = 1, 2, 3;\ I \neq J. \tag{5.2b}$$

Here c_{IJ}, for $I, J = 1, \ldots, 6$ are the low-frequency limit elastic constants, and

$$K = D - \tfrac{4}{3}G, \tag{5.3}$$

where

$$D = \tfrac{1}{3}(c_{11} + c_{22} + c_{33}), \quad G = \tfrac{1}{3}(c_{44} + c_{55} + c_{66}). \tag{5.4}$$

Coefficients $M\nu$ are dimensionless complex moduli: $\nu = 1$ is for the quasi-dilational mode, and $\nu = 2, 3, 4$ are for the shear waves. In (5.2a, b), M_δ is a shear modulus that can be chosen such that $\delta = 2$, 3 or 4. The mean stress $\Theta_\sigma = \Sigma_{I=1}^{3} \frac{1}{3}T_{II}$ can be expressed in terms of the mean strain $\Theta_\epsilon = \Sigma_{I=1}^{3} \frac{1}{3}S_{II}$ as

$$\Theta_\sigma = \tfrac{1}{3}(c_{J1} + c_{J2} + c_{J3})S_J + K\Theta_\epsilon(M_1 - 1), \tag{5.5}$$

which only depends on the first complex modulus involving quasi-dilatational dissipation mechanisms. Moreover, the deviatoric stresses are

$$T_I - \Theta_\sigma = \sum_{K=1}^{3} (\delta_{IK} - \tfrac{1}{3})c_{KJ}S_J + 2G(S_I - \Theta_\epsilon)(M_\delta - 1), \quad \text{for } I \leq 3, \tag{5.6a}$$

and

$$T_I = \sum_{J=1}^{3} c_{IJ}S_J + \sum_{J=4}^{6} c_{IJ}S_JM_{J-2}, \quad \text{for } I > 3, \tag{5.6b}$$

which depend on the complex moduli associated to the quasi-shear mechanisms. Eq. (5.1) gives the elasticity matrix of the generalized Hooke's law in the anisotropic-elastic limit when $M_\nu \to 1$, and gives the 3D isotropic-viscoelastic constitutive relation in the isotropic limit (Carcione et al., 1988). The stiffness matrix (5.1) generalizes that given in Carcione (1990), since here three relaxation functions instead of one are used to describe the anelastic properties of the shear modes. In this way, it is possible to control the quality factor along three preferred directions, like the principal axes of the anisotropic medium, for instance. The choice of the complex moduli depends on the symmetry system, since the attenuation symmetries follow the symmetry of the crystallographic form of the material. For isotropic, cubic and hexagonal media, two relaxation functions are necessary and sufficient to model the anelastic properties. For instance, for transversely isotropic media with symmetry axis in the z-direction, $M_\delta = M_2 = M_3$, and $p_{66} = \frac{1}{2}(p_{11} - p_{12})$ should be taken in order to preserve transverse isotropy.

For L_ν dissipation mechanisms, the theory assumes the following form for the complex moduli:

$$M_\nu(\omega) = \frac{1}{L_\nu} \sum_{l=1}^{L_\nu} \frac{1 + i\omega\tau_{\epsilon l}^{(\nu)}}{1 + i\omega\tau_{\sigma l}^{(\nu)}}, \quad \nu = 1, \ldots, 4, \tag{5.7}$$

where $\tau_{\epsilon l}^{(\nu)}$ and $\tau_{\sigma l}^{(\nu)}$ are relaxation times such that $\tau_{\epsilon l}^{(\nu)} \geq \tau_{\sigma l}^{(\nu)}$. Eq. (5.7) represents the complex modulus of a generalized a standard linear solid. Note that $\tau_{\epsilon l}^{(\nu)} \to \tau_{\sigma l}^{(\nu)}$ gives the low-frequency elastic limit where $M_\nu \to 1$ as mentioned before. When $L_\nu = 1$, $\forall \nu$, the quality factor corresponding to each modulus is given by

$$Q_\nu(\omega) = \frac{\text{Re}[M_\nu]}{\text{Im}[M_\nu]} = \frac{1 + \omega^2\tau_\epsilon^{(\nu)}\tau_\sigma^{(\nu)}}{\omega\left(\tau_\epsilon^{(\nu)} - \tau_\sigma^{(\nu)}\right)}, \tag{5.8}$$

whose minimum is located at $\omega_{0\nu} = 1/\sqrt{\tau_\epsilon^{(\nu)}\tau_\sigma^{(\nu)}}$ (Ben-Menahem and Singh, 1981).

Figure 1 illustrates a polar diagram of quality factor curves in a symmetry plane of an orthorhombic medium. Only one quadrant of the plane is displayed due to symmetry considerations. As can be seen, the values of the quality factors along the directions of the cartesian axes are determined by the one-dimensional quality factors Q_ν for the shear waves, and by simple functions of the stiffnesses for the qP wave.

The complex velocities are the key to obtain the attenuation properties. These velocities are now given for the symmetry planes of orthorhombic media, which include the tetragonal and

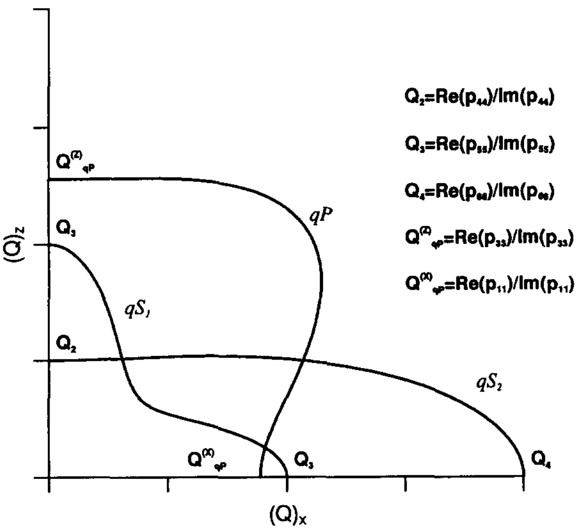

Fig. 1. Polar diagram of quality factor curves in a symmetry plane of an orthorhombic medium. Only one-quarter of the plane is displayed due to symmetry considerations. The values of the quality factors along the directions of the cartesian axes are determined by the one-dimensional quality factors Q_ν for the shear waves, and simple functions of the stiffnesses for the qP wave.

hexagonal systems analyzed in the examples. In the natural coordinate system, orthorhombic media have $p_{1J} = p_{2J} = p_{3J} = 0$, for $J \geq 4$, and $p_{45} = p_{46} = p_{56} = 0$, and nine independent stiffness constants. For these media, the eigenvalues of the Christoffel matrix (3.5) in the symmetry planes are as follows.

