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ORIGINAL CONTRIBUTION

On the Kramers-Kronig relations

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Abstract

We provide a new derivation of the Kramers-Kronig relations on the basis of the Sokhotski-Plemelj equation with detailed mathematical justifications. The relations hold for a causal function, whose Fourier transform is regular (holomorphic) and square-integrable. This implies analyticity in the lower complex plane and a Fourier transform that vanishes at the high-frequency limit. In viscoelasticity, we show that the complex and frequency-dependent modulus describing the stiffness does not satisfy the relation but the modulus minus its high-frequency value does it. This is due to the fact that despite its causality, the modulus is not square-integrable due to a non-null instantaneous response. The relations are obtained in addition for the wave velocity and attenuation factor. The Zener, Maxwell, and Kelvin-Voigt viscoelastic models illustrate these properties. We verify the Kramers-Kronig relations on experimental data of sound attenuation in seabottoms sediments.

Keywords Kramers-Kronig relations · Sokhotski-Plemelj equation · Causality · Viscoelasticity · Waves · Zener model

Introduction

Viscoelastic attenuation and velocity dispersion is important in fields that involve wave propagation. This led to significant research in seismology, seismic wave propagation, and imaging in the oil and gas industry, non-destructive industrial evaluation based on ultrasonic waves, and medical imaging (e.g., Toksöz and Johnston 1981). The correct rheological equation is essential to describe the physics. Mechanical models provide the building blocks of the constitutive equation. Two basic elements are required: weightless springs—no inertial effects are present—that represent the elastic solid, and dashpots, consisting of loosely fitting pistons in cylinders filled with a viscous fluid. The Zener model, which combines a spring and a Kelvin-Voigt element (spring and dashpot connected in parallel), is the most suitable model for rocks and metals (Zener 1948; Liu et al. 1976). In addition, this model can be applied to electromagnetism, since the Debye model used to describe the behavior of dielectric materials is mathematically equivalent to the Zener model (Carcione 1999).

Proper models should satisfy the Kramers-Kronig relations. These relations are known from the beginning of the twentieth century from the works of Kronig and Kramers (1926; 1927), who developed them in the theory of electromagnetic wave propagation, showing the interrelation between the real and imaginary parts of the complex susceptibility. Later, it has been shown that the relations are of general nature and can be applied to a variety of systems, such as electrical, mechanical, and acoustical under certain conditions of causality, linearity, regularity, and square-integrability, which are the characteristics of real linear physical systems (Nussenzveig 1972; Stastna et al. 1985). The relations can therefore be used to connect the real and imaginary parts of the relevant frequency response function and provide a powerful tool for experimental and theoretical investigations of materials. For example, in wave propagation in anelastic media, knowing the phase velocity as a function of frequency, one can obtain the dependence of the attenuation factor, having important practical applications (Wang 2007). Due to the difficulty in computing Hilbert transforms, because the properties are not known for all frequencies, approximate (local) Kramers-Kronig relations were developed. These local relations relate the damping properties at one frequency to the rate of

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frequency variation of the dynamic modulus. Pritz (1999) studied their accuracy using a fractional Zener model.

In this work, we provide a complete derivation of the relations using the Sokhotski-Plemelj equation, showing explicitly what are the conditions for the relations to hold. Incidentally, we note that dispersion relations, and especially Kramers-Kronig formulae, have been considered an interesting topic even outside the community of physicists. For example, mathematicians have pointed out that their mathematical foundations rely upon Titchmarsh's theorem, which in turn is a synthesis of the Paley-Wiener theorem and the Marcel Riesz theorem (Labuda and Labuda 2014). Moreover, dispersion relations are one of the few important topics of "mature physics" in which the paradigm of causality plays a crucial role (Frisch 2009).

The Kramers-Kronig relations have a wide application in elastodynamics and electromagnetism. Our simple derivation of the relations yields more physical insight on the conditions for their validity, namely, causality, and smoothness. Moreover, assuming that the complex modulus satisfies the relations, their counterparts for attenuation and phase velocity are also deduced. Finally, a case study with experimental data confirms the intuitive view that viscoelastic models satisfying the relations better fit data over a large frequency band.

