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Reflection and Transmission of Anti-Plane Shear Waves in Anisotropic Media*

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Abstract

For special symmetries (including monoclinic), the 3D equations of linear (visco-) elasticity admit anti-plane (i. e. linearly polarized) shear (i. e. divergence-free) solutions, which therefore depend on two coordinates only in a suitable reference system. The propagation conditions for these waves in transversely isotropic media are obtained here using coordinate-free notation. Special attention is paid to monochromatic inhomogeneous waves, in which propagation and attenuation directions differ. Moreover, the problem of reflection and transmission at a welded plane surface is considered. The assumed continuity of displacement and traction yields a generalized Snell's law, together with reflection and refraction coefficients. Numerical computations, with geophysically meaningful parameters, illustrate the main differences between the elastic and the dissipative materials.

1 Introduction

There is a vast literature on waves in anisotropic elastic media, e. g. [1], and in isotropic viscoelastic media, e. g. [2] and [3]; but only a few scattered papers take into consideration both anisotropy and dissipation, e. g. [4], [5], [6] and [7]. The aim of this work is to contribute to fill this gap starting with

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a study of anti-plane shear waves, which are physically important in spite of their relatively simple mathematical structure.

2 Theory

2.1 Unbounded homogeneous medium

2.1.1 Basic equations

The equations of linear elastodynamics are, in coordinate-free notation:

$$\text{(momentum equation)} \quad \mathrm{Div}\, T + \mathbf{b} = \rho\, \ddot{\mathbf{u}}, \tag{1}$$

$$\text{(constitutive law)} \quad T = \mathcal{C}\, E, \tag{2}$$

$$\text{(strain-displacement relation)} \quad E = \mathrm{sym}(\nabla \mathbf{u}), \tag{3}$$

where T is stress, $\mathbf{b}$ is body-force density, ρ is mass density, $\mathbf{u}$ is displacement, $\mathcal{C}$ is the elasticity operator, E is strain, and "sym" denotes the symmetric part of a tensor [8]. From Eq. (1)-(3) we get a single (3D vector) wave equation for displacement in the absence of body forces:

$$\mathrm{Div}\, \mathcal{C}\, (\mathrm{sym}\ \nabla \mathbf{u}) = \rho\, \ddot{\mathbf{u}}. \tag{4}$$

2.1.2 Anti-plane shear waves

Anti-plane waves are expressed by the ansatz

$$\mathbf{u}[t, \mathbf{x}] = u[t, \mathbf{x}]\ \hat{\mathbf{u}}, \tag{5}$$

where $\mathbf{x}$ is space position, and $\hat{\mathbf{u}}$ is a fixed direction (unit vector). By definition, *shear* waves fulfill the zero-divergence condition

$$\mathrm{Div}\ \mathbf{u} = 0. \tag{6}$$

From (5) and (6) we get that *anti-plane shear* waves verify

$$\mathrm{trc}\ E = 0. \tag{7}$$

Substituting ansatz (5) into wave equation (4) yields three scalar constraints for the single scalar unknown u: the system, in general, is overdetermined and so it has no solution. However, for special rheologies, the three scalar equations in (4) reduce to a single one if polarization $\hat{\mathbf{u}}$ is suitably chosen. The range of materials for which this happens is slightly wider than the monoclinic class [9]; here, for simplicity, we confine our analysis to transversely isotropic media, whose elastic constitutive law is given by

$$\begin{aligned} T \;&= [c_{12}\mathrm{tr}E - (c_{12}-c_{13})\hat{\mathrm{e}}\cdot(E\hat{\mathrm{e}})]\, I \\ &+(c_{11}-c_{12})E \\ &-(c_{11}-c_{12}-2c_{44})\ 2\ \mathrm{sym}\,[\hat{\mathrm{e}}\otimes(E\hat{\mathrm{e}})] \\ &-[(c_{12}-c_{13})\mathrm{tr}E-(c_{11}+c_{33}-2c_{13}-4c_{44})\hat{\mathrm{e}}\cdot(E\hat{\mathrm{e}})]\,\hat{\mathrm{e}}\otimes\hat{\mathrm{e}} \end{aligned} \tag{8}$$