In the XZ-plane:

$$\rho V_{1(2)}^2 = \tfrac{1}{2}\left(p_{55} + p_{11} l_x^2 + p_{33} l_z^2 \pm E\right),$$

$$\rho V_3^2 = p_{66} l_x^2 + p_{44} l_z^2, \qquad (5.9a)$$

where

$$E = \sqrt{\left[(p_{33} - p_{55}) l_z^2 - (p_{11} - p_{55}) l_x^2\right]^2 + 4(p_{13} + p_{55})^2 l_x^2 l_z^2}. \qquad (5.9b)$$

In the YZ-plane:

$$\rho V_{1(2)}^2 = \tfrac{1}{2}\left(p_{44} + p_{22} l_y^2 + p_{33} l_z^2 \pm E\right),$$

$$\rho V_3^2 = p_{66} l_y^2 + p_{55} l_z^2, \qquad (5.10a)$$

where

$$E = \sqrt{\left[(p_{33} - p_{44}) l_z^2 - (p_{22} - p_{44}) l_y^2\right]^2 + 4(p_{23} + p_{44})^2 l_y^2 l_z^2}. \qquad (5.10b)$$

In the XY-plane:

$$\rho V_{1(2)}^2 = \tfrac{1}{2}\left(p_{66} + p_{11} l_x^2 + p_{22} l_y^2 \pm E\right),$$

$$\rho V_3^2 = p_{55} l_x^2 + p_{44} l_y^2, \qquad (5.11a)$$

where

$$E = \sqrt{\left[(p_{22} - p_{66}) l_y^2 - (p_{11} - p_{66}) l_x^2\right]^2 + 4(p_{12} + p_{66})^2 l_x^2 l_y^2}. \qquad (5.11b)$$

In relatively weak anisotropic media, V_1 is the velocity of the qP wave (+ sign), while V_2 (− sign) and V_3 correspond to the shear waves, with the second one a pure mode. In complex materials,

Table 1
Material properties

Medium	Elasticities (GPa)									density (kg/m^3)
	c_{11}	c_{12}	c_{13}	c_{22}	c_{23}	c_{33}	c_{44}	c_{55}	c_{66}	ρ
Clay shale	66.6	19.7	39.4	66.6	39.4	39.9	10.9	10.9	23.4	2590.0
TeO_2	55.7	51.2	21.8	55.7	21.8	105.0	26.5	26.5	65.9	5990.0

	Relaxation times (s)							
	$\tau_\epsilon^{(1)}$	$\tau_\sigma^{(1)}$	$\tau_\epsilon^{(2)}$	$\tau_\sigma^{(2)}$	$\tau_\epsilon^{(3)}$	$\tau_\sigma^{(3)}$	$\tau_\epsilon^{(4)}$	$\tau_\sigma^{(4)}$
Clay shale	8.00×10^{-3}	7.49×10^{-3}	8.00×10^{-3}	7.25×10^{-3}	8.00×10^{-3}	7.25×10^{-3}	8.00×10^{-3}	7.25×10^{-3}
Q_v	30		20		20		20	
TeO_2	6.40×10^{-7}	6.33×10^{-7}	6.40×10^{-7}	6.31×10^{-7}	6.40×10^{-7}	6.31×10^{-7}	6.40×10^{-7}	6.27×10^{-7}
Q_v	200		150		150		100	

like for instance tellurium dioxide (tetragonal symmetry), this identification does not apply since along the same wavefront the wave may change from quasi-compressional to quasi-shear or viceversa. Note that in the simplest case of hexagonal symmetry, the qSV wave corresponds to V_2 in the XZ and YZ planes, and it is a pure mode defined by V_3 in the XY-plane.

6. Examples

The attenuation and quality factor surfaces are computed here for two media of hexagonal and tetragonal symmetries (clay shale and tellurium dioxide, respectively), whose non-zero elasticities, densities and relaxation times are given in Table 1. The stiffness matrix of orthorhombic media acquires tetragonal symmetry if $p_{11} = p_{22}$, $p_{23} = p_{13}$, and $p_{44} = p_{55}$, and hexagonal if, in addition, $p_{66} = \frac{1}{2}(p_{11} - p_{12})$. As mentioned before, hexagonal media require only two complex moduli, and for tetragonal media the condition $p_{44} = p_{55}$ implies a maximum of three complex moduli since $M_2 = M_3$ but $p_{66} = c_{66} M_4$ is an independent stiffness. Each complex modulus has one dissipation

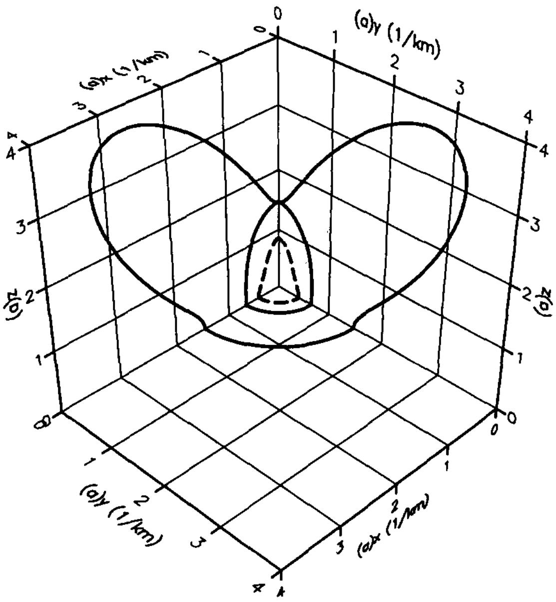

Fig. 3. Sections of the attenuation factor surfaces at 20 Hz for clay shale across three mutually perpendicular planes where the symmetry axis coincides with the vertical axis. The dotted line corresponds to the quasi-compressional wave. Only one octant of the sections is displayed from symmetry considerations.

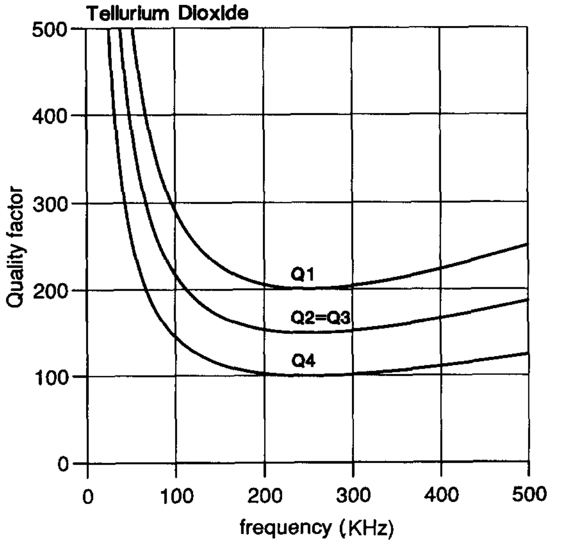

Fig. 2. One-dimensional quality factors Q_ν as a function of frequency for tellurium dioxide. The minimum values are given in Table 1.

mechanism, with relaxation minimum at 20 Hz in clay shale and 250 kHz in tellurium dioxide. The minimum value of the one-dimensional quality factor for each single mechanism is indicated in Table 1. Fig. 2 illustrates these quality factors as a function of frequency for tellurium dioxide. For clayshale, the symmetry constraints imply $\delta = 2$, while for tellurium dioxide $\delta = 2$ is our choice.

