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RESEARCH PAPER

MODELING EXTREME-EVENT PRECURSORS WITH THE FRACTIONAL DIFFUSION EQUATION

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Abstract

Extreme catastrophic events such as earthquakes, terrorism and economic collapses are difficult to predict. We propose a tentative mathematical model for the precursors of these events based on a memory formalism and apply it to earthquakes suggesting a physical interpretation. In this case, a precursor can be the anomalous increasing rate of events (aftershocks) following a moderate earthquake, contrary to Omori's law. This trend constitute foreshocks of the main event and can be modelled with fractional time derivatives. A fractional derivative of order $0 < \nu < 2$ replaces the first-order time derivative in the classical diffusion equation.

We obtain the frequency-domain Green's function and the corresponding time-domain solution by performing an inverse Fourier transform. Alternatively, we propose a numerical algorithm, where the time derivative is computed with the Grünwald-Letnikov expansion, which is a finite-difference generalization of the standard finite-difference operator to derivatives of fractional order. The results match the analytical solution obtained from the Green function. The calculation requires to store the whole field in the computer memory since anomalous diffusion "remembers the past".

MSC 2010: Primary 35R11; Secondary 86A15, 86A17, 86-08

Key Words and Phrases: extreme events, precursors, forecast, earthquakes, Omori's law, memory, fractional derivatives

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1. Introduction

Experimental evidence has been presented that in some cases anomalous aftershock or swarm sequences of earthquakes may be precursors of an incoming strong earthquake [7],[24]. The same type of phenomenon, obviously related to the appropriate indicator, has been suggested as precursor in the forecast of financial crises of banks [28], in economy [42] and, in general, in socio-economic events [25]. Another particular indicator is the anomalous duration of the aftershocks of a moderate earthquake, specifically the extended duration of foreshocks, which may have an increase in rate contrary to Omori's law [6], [20], [38], could be precursor of a possible future stronger earthquake. In this work, we attempt to model mathematically this phenomenon in order to give an insight to its meaning. We consider an impulse and a constant input flow of events which diffuses in the medium according to the Fourier law. The constitutive equation of the flow is modified to include a mathematical formalism which represents the memory of the medium.

The basic notion of memory functions is widely recognized in several fields of science. Memory can be represented mathematically by using fractional order derivatives. For instance, the diffusion equation has been generalized by using these derivatives, [30]. Mainardi et al. [33] have used the time-fractional derivative of distributed order between 0 and 1, in both the Riemann-Liouville and the Caputo sense, [3]. Fractional derivatives have been thoroughly studied by many authors, e.g. [27], [39], [26], [15], [32]. This mathematical tool has been applied in many fields such as theoretical physics [36], biology [13], medicine [16], diffusion [31], plasmas in bounded domains [1], geophysics [21] and in plasma turbulence [14]. Jiao et al. [22] have thoroughly studied and extensively applied the fractional derivative of distributed order introduced by Caputo [2]. In seismology, Carcione et al. [9] and Carcione [12] described the anelastic behaviour of general materials over wide frequency ranges by using fractional derivatives, in particular considering propagation with constant-Q characteristics, [35].

The time-fractional diffusion-wave equation is obtained by replacing the first-order time derivative in the classical diffusion equation by a derivative of fractional order. The order ν of the time derivative can be any real number between 0 and 2; $\nu = 1$ gives the classical diffusion equation and $\nu = 2$ gives the wave equation. The range [0, 1] corresponds to dispersive anomalous sub-diffusion, while the range [1, 2] corresponds to generalized wave propagation. Several physical phenomena, besides fluid flow, can be described with the fractional diffusion equation. For instance, turbulent plasma, diffusion of carriers in amorphous photoconductors, diffusion in

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turbulent flow, vortex dynamics, the chaotic regime of the Josephson junction, a percolation model in porous media, fractal media, various biological phenomena and finance problems, e.g. [34], [40], [5].

In this work, we obtain semi-analytical solutions for homogeneous coefficients, using the Green function method. Caputo [4] obtained analytical solutions with a different approach, using the Laplace transform and Bromwich integration. Moreover, we compute numerical solutions by discretizing the spatial and time variables. In particular, grid methods are required to simulate wave propagation in heterogeneous realistic models [10], [41], [11]. Fractional derivatives are computed numerically with the Grünwald-Letnikov (GL) and central-difference approximations, which are extensions of the standard FD approximation for derivatives of integer order [18], [29]. Unlike the standard operator of differentiation, the fractional operator increases in length as time increases, since it must keep the memory effects.