where $\hat{\mathrm{e}}$ is a unit vector [10]. Now, let's assume that $\hat{\mathrm{e}}$ is orthogonal to $\hat{\mathbf{u}}$ and introduce $\hat{\mathrm{e}}_x = \hat{\mathbf{u}} \times \hat{\mathrm{e}}$; it follows that $\{\hat{\mathrm{e}}_x,\ \hat{\mathbf{u}},\ \hat{\mathrm{e}}\}$ is a right-handed reference frame, and therefore

$$E = (\hat{\mathrm{e}}_x\cdot\nabla u)\ \mathrm{sym}(\hat{\mathrm{e}}_x\otimes\hat{\mathbf{u}}) + (\hat{\mathrm{e}}\cdot\nabla u)\ \mathrm{sym}(\hat{\mathrm{e}}\otimes\hat{\mathbf{u}}), \tag{9}$$

$$\hat{\mathrm{e}}\cdot(E\hat{\mathrm{e}}) = 0, \tag{10}$$

$$2\ \mathrm{sym}\,[\hat{\mathrm{e}}\otimes(E\hat{\mathrm{e}})] = (\hat{\mathrm{e}}\cdot\nabla u)\mathrm{sym}(\hat{\mathrm{e}}\otimes\hat{\mathbf{u}}). \tag{11}$$

Substituting (7) and (9)-(11) into (8), we get the stress as

$$T = 2\ c_{66}(\hat{\mathrm{e}}_x\cdot\nabla u)\ \mathrm{sym}(\hat{\mathrm{e}}_x\otimes\hat{\mathbf{u}}) + 2\ c_{44}(\hat{\mathrm{e}}\cdot\nabla u)\ \mathrm{sym}(\hat{\mathrm{e}}\otimes\hat{\mathbf{u}}), \tag{12}$$

where $c_{66} = (c_{11}-c_{12})/2$, and then

$$\mathrm{Div}\ T = 2\ c_{66}\ \mathrm{sym}(\hat{\mathrm{e}}_x\otimes\hat{\mathbf{u}})\nabla(\hat{\mathrm{e}}_x\cdot\nabla u) + 2\ c_{44}\ \mathrm{sym}(\hat{\mathrm{e}}\otimes\hat{\mathbf{u}})\nabla(\hat{\mathrm{e}}\cdot\nabla u)$$

At this point it is convenient to introduce coordinates $\{x,y,z\}$ through $\mathbf{x} = x\hat{\mathrm{e}}_x + y\hat{\mathbf{u}} + z\hat{\mathrm{e}}$. Then, by (5) and (6), the displacement $\mathbf{u}$ does not depend on coordinate y, and

$$\mathrm{Div}\ T = (c_{66}\partial_{xx}u + c_{44}\partial_{zz}u)\hat{\mathbf{u}};$$

thus the vector wave equation (4) reduces to the single scalar equation

$$c_{66}\partial_{xx}u + c_{44}\partial_{zz}u = \rho\ \ddot{u}. \tag{13}$$

From now on, we focus our attention on plane waves, which in our case take the form

$$u[t,x,z] = \mathrm{Re}\left[U\ \mathrm{e}^{\mathrm{i}\ \omega\ t}\exp[-\mathrm{i}(k_x x + k_z z)]\right], \tag{14}$$

where U is the only complex variable. For these waves, Eq. (13) gives the propagation condition

$$c_{66}{k_x}^2 + c_{44}{k_z}^2 = \rho\,\omega^2. \tag{15}$$

This equation holds, in particular, for isotropic media, with $c_{66} = c_{44} \equiv \mu$. The above discussion referred to elastic materials, but it applies as well to viscoelastic media provided that the parameters c_{44}, c_{66}, and k_x, k_z are taken as complex: this is, in essence, the content of the correspondence principle [11].