6.1. Clay shale

Fig. 3 shows sections of the clay shale attenuation factor surfaces at 20 Hz across three mutually perpendicular planes where the symmetry axis coincides with the vertical axis. The inner curve (dotted line) corresponds to the quasi-compressional wave; then follow the quasi-shear waves, which have a kiss singularity at the symmetry axis. Only one octant of the sections is displayed from symmetry considerations. The corresponding three-dimensional surfaces are illustrated in Fig. 4, where (a) is the *qP* mode, (b) is the *qSV*, and (c) is the *SH* mode. Note that for clarity, the three surfaces have been plotted with different scales. The quality factor sections and surfaces for clay shale are displayed in Figs. 5 and 6, respectively. The inner curve in Fig. 5 corresponds to the *qSV* wave. As can be appreciated in the figures, the symmetry of the attenuation properties follows the symmetry of the crystal class. As mentioned before, dilatational energy dissipates according to the complex modulus M_1 with quality factor $Q_1 = 30$, as indicated in Table 1. However, the anelastic characteristics of the quasi-compressional wave depend also on M_2. In fact, at the symmetry axis, the *qP* quality factor becomes, by (4.8) and (5.9a), $Q(l_z = 1) = \mathrm{Re}[p_{33}]/\mathrm{Im}[p_{33}]$, while in the horizontal plane, $Q(l_x = 1) = \mathrm{Re}[p_{11}]/\mathrm{Im}[p_{11}]$. The relation between these two values by (5.2a) is

$$\frac{Q(l_z = 1)}{Q(l_x = 1)} = \frac{c_{33} - D + \mathrm{Re}[A]}{c_{11} - D + \mathrm{Re}[A]},$$

$$\text{where } A = KM_1 + \tfrac{4}{3}GM_2. \tag{6.1}$$

If we define $a = \mathrm{Re}[A] - D$, it can be verified that $a > 0$ and $a < c_{11}$, $a < c_{33}$, for realistic media ($a = 0$ in the elastic case). This implies that, whatever the ratio c_{33}/c_{11}, the ratio between quality factors given by (5.12) departs from unity more than the ratio c_{33}/c_{11}. It follows that the quality factor gives a better indicator of anisotropy than the elastic constants. Another important consequence of this analysis is that, when $c_{11} > c_{33}$, the *qP* wave attenuates more along the symmetry axis than in the plane of isotropy.

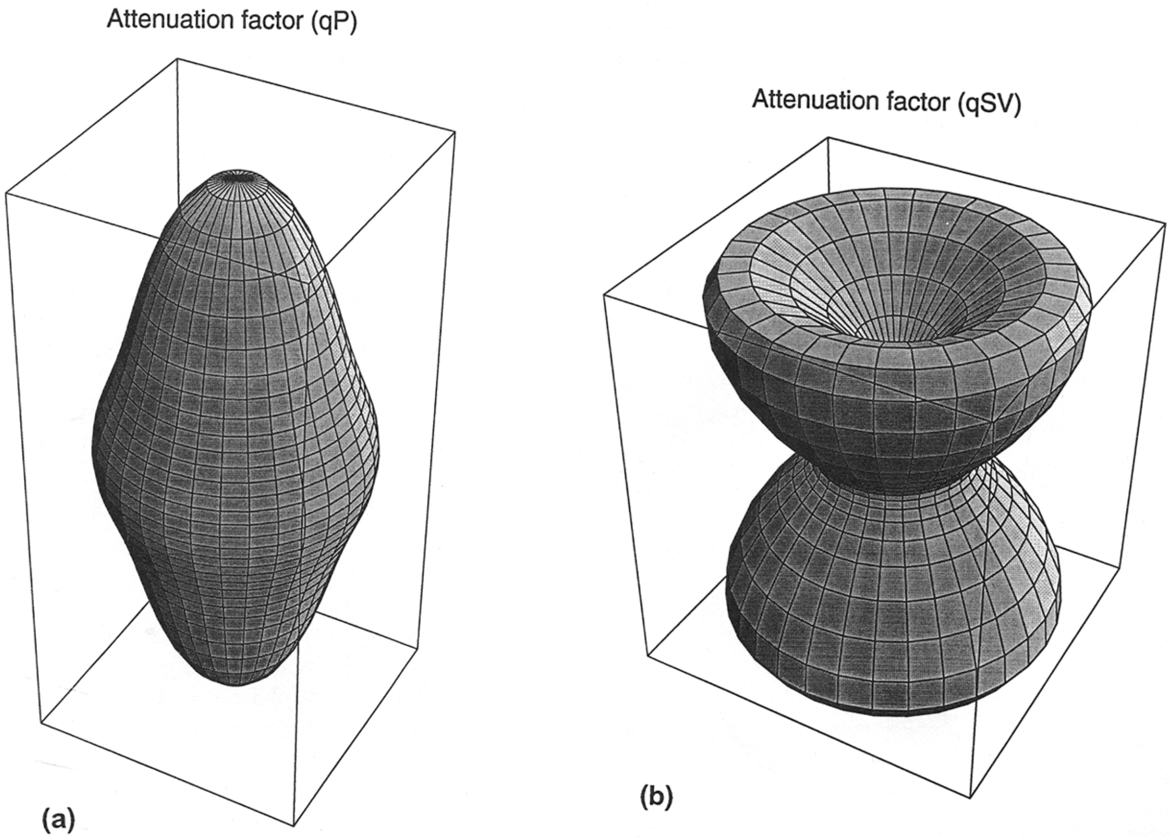

Fig. 4. Three-dimensional attenuation surfaces for clay shale at 20 Hz, where (a) is the *qP* mode, (b) is the *qSV*, and (c) is the *SH* mode. For clarity, the three surfaces have been plotted with different scales.