2. Diffusion with memory

The mathematical formalism used to model the flow of events consists in the Fourier equation modified with the application of a memory formalism, in the form of fractional order derivatives. The classical constitutive equation is

$$q(x,t) = -c\partial_x r(x,t), \tag{2.1}$$

where $r(x,t)$ represents the rate of events (or flow) at the point x and time t (seismicity rate in seismology), q is the spatial variation of the rate r at that point, $c(x)$ is a diffusion coefficient (with dimension m^2/s in the SI system), and ∂_x denotes a partial derivative with respect to the variable x. The additional equation

$$\partial_x q + \partial_t r = 0 \tag{2.2}$$

ensures conservation. The classical equations (2.1) and (2.2) lead to the diffusion equation

$$\partial_x c \partial_x r = \partial_t r. \tag{2.3}$$

We now introduce the effect of memory by assuming that $\partial_x r(x,t)$ in equation (2.1) is affected according to the relation

$$q(x,t) = -d(x)D^{1-\nu}\partial_x r(x,t), \tag{2.4}$$

where $\nu \in$ [0,2], here d is a pseudo-diffusion coefficient given by

$$d(x) = c(x)\omega_0^{\nu-1}, \tag{2.5}$$

where ω_0 is a reference frequency, and d has dimension of m^2/s$^\nu$ in the SI system. The fractional derivative is given by

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$$D^{\alpha} r(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial r/\partial \tau(x,\tau)}{(t-\tau)^{\alpha}} d\tau = \frac{1}{\Gamma(1-\alpha)} \frac{\partial r}{\partial t} * \frac{1}{t^{\alpha}}, \tag{2.6}$$

the Caputo fractional derivative of order α, see [39], [26], [15], [32], [22], where "$*$" denotes time convolution.

The operator D^{α} describes the perturbation of the present events (or flow of events) due to the previous events. The system "remembers" the past. The mathematical formalism defined by equation (2.6) is constructed with a weighted mean of the first-order derivative $r_{,\tau}(x,\tau)$ in the time interval $[0,t]$, which is a sort of feedback system, i.e., the values of $r_{,\tau}(x,\tau)$ at time τ far apart from t are given smaller weight than those at times τ closer to t. Hence, the weights are increasingly smaller with increasing time separation from the time t to imply that the effect of the past is fading with increasing time. Importantly, the weights multiplying the first-order derivative of $r(x,\tau)$ inside the integral appearing in equation (2.3) can be chosen in many ways.

3. The analytical solution

Combining equations (2.2) and (2.4) yields

$$D_t^{\nu} r(x,t) + s(x,t) = \partial_x d \partial_x r(x,t), \tag{3.1}$$

where we have added a source $s(x,t)$. By taking $s(x,t) = \delta(t)\delta(x)$, where δ represents Dirac's function, equation (3.1) gives the Green function (or fundamental solution) $g(x,t)$. The cases ν =1 and ν = 2 give the classical diffusion and wave equations, respectively. The time Fourier-Laplace transform of the Green function corresponding to (3.1) is given in Hanyga [19] (Eq. (A3)), with $s = \mathrm{i}\omega$ and A =1:

$$\widetilde{g}_{\nu}(x,\omega) = (\mathrm{i}\omega)^{-\nu/4} \sqrt{\frac{x}{2\pi d^{3/2}}}\, K_{-1/2}\left[(\mathrm{i}\omega)^{\nu/2} \frac{x}{\sqrt{d}}\right], \tag{3.2}$$

where K_{γ} denotes the MacDonald or modified Bessel function of order γ. Note that Hanyga [19] (Eq. 2.1) solves $\partial_t^{2\nu} u = A\Delta u + \mathcal{S}$, where u is the unknown variable, A is a constant and $\mathcal{S}$ is the source. Moreover, $\Phi = 1$ in his Eq. (A2) since the initial conditions are zero. Being $K_{\pm 1/2}(z) = \sqrt{\pi/(2z)}\exp(-z)$, solution (3.2) is related to that formerly derived by Mainardi [30], [31] for the homogeneous time fractional diffusion-wave equation equipped with impulsive initial-boundary conditions.