The Sokhotski-Plemelj equation

Consider the integral

$$\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty}\frac{\phi(\omega)}{\omega\pm\mathrm{i}\epsilon}d\omega, \tag{1}$$

where ϕ is a continuous function of the angular frequency ω. We can write as follows:

$$\frac{1}{\omega\pm\mathrm{i}\epsilon}=\frac{\mp\mathrm{i}\pi\epsilon}{\pi(\omega^2+\epsilon^2)}+\frac{\omega^2}{\omega(\omega^2+\epsilon^2)}. \tag{2}$$

Function $\epsilon/[\pi(\omega^2+\epsilon^2)]$ is a nascent delta function, so that in the limit $\epsilon\to 0^+$, it is equal to the Dirac delta function $\delta(\omega)$, while function $\omega^2/(\omega^2+\epsilon^2)$ is 1 in this limit. Using these results to compute limit (1), we obtain the formula

$$\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty}\frac{\phi(\omega)}{\omega\pm\mathrm{i}\epsilon}d\omega=\mp\mathrm{i}\pi\phi(\omega)+\mathcal{P}\int_{-\infty}^{\infty}\frac{\phi(\omega)}{\omega}d\omega, \tag{3}$$

where "$\mathcal{P}$" denotes the Cauchy principal value of the integral, since the argument can have a singularity. From Eq. 3, we can identify the Sokhotski-Plemelj operator as follows:

$$\frac{1}{\omega\pm\mathrm{i}\epsilon}=\mp\mathrm{i}\pi\delta(\omega)+\mathcal{P}\frac{1}{\omega}. \tag{4}$$

Equation 4 was obtained by Sokhotskii (1873) and rediscovered by Plemelj (1908).

The Kramers-Kronig relations

Let $g(t)$ be a causal function; then

$$g(t)=H(t)g(t), \tag{5}$$

where H is the Heaviside function. Taking the Fourier transform of Eq. 5 yields

$$\tilde{g}(\omega)=\frac{1}{2\pi}\tilde{H}(\omega)*\tilde{g}(\omega), \tag{6}$$

where the tilde means Fourier transform and the asterisk $(*)$ denotes convolution with respect to time, namely,

$$(g_1*g_2)(t)=\int_{-\infty}^{\infty}g_1(\tau)g_2(t-\tau)d\tau. \tag{7}$$

The transform convention is the following: $\tilde{g}(\omega)=\mathcal{F}(g)=\int g(t)\exp(-\mathrm{i}\omega t)dt$ and $g(t)=(2\pi)^{-1}\int\tilde{g}(\omega)\exp(\mathrm{i}\omega t)d\omega$. Other definitions have a different expression for (6) (e.g., Bracewell 1965). Using these definitions, the Fourier transform of the Heaviside function is as follows:

$$\begin{aligned}\tilde{H}(\omega)&=\lim_{\epsilon\to 0}\int_0^{\infty}\exp(-\epsilon t-\mathrm{i}\omega t)\mathrm{d}t\\&=\lim_{\epsilon\to 0}\left(\frac{1}{\mathrm{i}\omega+\epsilon}\right)=-\frac{\mathrm{i}}{\omega-\mathrm{i}0}.\end{aligned} \tag{8}$$

From the Sokhotski-Plemelj, Eqs. 4, 8 take the form as follows:

$$\tilde{H}(\omega)=\pi\delta(\omega)-\mathrm{i}\mathcal{P}\frac{1}{\omega}. \tag{9}$$

Replacing this transform into Eq. 6 yields

$$\tilde{g}(\omega)=-\frac{\mathrm{i}}{\pi}\left(\mathcal{P}\frac{1}{\omega}\right)*\tilde{g}(\omega). \tag{10}$$