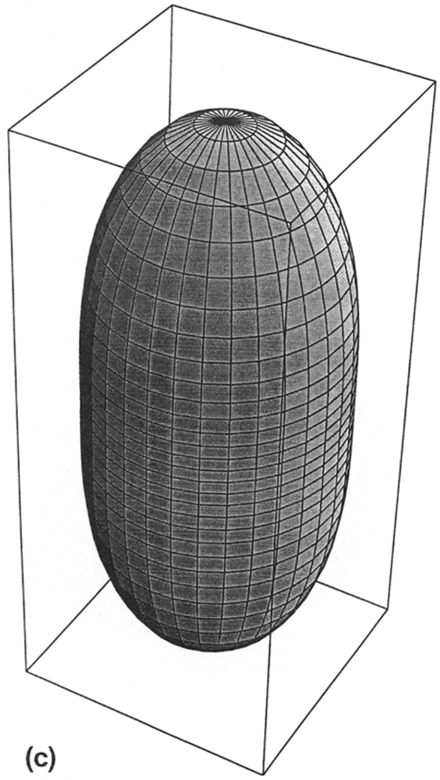
(c)

Fig. 4. (continued)

On the other hand, the quality factor of the *qSV* wave along the three axes is uniquely determined by the complex modulus M_2 since

$$Q = \frac{\mathrm{Re}[p_{55}]}{\mathrm{Im}[p_{55}]} = Q_2 \tag{6.2}$$

in virtue of Eqs. (5.1) and (5.8). Similarly, the *SH* mode has $Q = Q_2$ along the symmetry axis, and

$$Q = \frac{\mathrm{Re}[p_{66}]}{\mathrm{Im}[p_{66}]}, \quad \text{where} \quad p_{66} = c_{66} + G(M_2 - 1), \tag{6.3}$$

in the horizontal plane. For this wave,

$$\frac{Q(l_z = 1)}{Q(l_x = 1)} = \frac{G\,\mathrm{Re}[M_2]}{c_{66} + G(\mathrm{Re}[M_2] - 1)}, \tag{6.4}$$

where G is given by Eq. (5.4). Since $\mathrm{Re}[M_2] > 1$, it is straightforward to prove that when $c_{66} > c_{55}$, this ratio is less than 1, and the attenuation is higher along the symmetry axis. This is the case for clay shale, as can be seen in Table 1.

6.2. Tellurium dioxide

The attenuation and quality factor surfaces of tellurium dioxide are much more complicated than the clay shale surfaces. Actually, there exists one single surface that for a given direction determines three points which are the eigenvalues of the Christoffel matrix (3.5). Moreover, as mentioned before, the type of wave is not uniquely defined, even at symmetry planes where the eigenvalues are given by Eqs. (5.9), (5.10) and (5.11). However, the analysis of the eigenvectors is not within the scope of this work.

Figure 7a and 7b show sections of the attenuation surface across the symmetry planes, where the dotted line corresponds to the complex velocity V_1. Fig. 7b is a magnification of the region around the origin of Fig. 7a. As can be seen, one of the modes practically does not propagate around 45° in the horizontal plane, due to the strong dissipation. It can be shown that this mode

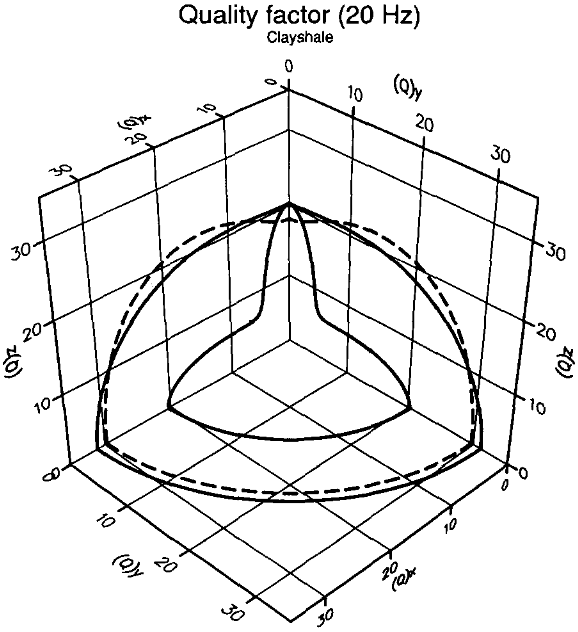

Fig. 5. Sections of the quality factor surfaces for clay shale (see Fig. 3).

behaves as a shear wave along that direction and is associated with a high value of the slowness (Auld, 1991). Sections of the quality factor surfaces are displayed in Fig. 8. As with the clay shale, the elasticities and the complex moduli control the values of the quality factors along the natural coordinate axes. For instance, since $c_{33} > c_{11}$, the quality factor of the mode represented