2.1.3 Energy flow

In linear elastodynamics, the total energy density and the (Umov-Poynting) intensity vector are defined by

$$\epsilon = \frac{1}{2}E \cdot T + \frac{1}{2}\rho\,\dot{\mathbf{u}} \cdot \dot{\mathbf{u}} \tag{16}$$

and

$$\mathbf{q} = -T\,\dot{\mathbf{u}}, \tag{17}$$

respectively. It is well known that they satisfy the energy balance equation

$$\dot{\epsilon} = -\mathrm{Div}\;\mathbf{q} + \mathbf{b} \cdot \dot{\mathbf{u}}, \tag{18}$$

as stated, e. g., in chapter 34 of Gurtin's book [8].

Assuming ansatz (14) and a transversely isotropic material, Eq. (17) becomes

$$\mathbf{q} = \omega \left(\mathrm{Re}\left[\mathrm{i}U\;\mathrm{e}^{\mathrm{i}\;\omega\;t}\exp[-\mathrm{i}(k_x x + k_z z)]\right]\right)^2 (c_{66}\;k_x\;\hat{\mathbf{e}}_x + c_{44}\;k_z\;\hat{\mathbf{e}}_z).$$

It is then clear that, for this kind of waves, the Poynting vector is easily computed from the wave parameters and the material constants. Moreover, the quotient between the Poynting vector components determines the grazing angle of $\mathbf{q}$: this angle

$$\psi \equiv \arctan\frac{|q_x|}{|q_z|} = \arctan\frac{c_{66}\;|k_x|}{c_{44}\;|k_z|} \tag{19}$$

is then a constant and, in anisotropic media, usually differs from the corresponding quantity for the propagation vector $\mathbf{k}$. On the other hand, the magnitude of Poynting's vector depends both on time and space.

The time average of the Poynting vector over the one-period interval $[0, 2\pi/\omega]$ is given by

$$< \mathrm{q} > = \frac{1}{2}\, \omega\, |U|^2\, (c_{66}\, k_x\, \hat{\mathrm{e}}_x + c_{44}\, k_z\, \hat{\mathrm{e}}_z)$$

and therefore is independent not only on time (which is obvious), but also on space (which is rather surprising). From this formula, and propagation condition (15), it follows that the magnitude of Poynting's vector may be expressed as

$$\| < \mathrm{q} > \| = \frac{1}{2}\, \omega\, |U|^2\, \sqrt{c_{44}\left(\rho\, \omega^2 - c_{66}\left(1 - \frac{c_{66}}{c_{44}}\right) {k_x}^2\right)}. \tag{20}$$

2.2 Two homogeneous half-spaces

Consider a wave $u_I[t, x, z] = U_I \exp[-\mathrm{i}(k_{Ix}x + k_{Iz}z - \omega_I\, t)]$, traveling in an elastic isotropic medium of density ρ_1, that is incident at a plane interface $z = 0$ with a transversely isotropic, viscoelastic medium of density ρ_2. We want to compute the reflected wave u_R and the transmitted wave u_T. The assumed continuity of displacement,

$$u_I + u_R = u_T \quad \text{at } z = 0, \tag{21}$$

has three main consequences: first, the frequencies of the three waves are the same; second, the x-components of the propagation vectors coincide (generalized Snell's law); third, the sum of incident and reflected amplitudes equals the transmitted wave amplitude, namely

$$U_I + U_R = U_T. \tag{22}$$

The assumed continuity of traction,

$$T_I\; \hat{\mathrm{e}} + T_R\; \hat{\mathrm{e}} = T_T\; \hat{\mathrm{e}} \quad \text{at } z = 0, \tag{23}$$

yields, in particular,

$$\mu(k_{Iz}\, U_I + k_{Rz}\, U_R) = c_{44} k_{Tz} U_T.$$

From these results, and propagation conditions of the form (15), the reflected and transmitted waves can be determined, given the incident wave, using the reflection and transmission coefficients

$$C_R \equiv \frac{U_R}{U_I} = C_T - 1, \tag{24}$$

$$C_T \equiv \frac{U_T}{U_I} = 2\left(1 + \frac{c_{44}}{\mu}\frac{k_{Tz}}{k_{Iz}}\right)^{-1}. \tag{25}$$