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3.1. Laplace transform inversion and Wright functions

Then, the Laplace transform in (3.2) explicitly reads

$$\widetilde{g}_\nu(x,s) = \frac{1}{2\sqrt{d}}\,\frac{\exp\left(-x\,s^{\nu/2}/\sqrt{d}\right)}{s^{\nu/2}}\,,$$

whose inversion yields [43] (p. 119) [17] (p. 394),

$$g_\nu(x,t) = \frac{t^{\nu/2-1}}{2\sqrt{d}}\,W_{-\nu/2,\nu/2}\left(-\frac{x}{\sqrt{d}\,t^{\nu/2}}\right)\,. \tag{3.3}$$

Here, W denotes the Wright function in general, defined as an entire function in the complex domain with $\lambda > -1$ and $\mu \in \mathbb{C}$:

$$W_{\lambda,\mu}(z) = \frac{1}{2\pi\mathrm{i}}\int_{Ha}\exp\left(\sigma + z\sigma^{-\lambda}\right)\frac{d\sigma}{\sigma^\mu} = \sum_{n=0}^{\infty}\frac{z^n}{n!\,\Gamma[\lambda n+\mu]}\,, \tag{3.4}$$

where Ha denotes the Hankel contour (a path in the complex plane which extends from $[\infty,\,\epsilon]$, around the origin counter clockwise and back to $[\infty,\,-\epsilon]$, where ϵ is an arbitrarily small positive number).

For $\nu = 1$ (standard diffusion equation), we recover the known solution

$$g_1(x,t) = \frac{1}{2\sqrt{\pi d\,t}}\exp\left(-\frac{x^2}{4d\,t}\right). \tag{3.5}$$

In particular, we are interested in a modified source at $x = 0$

$$s(x,t) = [\delta(t) + \eta H(t)]\delta(x)\,, \tag{3.6}$$

where $\eta < 1$ and H is the Heaviside function. We have

$$\widetilde{s}(x,\omega) = \left(1 + \frac{\eta}{\mathrm{i}\omega + \epsilon}\right)\delta(x), \tag{3.7}$$

where ϵ is a non-physical parameter to avoid the singularity in the inverse time Fourier-Laplace transform, such that $\epsilon t \ll 1$.

As a consequence one can compute the solution in the space-time domain by adding the contribution of the constant rate in the source, that is

$$r(x,t) = g_\nu(x,t) + \eta\int_0^t g_\nu(x,\tau)\,d\tau\,, \tag{3.8}$$

where, by inversion of the corresponding Laplace transform,

$$\int_0^t g_\nu(x,\tau)\,d\tau = \frac{t^{\nu/2}}{2\sqrt{d}}\,W_{-\nu/2,1+\nu/2}\left(-\frac{x}{\sqrt{d}\,t^{\nu/2}}\right)\,. \tag{3.9}$$

There are two alternatives to obtain the solution, i.e., we multiply equation (3.2) with the source (3.7) and perform an inverse Fourier transform or use the solution (3.8) in terms of the Wright functions (3.3) and (3.9) by

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performing a time convolution with the source. In both cases, the solution is semi-analytical. We choose the first option and to ensure a real time-domain solution, we consider an Hermitian frequency-domain solution.

4. Numerical algorithm

The most used time approximation in fractional calculus is the backward Grünwald-Letnikov (GL) derivative. The GL fractional derivative of a function f is

$$h^{\nu}\frac{\partial^{\nu} f(t)}{\partial t^{\nu}} \sim \sum_{j=0}^{J}(-1)^j \left(\begin{array}{c} \nu \\ j \end{array} \right) f(t-jh), \tag{4.1}$$

where h is the time step, and $J = t/h - 1$. The binomial coefficients can be defined in terms of Euler's Gamma function as

$$\left(\begin{array}{c} \nu \\ j \end{array} \right) = \frac{\Gamma(\nu+1)}{\Gamma(j+1)\Gamma(\nu-j+1)}$$

and can be calculated by a simple recursion formula

$$\left(\begin{array}{c} \nu \\ j \end{array} \right) = \frac{\nu-j+1}{j}\left(\begin{array}{c} \nu \\ j-1 \end{array} \right), \quad \left(\begin{array}{c} \nu \\ 0 \end{array} \right) = 1.$$

If ν is a natural number, we have the classical derivatives. In this case $J = \nu$ in equation (4.1). The GL approximation is of order $\mathrm{O}(h)$. The fractional derivative of f at time t depends on all the previous values of f. This is the memory property of the fractional derivative. In our calculations we consider the whole memory history since for $\nu < 1$ it is not possible to use the short-memory principle, i.e., less terms in the sum of equation (4.1), as can be used in the simulation of wave propagation. The waves "forget" the past but the diffusion fields "remember" it.