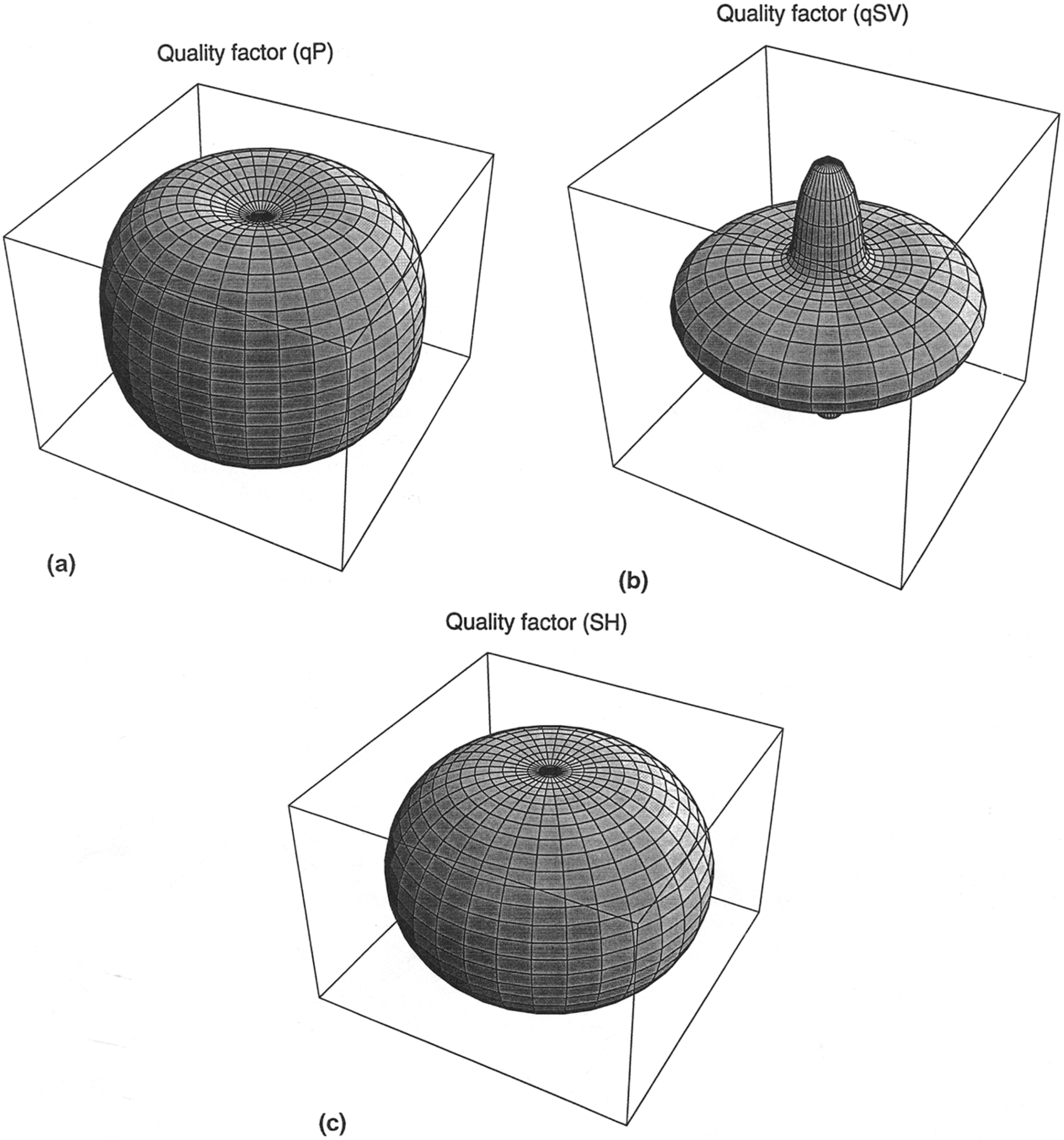

Fig. 6. Three-dimensional quality factor surfaces for clay shale (see Fig. 4).

with a dotted line is higher along the vertical axis than in the horizontal axes. The quasi-shear waves have their quality factor controlled by M_2 at the symmetry axis, where $Q = 150$ (see Table 1), and partially by M_4 along the horizontal axes, for which $Q_4 = 100$.

Sections of the attenuation and quality factor surfaces in several meridian planes (a), and inclined planes (b), are illustrated in Figs. 9 and 10, respectively. The meridian planes contain the z-axis, and are characterized by the longitude angle ϕ measured from the positive x-semiaxis, while the inclined planes contain the x-axis, and are characterized by the inclination α measured from the positive y-semiaxis. As is seen from Eqs. (3.4) and (3.5), the Christoffel matrix is a function of the propagation direction (l_x, l_y, l_z). However, using any computer algebra software, it is easily seen that the characteristic polynomial of the Christoffel matrix is not affected by a rotation of the propagation direction through an angle $\frac{1}{2}\pi$ about the z-axis. Hence, the attenuation and

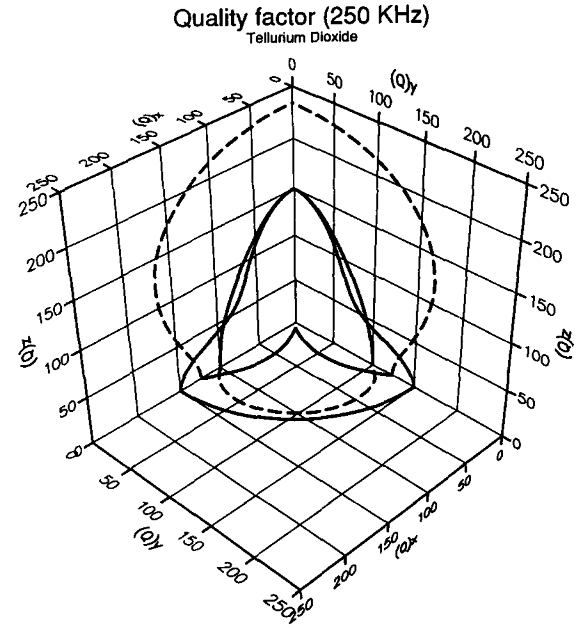

Fig. 8. Sections of the quality factor surfaces for tellurium dioxide (see Fig. 6).

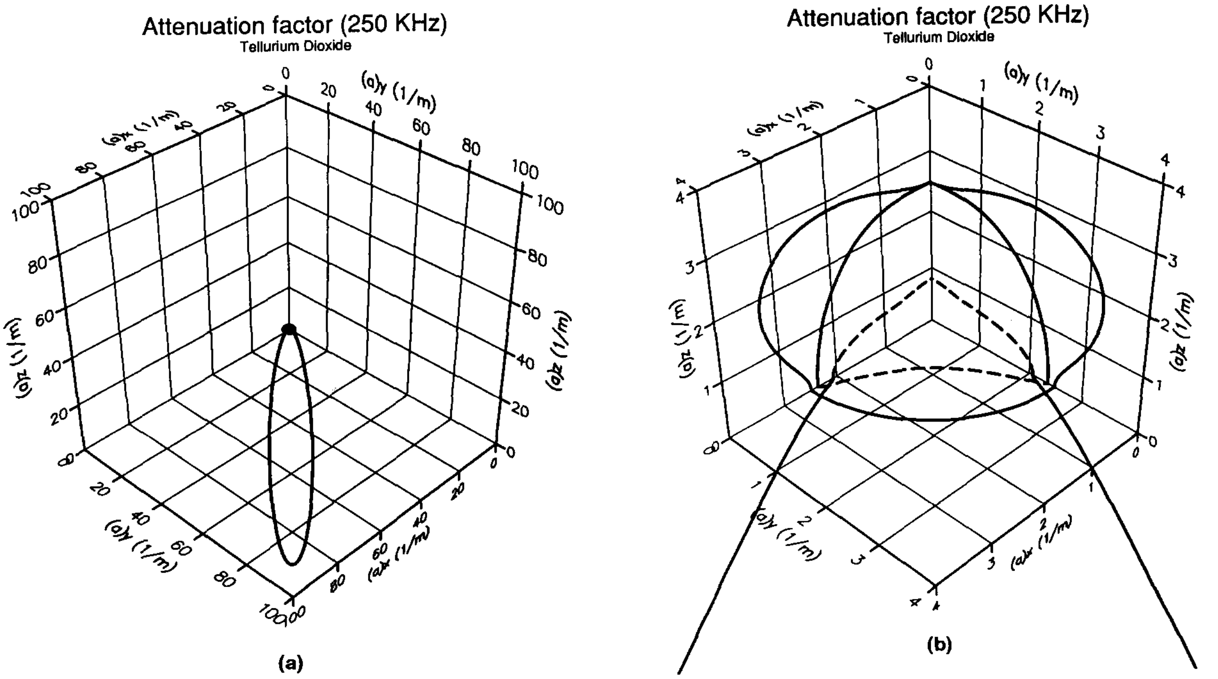

Fig. 7. Sections of the attenuation factor surfaces at 250 kHz for tellurium dioxide across three mutually perpendicular planes where the symmetry axis coincides with the vertical axis. In Figure (b), the dotted line corresponds to the complex velocity V_1.