Since the incidence medium is elastic, we may write the propagation vector $\mathbf{k}_I$ in terms of the incidence angle $\theta_I = \arctan|k_{Ix}/k_{Iz}|$ as $k_{Ix} = k_I \sin\theta_I$ and $k_{Iz} = -k_I \cos\theta_I$, with ${k_I}^2 = \rho_1\omega^2/\mu$; hence, by eliminating frequency from the propagation conditions for the two media, we get

$$\left(\frac{k_{Tz}}{k_{Iz}}\right)^2 = \frac{\mu}{c_{44}}\left(\frac{\rho_2}{\rho_1} - \frac{c_{66}}{\mu}\sin^2\theta_I\right)\frac{1}{\cos^2\theta_I}.$$

Thus, the transmission coefficient C_T depends on the incident wave through the incidence angle only.

When the transmission medium is viscoelastic, one has to assume that the transmitted plane wave is inhomogeneous, i.e., that the propagagation vector $\mathbf{k}$ is complex with non-parallel real and imaginary parts. We express it as $\mathbf{k} = \vec{\kappa} - \mathrm{i}\,\vec{\alpha}$, where $\vec{\kappa}$ and $\vec{\alpha}$ are the real propagation and attenuation vectors, respectively. Because of the generalized Snell's law, the angle between the real attenuation vector and the normal to the interface is zero. Moreover, the angle of transmission (i.e., the angle between the real propagation vector and the normal to the interface) depends on the incident wave through the incidence angle only, since

$$\left(\frac{k_{Tx}}{k_{Tz}}\right)^2 = \frac{c_{44}}{\mu}\sin^2\theta_I\left(\frac{\rho_2}{\rho_1} - \frac{c_{66}}{\mu}\sin^2\theta_I\right)^{-1}.$$

If c_{44} and c_{66} are real, and $\rho_2/\rho_1 < c_{66}/\mu$, then for incidence angles greater than the total reflection angle

$${\theta_I}^{TOT} = \arcsin\sqrt{\frac{\rho_2}{\rho_1}\frac{\mu}{c_{66}}}$$

the transmitted wave becomes evanescent.

Medium		ρ	μ	c_{44}	c_{66}
Incidence (elastic)		2000	10.58	–	–
Transmission	elastic	2500	–	44.1	19.6
	viscoelastic	2500	–	64.8 + 4.03 i	28.8 + 1.79 i

Table 1: *Numerical values of the material parameters used in the examples, with densities expressed in* kg/m^3 *and stiffnesses in* GPa.

From Eq. (19) and the propagation condition for the incidence medium, we have that $\psi_R = \psi_I = \theta_R = \theta_I$. Likewise, for an elastic transmission medium, we get that $\tan\psi_T = (c_{66}/c_{44})\tan\theta_T$; therefore, $\psi_T = \theta_T$ when both media are elastic isotropic. Finally we note that, because of Eq. (20), the magnitudes of the average Poynting vectors of the reflected and transmitted waves depend on the incident wave only through its angular frequency ω and its horizontal wavenumber k_x; thus, by the generalized Snell's law, they are all equal to the magnitude of the average incident Poynting vector, given by

$$\| < \mathbf{q}_I > \| = \frac{1}{2}\,\omega^2\,|U_I|^2\,\sqrt{\mu\,\rho_1}.$$

3 Examples

We consider here two correlated situations of geophysical interest: in both cases the incidence medium is the same (isotropic elastic) and the transmission medium is transversely isotropic, but the latter is elastic in the first example and viscoelastic in the second, as shown in Table 1. The Lamé parameter $\mu = \rho_1\,{c_S}^2$ corresponds, here, to a shear wave velocity $c_S = 2300$ m/s. Assume, for clarity, that the interface is horizontal. Then, in the elastic case, the stiffnesses $c_{44} = \rho_2\,{c_V}^2$ and $c_{66} = \rho_2\,{c_H}^2$ in Table 1 correspond to a vertical shear wave velocity $c_V = 4200$ m/s and to a horizontal shear wave velocity $c_H = 2800$ m/s, respectively.