The time discretization of equation (3.1) at ndt using the GL derivative is

$$D^{\nu} r_{n-1} + s_{n-1} = \partial_x d \partial_x r_{n-1}. \tag{4.2}$$

where

$$h^{\nu} D^{\nu} r_{n-1} = r_n + \sum_{j=1}^{J}(-1)^j \left(\begin{array}{c} \nu \\ j \end{array} \right) r_{n-j}. \tag{4.3}$$

Combining equations (4.2) and (4.3), r_n can be computed from its past values r_{n-j} as

$$r_n = h^{\nu}(\partial_x d \partial_x r_{n-1} - s_{n-1}) - \sum_{j=1}^{J}(-1)^j \left(\begin{array}{c} \nu \\ j \end{array} \right) r_{n-j}. \tag{4.4}$$

The accuracy and stability of this algorithm are analyzed in Appendix A.

The source is implemented as

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$$s_n = \frac{1}{h} + \eta H_n(t), \quad n = 0, 1 \ldots, \tag{4.5}$$

where $1/h$ and H_n are the discrete representations of the delta and Heaviside functions, respectively.

The spatial derivatives are calculated with the Fourier pseudospectral method by using the fast Fourier transform (FFT), [8]. The Fourier method has spectral accuracy for band-limited signals. Then, the results are not affected by spatial numerical dispersion. In the case of inhomogeneous media, the algorithm employs the staggered Fourier method. Staggered operators evaluate derivatives between grid points. For instance, if Δx is the grid (cell) size and k_1 is the wavenumber component, a phase shift $\exp(\pm \mathrm{i} k_1 \Delta x/2)$ is applied when computing the x-derivative. Then, $\partial_x d \partial_x$ is calculated as $D_x^- d D_x^+$, where $D_x^\pm$ is the discrete operator and $\pm$ refers to the sign of the phase shift. The spatial differentiation requires the interpolation of the material properties at half grid points.

5. Application to earthquakes

It is known that many aftershock sequences can be described with Omori's law, by which after a main shock the event rate of aftershocks decays according to

$$r(t) = \frac{r_0 \zeta^u}{(t+\zeta)^u}, \tag{5.1}$$

where ζ and u are constants, and $r(t)$ denotes the frequency of aftershocks occurring in a unit time interval [44]. Omori found this law by studying the aftershocks of the Nobi earthquake, which occurred in 1891 and had magnitude 8. A 100 years fit of the data gives the curve shown in **Figure 1**, where $u = 1$ and $\zeta = 0.797$ days, [23].

We now consider the event precursors consisting in the anomalous duration of the aftershocks of a moderate event, specifically the extended duration of aftershocks which may have an increase in rate contrary to Omori's law. First, we compare the analytical and numerical solutions at x = 100 km for $\eta = 0$ (an impulse), with $\nu = 0.9$, c = 3000 km/yr^2, ω_0 = 2 π/yr and a maximum time $t_{\max}$ = 10 yr. The analytical solution is computed with a FFT of length $N = 2^{20}$ points and a time step dt = 60 day, while the numerical method uses N_x = 165 grid points, D_x = 10 km, h = 0.073 day and the whole history of the seismicity rate to compute the fractional derivative. This comparison is shown in **Figure 2**, where the symbols correspond to the numerical solution. The match is excellent.

Next, we compare solutions for an impulse plus a constant rate using the same parameters and $\eta = 0.\ 2$. **Figure 3** shows that the agreement is very good. These comparisons allows us to verify that both the analytical and numerical solutions are correct. The trend shown in Figure 3 agrees very well with the seismicity rates of Hayward fault reported by Parsons [37], which have an almost linear behaviour with time (see his Figure 2).

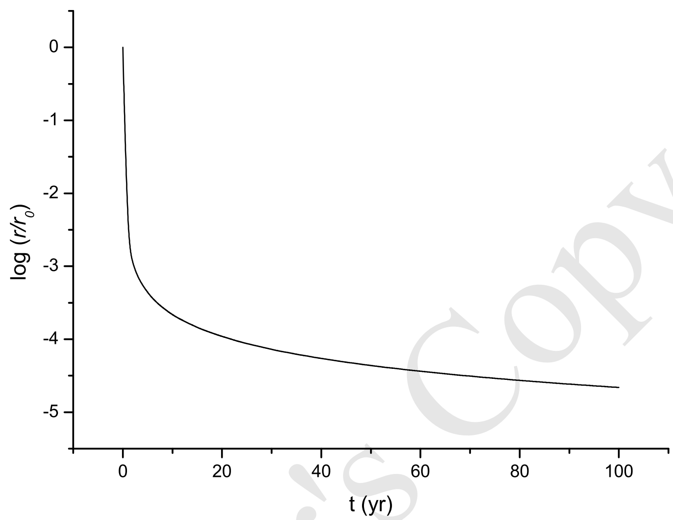

Fig. 1: Decay of aftershocks after the Nobi earthquake represented with Omori's empirical law.