quality factor surfaces are symmetric with respect to this rotation and its multiples. It follows that Figs. 9 and 10 completely describe the situation, although the longitude angle varies only in the first quadrant and not in the whole [0, 2π] interval. Moreover, a similar computation yields sym-

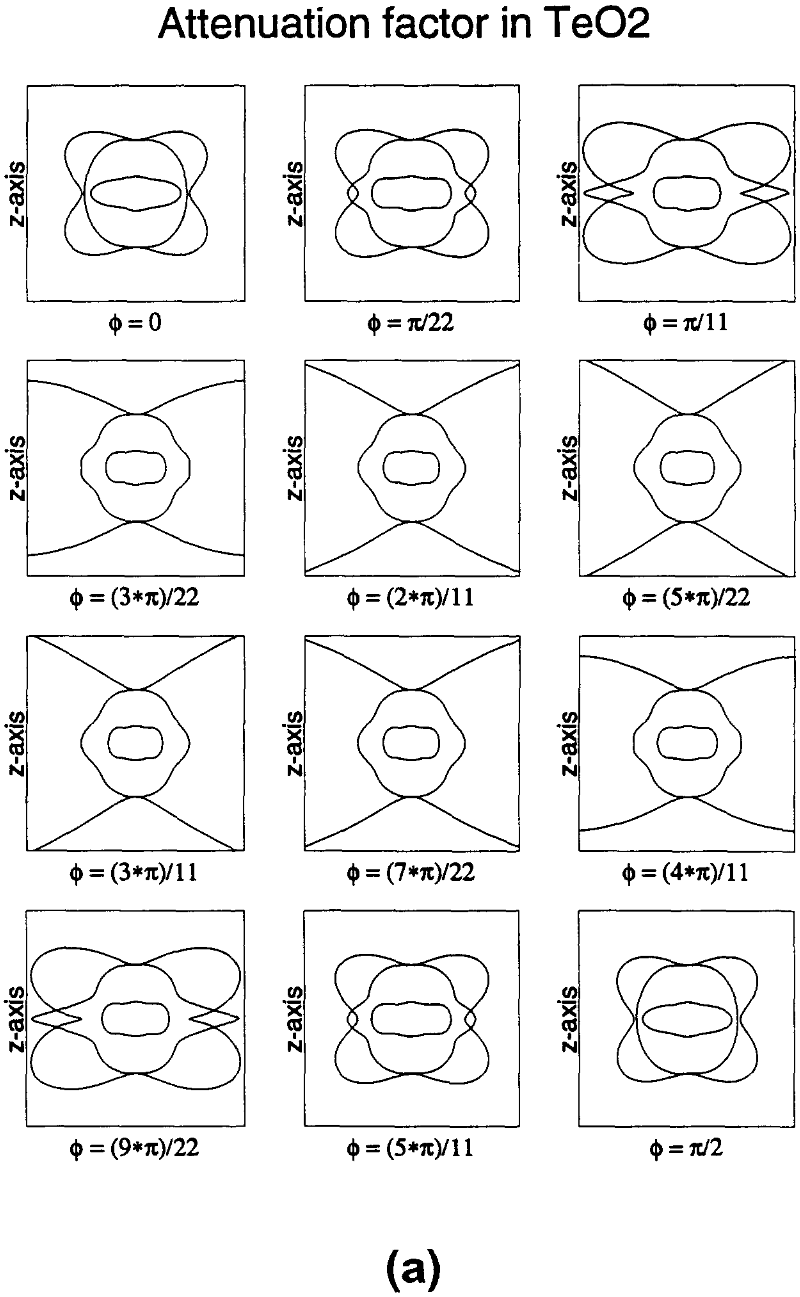

Fig. 9. Sections of the attenuation surfaces in meridian planes (a), and inclined planes (b), at 250 kHz for tellurium dioxide.

metry with respect to the $x = y$ plane, as well as with respect to the coordinate planes. Hence, Figs. 9 and 10 are also adequate to describe the dependence with respect to the inclination of the section plane.

7. Conclusion

Wave attenuation, as wave velocity, is a property that can be used for the characterization of anisotropic media. We obtained closed expres-