In the viscoelastic case, a standard linear solid constitutive law has been assumed (cf. [11] and [13]). Then, the complex stiffness c_{44} may be expressed in terms of the more directly measurable quantity Q_{0V} (minimum quality factor of vertical shear waves) as

$$c_{44} = \rho_2 \, {c_V}^2 \, \frac{2\pi \left(+1 + \sqrt{1 + Q_{0V}^2} \right) - \mathrm{i} \, Q_{0V}}{2\pi \left(-1 + \sqrt{1 + Q_{0V}^2} \right) - \mathrm{i} \, Q_{0V}}.$$

A similar formula holds for c_{66}. Choosing $Q_{0H} = Q_{0V} = 5$ leads to the values in Table 1. Fig. 1 shows cartesian plots of the transmission angle and of the reflection coefficient for the two geological structures described above: in both cases the incidence medium is the same (elastic isotropic), but the transmission medium is either elastic isotropic (left part of Fig. 1) or viscoelastic and transversely isotropic (right part of Fig. 1). The total reflection angle is 55 degrees: for incidence angles greater than that, the transmission angle is $\pi/2$, the magnitude of the reflection coefficient is 1, and its argument grows from 0 to 180 degrees (see left part of Fig. 1).

4 Conclusion

We have computed the reflected and transmitted waves due to an incident anti-plane shear wave, assuming an elastic isotropic incidence medium and a transversely isotropic transmission medium. The analysis for a viscoelastic transmission medium follows closely that for an elastic medium, via the correspondence principle. The most evident conclusion that can be drawn from our numerical examples (see Fig. 1) is that discontinuities and corner points appear only in the purely elastic case.