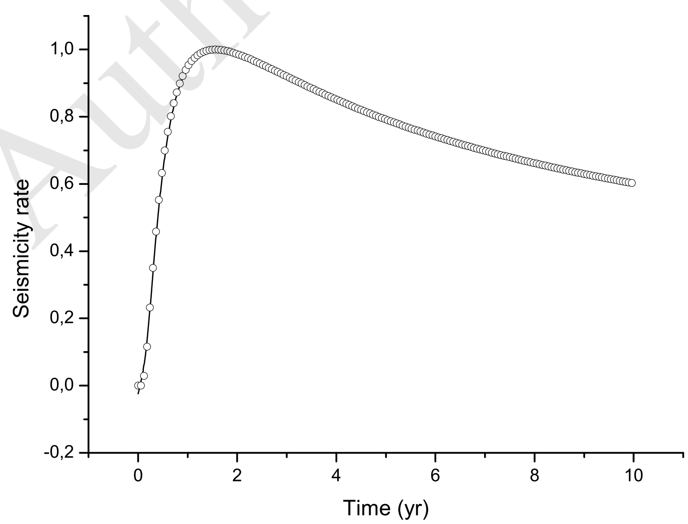

Fig. 2: Comparison of the seismicity rate calculated analytically (solid line) and numerically (symbols). The source is an impulse function and the order of the fractional derivative is $\nu = 0.9$.

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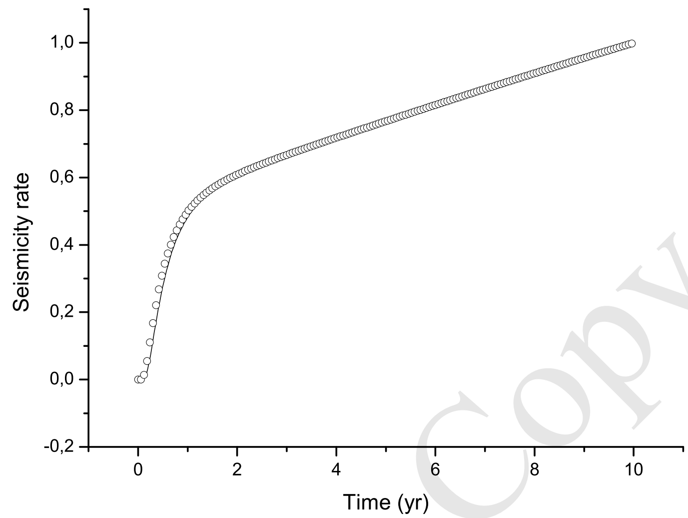

Fig. 3: Comparison of the seismicity rate calculated analytically (solid line) and numerically (symbols). The source is an impulse plus a constant rate with $\eta = 0.2$.

Several cases are shown in **Figure 4**, where the seismicity rates agree with or show opposite behaviour to Omori's law. The source is an impulse plus a constant rate (steady flow) and the curves are normalized with respect to the maximum value of the case $\nu = 0.5$ and $\eta = 0.05$. In some cases it is seen that the flow is decreasing or increasing depending on the value of η which quantifies the amount of steady flow at the origin (see Figure 4a). In Figure 4b, we can verify that the order of the time derivative models different behaviours in agreement or disagreement with Omori's law.

Figure 5 shows the seismicity rate at a larger time for different values of the order of the fractional derivative. The curves, which correspond to a delta function ($\eta = 0$), follow Omori's law, showing that different rates can be described with different fractional orders.

6. Conclusions

We have modeled flow of events representing precursors of a main event showing different trends, where the model is based on fractional time derivatives. The analytical solution is obtained with an inverse Fourier transform and considers as sources an impulse plus a constant rate of events. Alternatively, a solution has been computed with a numerical integration based on the Grünwald-Letnikov derivative and a spatial discretization, where the