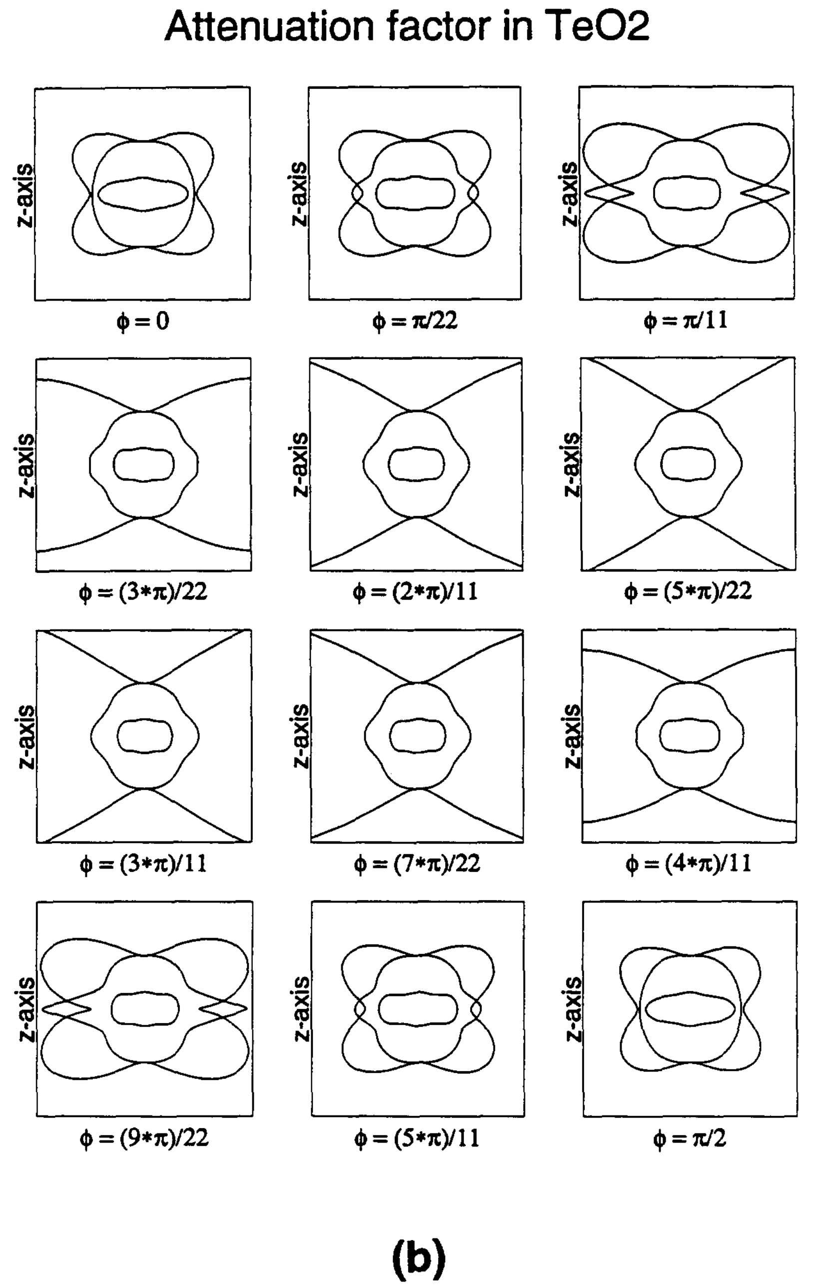

Fig. 9 (continued)

sions of measurables quantities, like the attenuation and the quality factor, in terms of the complex velocities of the medium. In this way, these material properties can be determined from the analysis of the propagation of homogeneous plane waves (Hosten et al., 1987; Arts et al., 1992). For the constitutive rheology under consideration, the attenuation surfaces are strongly dependent on

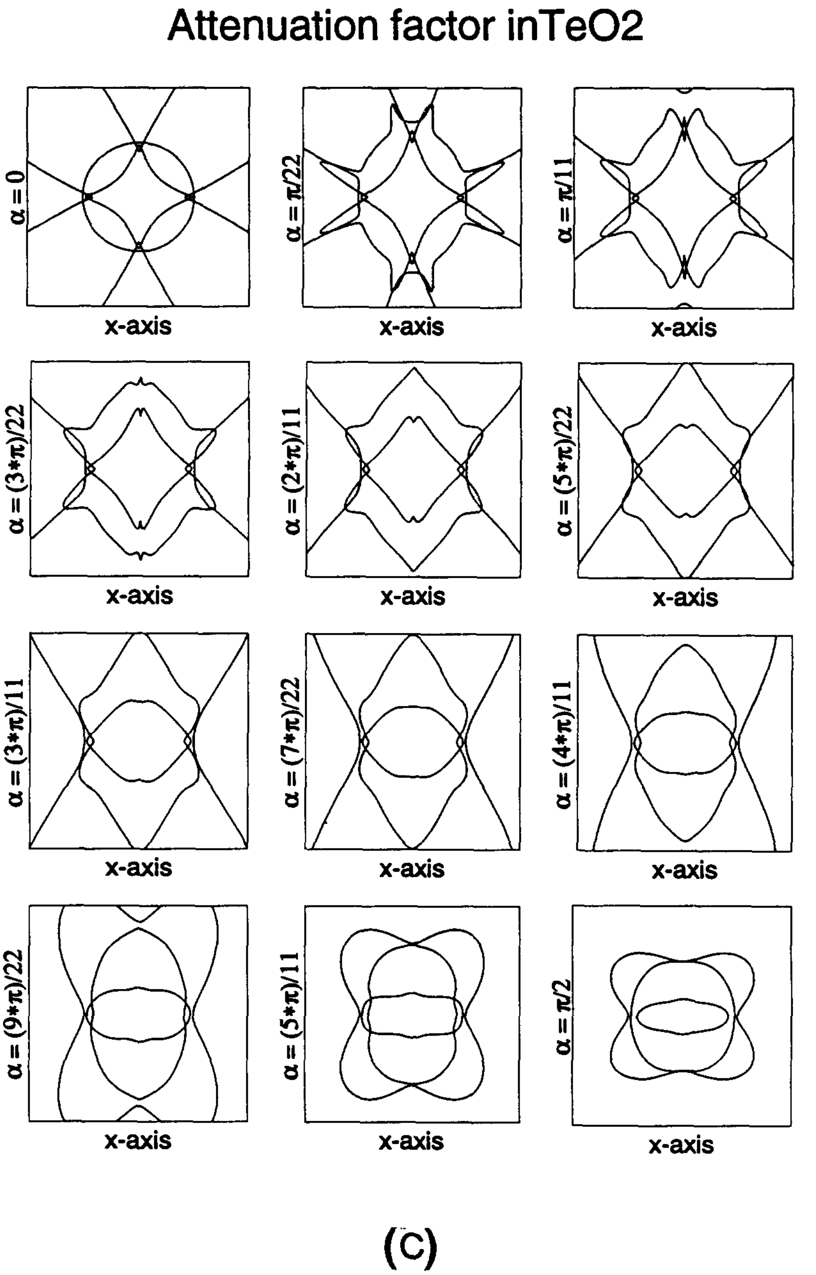

Fig. 9 (continued)