Separating real and imaginary parts,

$$\tilde{g}=\tilde{g}_{\mathrm{R}}+\mathrm{i}\tilde{g}_{\mathrm{I}}, \tag{11}$$

we obtain from Eq. 10:

$$\begin{aligned}\tilde{g}_{\mathrm{R}}&=\frac{1}{\pi}\left(\mathcal{P}\frac{1}{\omega}\right)*\tilde{g}_{\mathrm{I}},\\ \tilde{g}_{\mathrm{I}}&=-\frac{1}{\pi}\left(\mathcal{P}\frac{1}{\omega}\right)*\tilde{g}_{\mathrm{R}},\end{aligned} \tag{12}$$

i.e.,

$$\begin{aligned}\tilde{g}_{\mathrm{R}}(\omega)&=\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\tilde{g}_{\mathrm{I}}(\omega')}{\omega-\omega'}d\omega',\\ \tilde{g}_{\mathrm{I}}(\omega)&=-\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\tilde{g}_{\mathrm{R}}(\omega')}{\omega-\omega'}d\omega',\end{aligned} \tag{13}$$

which are a formulation of the Kramers-Kronig relations (Golden and Graham 1988).

Since the Hilbert transform is defined as follows:

$$\mathcal{H}[\phi(\omega)]=\frac{1}{\pi}\left(\mathcal{P}\frac{1}{\omega}\right)*\phi(\omega), \tag{14}$$

the Kramers-Kronig relations (13) may be written compactly as follows:

$$\widetilde{g}_R = \mathcal{H}(\widetilde{g}_I),\\ \widetilde{g}_I = -\mathcal{H}(\widetilde{g}_R). \tag{15}$$

Each of Eq. 15 is equivalent to the other; indeed, the inverse of Hilbert transform $\mathcal{H}$ is just $-\mathcal{H}$. Causality implies analyticity in the lower complex plane. This is a necessary, but not sufficient condition for the validity of Kramers-Kronig relations and regularity requirements are needed, which refers to how smooth the function is in that domain. A smooth function is a function that has derivatives of all orders. It is said that the function is holomorphic, i.e., it is infinitely differentiable. If a function is holomorphic, it is analytic. In addition, the function must be square-integrable. This ensures that $\widetilde{g}$ decreases sufficiently rapidly at infinity; otherwise, $\widetilde{g}_R$ and $\widetilde{g}_I$ can be completely unrelated, as shown by the example: $\widetilde{g}(\omega) = a + \mathrm{i}b$ (a complex constant). The concept of regularity and square-integrability is widely treated by Nussenzveig (1960, 1972). For example, if $g = \delta(t)$ then $\widetilde{g}_R(\omega) = 1$ is nonzero while $\widetilde{g}_I(\omega)$ is zero, which clearly implies that (13) cannot hold in this case.

Viscoelasticity and wave propagation

The viscoelastic constitutive law may be expressed by Boltzmann's superposition principle in the form

$$\sigma = \dot{\psi} * \epsilon \tag{16}$$

(Golden and Graham 1988), where σ is stress, ψ is the time-dependent relaxation function, ϵ is strain, and a dot over a symbol denotes differentiation with respect to time. Equation 16 yields that

$$\dot{\epsilon} = \delta(t) \quad \text{implies} \quad \sigma = \psi * \delta = \psi. \tag{17}$$

Thus, ψ is the response to an impulsive forcing. Since δ is zero for negative times, the physical requirement of causality implies that ψ also is zero for negative times; therefore, its Fourier transform $\widetilde{\psi}$ satisfies in principle the Kramers-Kronig relations under additional conditions of regularity and square-integrability.