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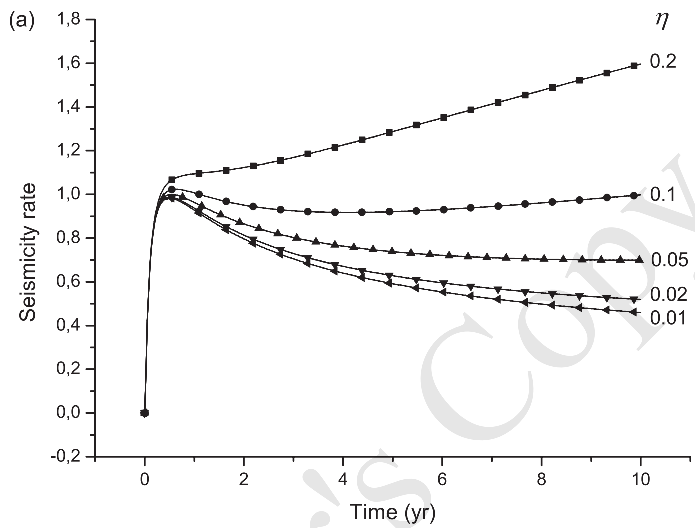
spatial derivatives are calculated with the Fourier pseudospectral method. Both methods yields similar results.



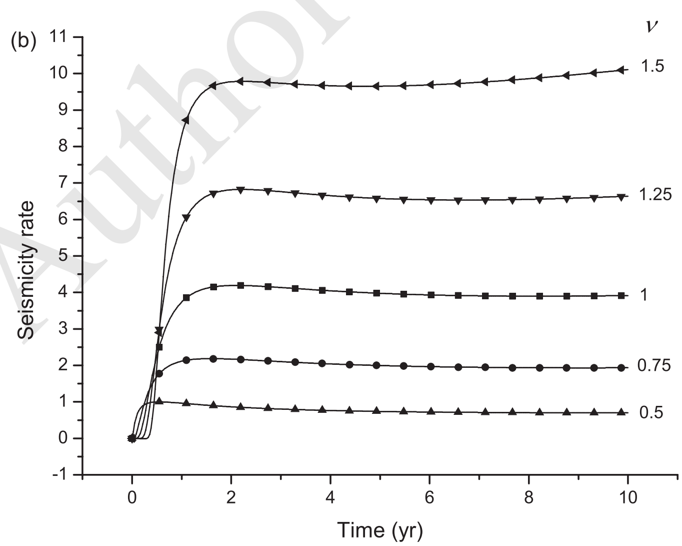


Fig. 4: Seismicity rates (normalized) for $\nu = 0.5$ (a) and $\eta = 0.05$ (b). The source is an impulse plus a constant rate.

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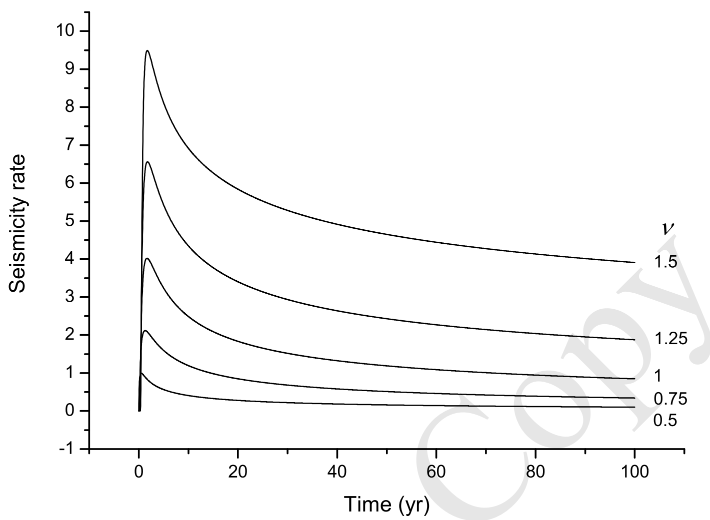

Fig. 5: Seismicity rates (normalized) for $\eta = 0$. The source is an impulse.

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Anomalous sequences occur if the background flow increases, contrary to Omori's law in the case of earthquakes. This could be the precursor of an incoming event, which is probably large if the increasing flow has lasted enough time or the rate is sufficiently high. The sequences are represented by aftershocks, i.e., an earthquake at t_0 relaxes the state of the seismicity rate, which decreases less rapidly after t_0. If the background flow rate has lasted enough time, the incoming earthquake could be large due to the accumulation of elastic energy, implying in this interpretation that the rate of flow and stress seem equivalent.