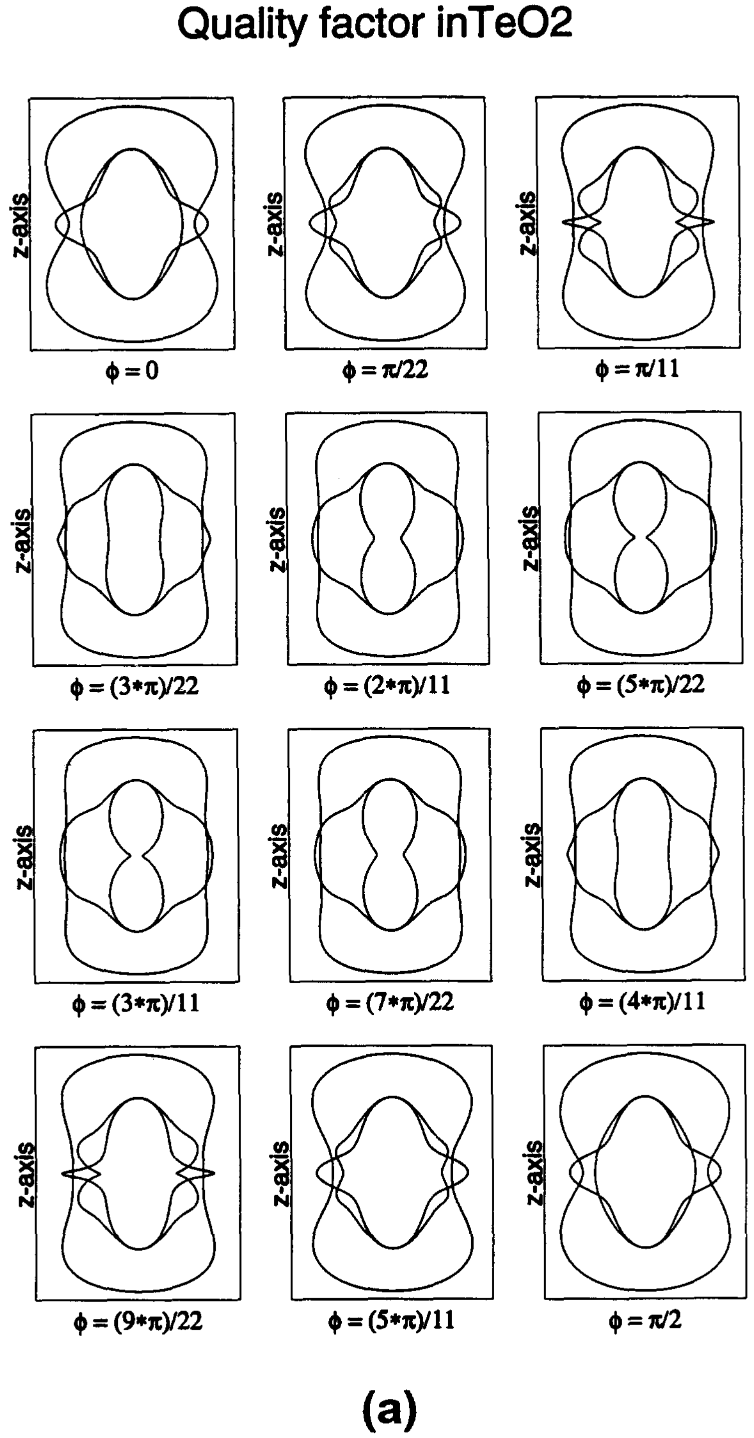

Fig. 10. Sections of the quality factor surfaces (see Fig. 9).

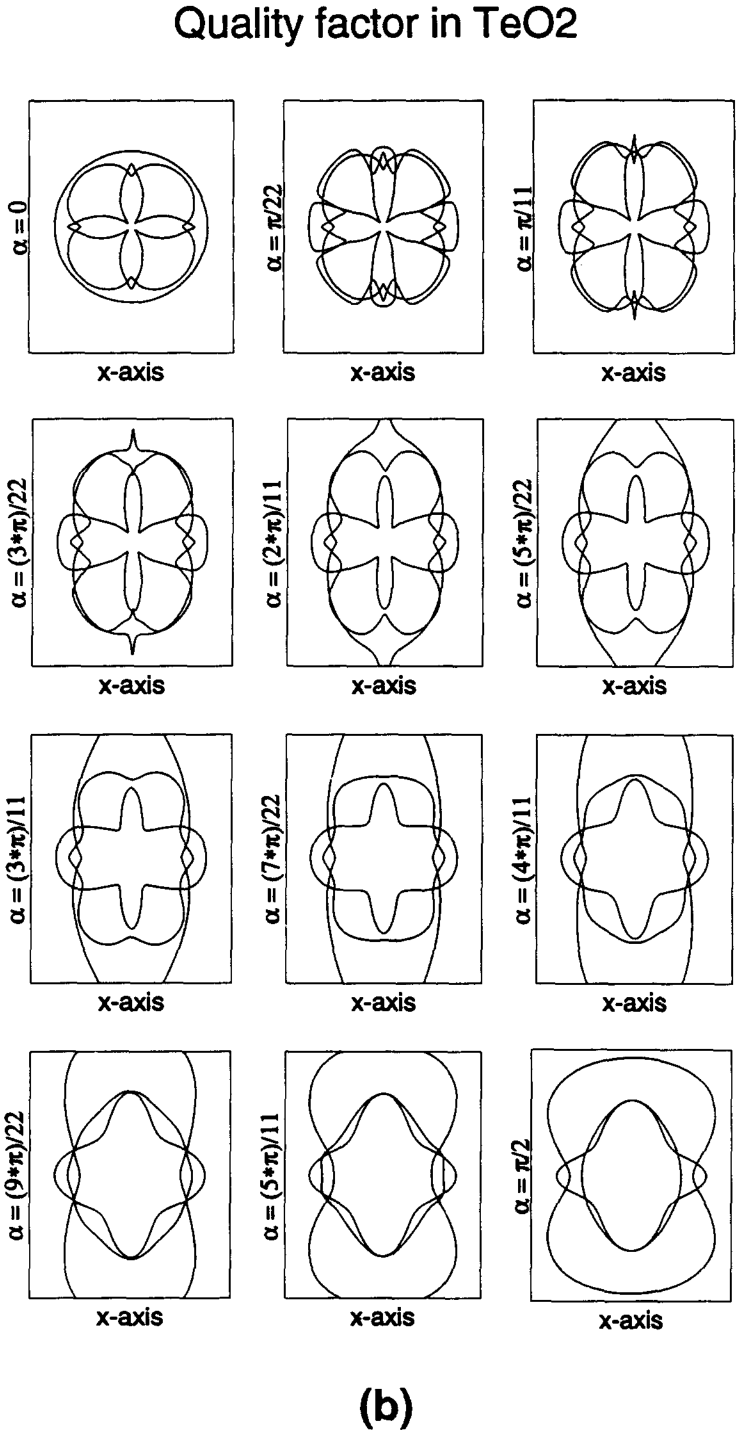

Fig. 10. (continued)

the values of the elasticities, and seem to provide a better indicator of anisotropy. For example, in clay shale $c_{11} > c_{33}$ implies that the dissipation along the symmetry axis is higher than in the plane of isotropy. The same applies to the *SH* wave if $c_{66} > c_{55}$. Moreover, the theory predicts that in tellurium dioxide, one of the shear modes does not propagate along a given direction due to the strong dissipation. This effect could be confirmed by laboratory experiments or by numerical simulations. In particular, the present constitutive relation can be easily implemented in time-domain computations of transient wavefields (Carcione, 1990). We worked out our analysis for a particular frequency. However, the dependence of the stiffnesses on the frequency can be obtained from the frequency dependence of the attenuation and quality factor. For this purpose, the constitutive model provides an arbitrary frequency dependence, and gives elastic behaviour at the low and high frequency limits.

In conclusion, we presented a linear 3D rheological model, as general as possible, where the anelastic properties of the different wave modes can be conveniently defined along preferred directions, and gave the expressions of measurable quantities in terms of the material properties. The theory can be used either for matching experimental data for material characterization, or for predicting directional attenuation behaviour of anisotropic media.

Acknowledgments

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