In the frequency domain, constitutive law (16) becomes

$$\widetilde{\sigma} = M\,\widetilde{\epsilon}, \quad M(\omega) = \mathcal{F}(\dot{\psi}) = \int_{-\infty}^{\infty} \dot{\psi}(t)\exp(-\mathrm{i}\omega t)\mathrm{d}t, \tag{18}$$

where the complex modulus $M(\omega)$ is the Fourier transform of the relaxation rate $\dot{\psi}$. We have seen that ψ is causal and $\widetilde{\psi}$ may satisfy the Kramers-Kronig relations. Thus, $\dot{\psi}$ is also causal, and one may expect that M satisfies the Kramers-Kronig relations as well; but we shall now see that it is not exactly so. Indeed, ψ may be factored as $\psi = gH$, where g is a smooth function, so that

$$\dot{\psi} = g\dot{H} + \dot{g}H = g\delta + \dot{g}H = g(0)\delta + \dot{g}H. \tag{19}$$

Fourier transforming this equation, we get

$$M = g(0) + \mathcal{F}(\dot{g}H), \tag{20}$$

where $g(0) = M(\infty)$ by Parseval's theorem (Bracewell 1965). If

$$\eta = \dot{g}H = \dot{\psi} - g(0)\delta \tag{21}$$

is causal and smooth, its Fourier transform $\mathcal{F}(\eta) = M - g(0) = M - M(\infty)$ satisfies Kramers-Kronig relations, but M does not, since we have seen above that the Fourier transform of the δ function in Eq. 19 does not satisfy the Kramers-Kronig relations. Therefore, we conclude, because of the linearity of Kramers-Kronig relations, that M does not satisfies the Kramers-Kronig relations, unless $M(\infty) = 0$.

Another argument is that a necessary condition for the Kramers-Kronig relations (15) to hold is that the frequency-domain complex function $\widetilde{g}(\omega)$ should approach zero as ω approaches ∞; $M(\omega) - M(\infty)$ does it, but $M(\omega)$ is equal to $M(\infty)$. Then, the Kramers-Kronig relations for stiffness are as follows:

$$M_R - M(\infty) = \mathcal{H}(M_I),\\ M_I = -\mathcal{H}[M_R - M(\infty)]. \tag{22}$$

Therefore, if the real and imaginary parts of $M - M(\infty)$ are Hilbert transform pairs, the relaxation function is causal. On the other hand, causality implies that the frequency response function is analytic on the lower half complex ω-plane, i.e., this function is infinitely differentiable and cannot have poles there. Any pole should be located in the upper half plane. Let us put this in mathematical language. A causal function $\beta(t)$ has the Fourier transform

$$\widetilde{\beta}(\omega) = \int_0^{\infty} \beta(t)\exp(-\mathrm{i}\omega t)\mathrm{d}t. \tag{23}$$

If $\omega = \omega_R + \mathrm{i}\omega_I$, we have that for the lower half plane ($\omega_I \le 0$)

$$\widetilde{\beta}(\omega) = \int_0^{\infty} \beta(t)\exp(-\mathrm{i}\omega_R t)\exp(-|\omega_I|t)\mathrm{d}t \tag{24}$$

and the integral has a finite value because the factor $\exp(-|\omega_I|t)$ tends to zero for $t \to \infty$. Moreover, on differentiating with respect to ω this factor ensures that all the derivatives of $\widetilde{\beta}(\omega)$ are also finite. This means that the function is analytic in the lower half ω-plane. This property implies that there are no poles on this plane, because the function would be singular at a pole. It also implies that there are no branch points on this plane so that a causal function is single valued there. For $M(\infty) \neq 0$, function M is analytic but it is not square-integrable, while $M - M(\infty)$ is analytic and square-integrable.

Square-integrability of $M(\omega)$ along the real axis of the ω-plane implies

$$\int_{-\infty}^{\infty} |M(\omega)|^2 d\omega < C, \tag{25}$$

where C is a constant (Nussenzveig 1972). Square-integrability is equivalent to $M(\omega) \to 0$, for $|\omega| \to \infty$ ($\pi \geq \arg(\omega) \geq 0$). In most cases, the square-integrability condition cannot be satisfied, but rather the weaker condition that $|M(\omega)|$ is bounded is satisfied, i.e., $|M(\omega)|^2 < C$ is verified. A lossless medium and the Maxwell and Zener models satisfy this weak condition, but the Kelvin-Voigt and constant-Q models do not. In fact, in the case of the Zener model, M satisfies the weak condition and $M - M(\infty)$ is square-integrable. A constant-Q model has $M(\omega) \propto \omega^{2\gamma}$, where $0 < \gamma < 1/2$ (Carcione 2014, Eq. 2.212) and does not satisfy the conditions. All these concepts will be clear in the example below.