Appendix A. Stability and accuracy

Here, we analyse the numerical stability and accuracy of the numerical discretization. The kernel exp(ikx) replaced in equation (4.4) gives

$$r_n = -h^\nu d\ k^2 r_{n-1} - \sum_{j=1}^{J} (-1)^j \left(\begin{array}{c} \nu \\ j \end{array} \right) r_{n-j}. \tag{A.1}$$

Let us assume the relation

$$r_j = A r_{j-1}, \tag{A.2}$$

where A is the amplification factor. Then

$$A = -h^\nu d\ k^2 - \sum_{j=1}^{J} (-1)^j \left(\begin{array}{c} \nu \\ j \end{array} \right) A^{1-j}. \tag{A.3}$$

The von Neumann condition for stability implies

$$\max|A| \leq 1. \tag{A.4}$$

In particular, setting $A = -1$ and $k = \pi/\Delta x$, the Nyquist wavenumber, we obtain the following stability condition:

$$h \leq \left\{ \frac{\Delta x^2}{\pi^2 d} \left[1 + \sum_{j=1}^{J} \left(\begin{array}{c} \nu \\ j \end{array} \right) \right] \right\}^{1/\nu}. \tag{A.5}$$

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References

[1] O.P. Agrawal, Solution for a fractional diffusion-wave equation defined in a bounded domain. *Nonlinear Dynamics* **29** (2002), 145–155.

[2] M. Caputo, Linear models of dissipation whose Q is almost frequency independent-II. *Geophysical Journal of the Royal Astronomical Society* **3** (1967), 529–539; Reprinted in: *Fract. Calc. Appl. Anal.* **11**, No 1 (2008), 3–14.

[3] M. Caputo, Distributed order differential equations modelling dielectric induction and diffusion. *Fract. Calc. Appl. Anal.* **4**, No 4 (2001), 421–442.

[4] M. Caputo, Mathematical modelling of an extreme-event precursor. *Accademia delle Scienze di Ferrara* **85** (2008), 201–209.

[5] M. Caputo, The convergence of economic developments. *Studies in Non Linear Dynamics* & *Econometrics* **16**, No 2 (2012), 1558–3708.

[6] M. Caputo, R. Console, A.M. Gabrielov, V.I. Keilis-Borok, T.V. Sidorenko, Long term premonitory seismicity patterns in Italy. *Geophysical J. of the Royal Astronomical Society* **75**, No 1 (1983), 71–77.

[7] M. Caputo, P. Gasperini, V.I. Keilis-Bork, L. Marcelli, I. Rotwain, Earthquake's swarms as forerunners of strong earthquakes in Italy. *Annali di Geofisica* **30**, No 3-4 (1977), 269–283.

[8] J.M. Carcione, *Wave Fields in Real Media: Wave Propagation in Anisotropic, Anelastic, Porous and Electromagnetic Media*. Handbook of Geophysical Exploration, Vol. 38, Elsevier (2nd Ed., revised and extended) (2007).

[9] J.M. Carcione, F. Cavallini, F. Mainardi, A. Hanyga, Time-domain seismic modeling of constant Q-wave propagation using fractional derivatives. *Pure and Applied Geophysics* **159** (2002), 1719–1736.

Author's Copy

[10] J.M. Carcione, G. Herman, F.P.E. ten Kroode, Seismic modeling. *Geophysics* **67** (2002), 1304–1325.

[11] J.M. Carcione, C. Morency, J.E. Santos, Computational poroelasticity – A review. *Geophysics* **75** (2010), A229–A243.

[12] J.M. Carcione, Theory and modeling of constant-Q P- and S-waves using fractional time derivatives. *Geophysics* **74** (2009), T1–T11.

[13] F. Cesarone, M. Caputo, C. Cametti, Memory formalism in the passive diffusion across a biological membrane. *J. Membrane Science* **250** (2005), 79–84.

[14] D. del-Castillo-Negrete, B. A. Carreras, V. E. Lynch, Fractional diffusion in plasma turbulence. *Phys. Plasmas* **11** (2004), 3854–3864.

[15] K. Diethelm, *The Analysis of Fractional Differential Equations. An Application Oriented Exposition using Differential Operators of Caputo Type*. Lecture Notes in Mathematics No 2004, Springer, Heidelbereg (2010).

[16] M. El-Shahed, A fractional calculus model of semilunar heart valve vibrations. *Proceedings of DETC03*, Chicago (2003).

[17] R. Gorenflo, Yu. Luchko, F. Mainardi, Analytical properties and applications of the Wright function. *Fract. Calc. Appl. Anal.* **2**, No 4 (1999), 383–414.

[18] A.K. Grünwald, Über "begrenzte" Derivationen und deren Anwendung. *Zeitschrift für Angewandte Mathematik und Physik* **12** (1867), 441–480.

[19] A. Hanyga, Multidimensional solutions of time-fractional diffusion-wave equations. *Proc. of the Royal Society of London* **A458** (2002), 933–957.