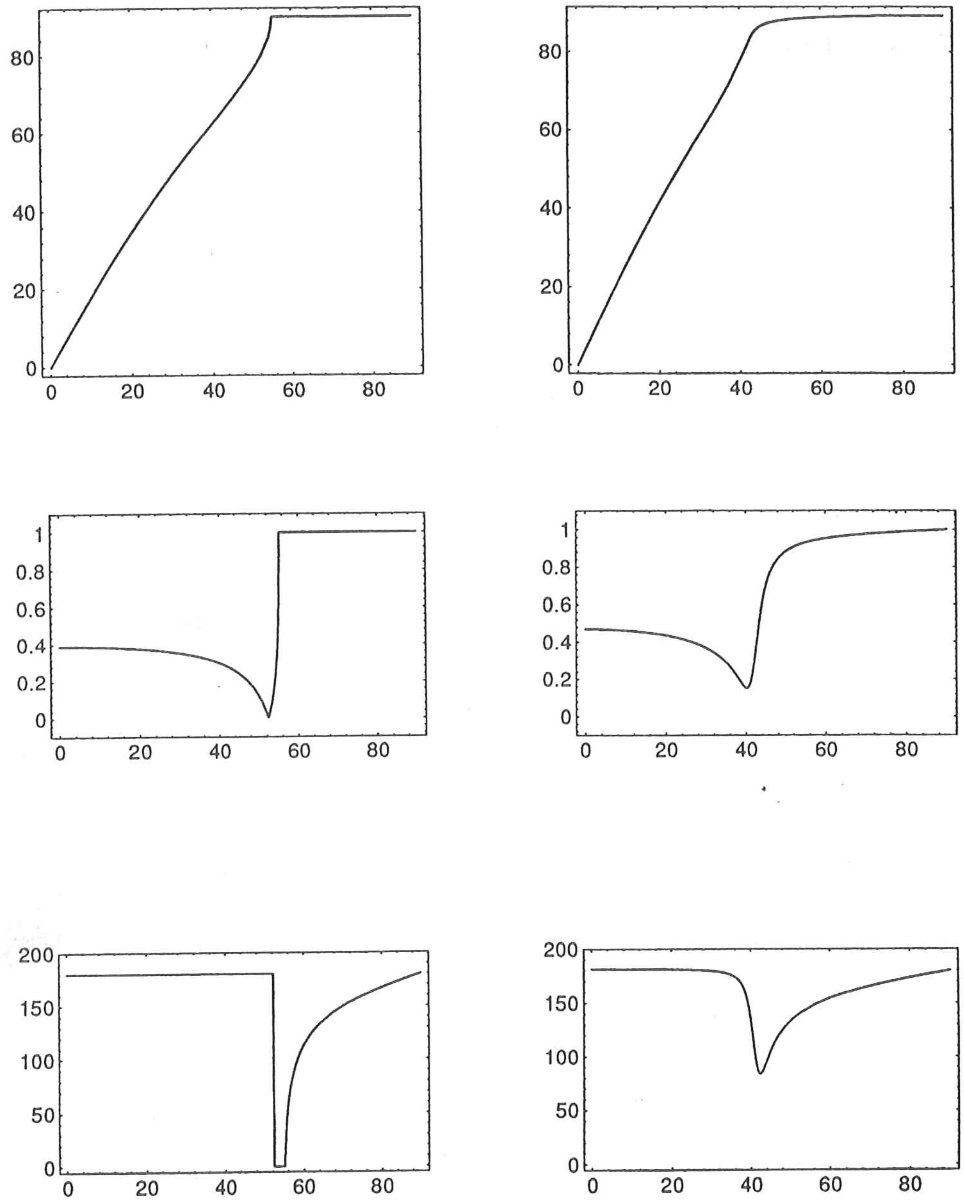

Figure 1: Cartesian plots of transmission angle (top), magnitude of reflection coefficient (center), and argument of reflection coefficient (bottom) for an elastic/elastic interface (left) and an elastic/viscoelastic interface (right).

References

[1] Fedorov, F. I., 1968, *Theory of elastic waves in crystals*: Plenum Press.

[2] Caviglia, G., and A. Morro, 1992, *Inhomogeneous waves in solids and fluids*: World Scientific, Singapore.

[3] Ainslie, M. A., and P. W. Burns, 1995, Energy-conserving reflection and transmission coefficients for a solid-solid boundary: *J. Acoust. Soc. Am.*, **98**, pp. 2836-2840.

[4] Le, L. H. T., E. S. Krebes, and G. E. Quiroga-Goode, 1994, Synthetic seismograms for *SH* waves in anelastic transversely isotropic media: *Geophys. J. Int.*, **116**, pp. 598–604.

[5] Carcione, J. M., and F. Cavallini, 1995, Energy balance and fundamental relations in anisotropic viscoelastic media: *Wave Motion*, **18**, pp. 11–20.

[6] Romeo, M., 1994, Inhomogeneous waves in anisotropic dissipative solids: *Q. Jl. Mech. appl. Math.*, **47**, pp. 481-491.

[7] Caviglia, G., and A. Morro, 1995, Two approaches to wave propagation in dissipative anisotropic solids, in *Mathematical and Numerical Aspects of Wave Propagation*, G. Cohen ed., SIAM, pp. 780-782.

[8] Gurtin, M. E., 1981, *An introduction to continuum mechanics*: Academic Press.

[9] Horgan, C. O., 1995, Anti-plane shear deformations in linear and non-linear solid mechanics: *SIAM Review*, **37**(1), pp. 53–81.

[10] Chadwick, P., 1989, Wave propagation in transversely isotropic elastic media I. Homogeneous plane waves: *Proc. R. Soc. Lond. A*, **422**, pp. 23–66.

[11] Ben-Menahem, A., and S. J. Singh, 1981, *Seismic waves and sources*: Springer-Verlag.

[12] Leitman, M. J. and G. M. C. Fisher, 1984, The linear theory of viscoelasticity: *In* C. Truesdell (Ed.), Mechanics of solids III, Springer-Verlag.

[13] Casula, G., and J. M. Carcione, 1995, Generalized mechanical model analogies of linear viscoelastic behaviour: *Boll. Geofis. Teor. Appl.*, **34**, pp. 235–256.