Kramers-Kronig relations for velocity and attenuation

The Kramers-Kronig relations can be applied to wave propagation in anelastic media, where the complex slowness of plane waves can be written as follows:

$$s = \frac{1}{c} = \frac{1}{c_p} - \mathrm{i}\frac{\alpha}{\omega}, \tag{26}$$

where c_p is the wave phase velocity and α is the attenuation factor. Since

$$s = \sqrt{\frac{\rho}{M}}, \tag{27}$$

where ρ is the mass density, is hermitian, let us identify $1/c_p - 1/c_\infty$ with $\tilde{g}_{\mathrm{R}} = [M_{\mathrm{R}} - M(\infty)]$ and $-\alpha/\omega$ with $\tilde{g}_{\mathrm{I}} = M_{\mathrm{I}}$, where $c_\infty = c_p(\omega = \infty)$ is the unrelaxed velocity. Performing the same mathematical developments to obtain Eq. 13, we get the following:

$$\frac{1}{c_p(\omega)} - \frac{1}{c_\infty} = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\alpha(\omega')d\omega'}{\omega'(\omega - \omega')} \tag{28}$$

and

$$\alpha(\omega) = \frac{\omega}{\pi}\,\mathcal{P}\int_{-\infty}^{\infty} \left(\frac{1}{c_p(\omega')} - \frac{1}{c_\infty}\right)\frac{d\omega'}{\omega - \omega'}. \tag{29}$$

Another form of these relations can be obtained as follows. Let us define $\bar{s} = s - s_\infty$, where $s_\infty = 1/c_\infty$. Because of the hermitian property $\bar{s}(-\omega) = \bar{s}^*(\omega)$, Eq. 10 implies

$$\bar{s} = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\bar{s}d\omega'}{\mathrm{i}(\omega - \omega')}. \tag{30}$$

Splitting the integral into two from 0 to ∞ and $\bar{s}$ into its real and imaginary parts, we obtain the following:

$$\bar{s} = \frac{2}{\pi}\mathcal{P}\int_0^{\infty} \frac{\bar{s}_{\mathrm{I}}\omega' - \mathrm{i}\bar{s}_{\mathrm{R}}\omega}{\omega^2 - \omega'^2}d\omega'. \tag{31}$$

Then,

$$\frac{1}{c_p(\omega)} - \frac{1}{c_\infty} = -\frac{2}{\pi}\mathcal{P}\int_0^{\infty} \frac{\alpha(\omega')d\omega'}{\omega^2 - \omega'^2} \tag{32}$$

and

$$\alpha(\omega) = \frac{2\omega}{\pi}\,\mathcal{P}\int_0^{\infty} \left(\frac{1}{c_p(\omega')} - \frac{1}{c_\infty}\right)\frac{\omega d\omega'}{\omega^2 - \omega'^2}. \tag{33}$$

On the other hand, in terms of the quality factor

$$Q = \frac{|\omega|}{2\alpha c_p} \tag{34}$$

(valid for low-loss solids, i.e., $Q \gg 1$) (Carcione 2014; p. 79), the Kramers-Kronig relations (28) and (29) become

$$\frac{1}{c_p(\omega)} - \frac{1}{c_\infty} = -\frac{1}{2\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{|\omega'|}{\omega' c_p(\omega')Q(\omega')}\frac{d\omega'}{\omega - \omega'} \tag{35}$$

and

$$\frac{1}{Q(\omega)} = \frac{2\omega c_p(\omega)}{\pi|\omega|}\,\mathcal{P}\int_{-\infty}^{\infty} \left(\frac{1}{c_p(\omega')} - \frac{1}{c_\infty}\right)\frac{d\omega'}{\omega - \omega'}. \tag{36}$$

Velocity dispersion, i,e., the difference between the low- and high-frequency phase velocities increases for increasing attenuation. The next example illustrates this relation.