[20] A. Helmstetter, D. Sornette, J. Grasso, Mainshocks are aftershocks of conditional foreshocks: How do foreshock statistical properties emerge from aftershock laws. *J. Geophys. Res.* **108** (B1) (2003), 2046; doi:10.1029/2002JB001991.

[21] G. Iaffaladano, M. Caputo, S. Martino, Experimental and theoretical memory diffusion of water in sand. *Hydrology amd Earth System Sciences* **10** (2006), 93–100.

[22] Z. Jiao, Y. Chen, I. Podlubny, *Distributed-Order Dynamic Systems: Stability, Simulation, Applications and Perspectives*. Springer (2012).

[23] H. Kanamori, E. E. Brodsky, The physics of earthquakes. *Rep. Prog. Phys.* **67** (2004), 1429–1496.

[24] V.I. Keilis-Borok, A.A. Soloviev, *Non Linear Dynamics of the Lithosphere and Earthquake Prediction*. Springer, Heidelberg (2003).

Author's Copy

[25] V.I. Keilis-Borok, A.A. Soloviev, C.B. Allégre, A.N. Sobolevskii, M.D. Intrilligator, Patterns of macroeconomic indicators preceding the unemployment rise in Western Europe and USA. *Pattern Recognition* **38**, No 3 (2005), 423–435.

[26] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam (2006).

[27] V. Kiryakova, *Generalized Fractional Calculus and Applications*. Longman Sci. & Techn., Harlow and J. Wiley & Sons, New York (1994).

[28] J. Kolari, M. Caputo, D. Wagner, Trait recognition: An alternative approach to early warning systems in commercial banking. *J. of Business Finance and Accounting* **23** (1996), 1415–1434.

[29] A.V. Letnikov, Theory of differentiation of fractional order. *Matematiceskij Sbornik* **3** (1868), 1–68 (in Russian).

[30] F. Mainardi, Fractional diffusive waves in viscoelastic solids. In: J.L. Wegner and F.R. Norwood (Eds.), *IUTAM Sympos. on Nonlinear Waves in Solids*, ASME Book No AMR 137, Fairfield – NJ (1995), 93–97.

[31] F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena. *Chaos, Solitons and Fractals* **7**, No 9 (1996), 1461–1477.

[32] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*. World Scientific Publishing (2010).

[33] F. Mainardi, A. Mura, G. Pagnini, R. Gorenflo, Time-fractional diffusion of distributed order. *J. of Vibration and Control* **14** (2008), 1267–1290.

[34] F. Mainardi, M. Raberto, R. Gorenflo, E. Scalas, Fractional calculus and continuous-time finance II: The waiting-time distribution. *Physica A* **287** (2000), 468–481.

[35] F. Mainardi, M. Tomirotti, Seismic pulse propagation with constant Q and stable probability distributions. *Annali di Geofisica* **40** (1997), 1311–1328.

[36] M. Naber, Distributed order fractional subdiffusion. *Fractals* **12**, No 1 (2004), 23–32.

[37] T. Parsons, Global Omori law decay of triggered earthquakes: Large aftershocks outside the classical aftershock zone. *J. Geophys. Res.* **107**, B9 (2002), 2199; doi:10.1029/2001JB000646.

[38] Z. Peng, J.E. Vidale, M. Ishii, A. Helmstetter, Seismicity rate immediately before and after main shock rupture from high-frequency waveforms in Japan. *J. Geophys. Res.* **112** (2007), B03306; doi:10.1029/2006JB004386.

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[39] I. Podlubny, *Fractional Differential Equations.* Academic Press, San Diego (1999).
[40] E. Scalas, R Gorenflo, F. Mainardi, Fractional calculus and continuous-time finance. *Physica A: Statistical Mechanics and its Applications* **284**, No 1 (2000), 376–384.
[41] G. Seriani, E. Priolo, J. M. Carcione, E. Padovani, High-order spectral element method for elastic wave modeling: High-order spectral element method for elastic wave modeling. In: *The 62nd Ann. Internat. Mtg., Soc. Expl. Geophys.*, Expanded Abstracts (1992), 1285–1288.
[42] D. Sornette, *Critical Phenomena in Natural Sciences.* Springer Verlag, Heidelberg (2002).
[43] B. Stanković, On the function of E.M. Wright, *Publ. de l'lnst. Mathématique, Beograd, Nouvelle Ser.* **10**, No 24 (1970), 113–124.
[44] T. Utsu, Y. Ogata, R.S. Matsuura, The centenary of the Omori formula for a decay law of aftershock activity. *J. Phys. Earth* **43** (1995), 1–33.

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