Example. The Zener model

A classical model of viscoelastic behavior is the Zener model (Fig. 1 shows the three basic mechanical models), which is defined by the differential equation

$$\sigma + \tau\dot{\sigma} = M_0\epsilon + M_\infty\tau\dot{\epsilon}, \tag{37}$$

or, equivalently,

$$\psi = gH,\ \dot{\psi} = M_\infty\delta + \frac{1}{\tau}(M_0 - g)H,$$

$$M(\omega) = M_\infty - \frac{M_\infty - M_0}{1 + \mathrm{i}\omega\tau}, \tag{38}$$

$$g(t) = M_0 + (M_\infty - M_0)\exp\left(-\frac{t}{\tau}\right) \tag{39}$$

and

$$\eta(t) = -\frac{1}{\tau}(M_\infty - M_0)\exp\left(-\frac{t}{\tau}\right)H, \tag{40}$$

where τ is a relaxation time, $M_0 = M(0)$ and $M_\infty = M(\infty)$ and $M_\infty \geq M_0$ holds (e.g., Carcione 2014). Note that $M - M_\infty$ satisfies condition (25) because, in this case, the integral in Eq. 25 is equal to $\pi(M_\infty - M_0)^2/\tau$. The Zener model is shown to satisfy the Kramers-Kronig relations in Appendix.

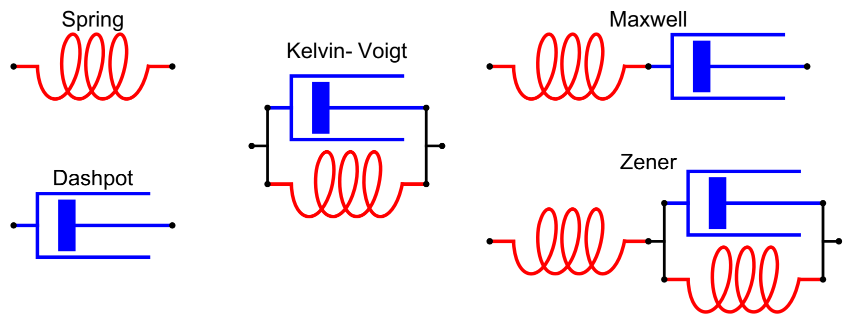

Fig. 1 Maxwell, Kelvin-Voigt and Zener models consisting of springs and dashpots

Function $(M - M_\infty)(\omega)$ has a unique pole in the upper half ω-plane, i.e., at

$$\omega = \frac{\mathrm{i}}{\tau}, \tag{41}$$

and therefore it is analytic in the lower half ω-plane as required by causality. Its inverse Fourier transform $\eta(t)$ is causal and smooth for $t > 0$, since it is basically an exponential function of time.

The quality factor is defined as follows:

$$Q(\omega) = \frac{M_R}{M_I} = \frac{M_0 + M_\infty(\omega\tau)^2}{\omega\tau(M_\infty - M_0)} \tag{42}$$

Carcione (2014, p. 91), which has a minimum at

$$Q_0 = \frac{2\sqrt{M_\infty M_0}}{M_\infty - M_0} = \frac{2c_\infty c_0}{c_\infty^2 - c_0^2} \tag{43}$$

Carcione (2014, p. 96), where we have defined the phase velocities at zero and infinite frequency as c_0 and c_∞, such that $M_0 = \rho c_0^2$ and $M_\infty = \rho c_\infty^2$, where ρ is the mass density. Then, it is easy to show that the amount of velocity dispersion is as follows:

$$\Delta c = c_\infty - c_0 = c_0 \left(Q_0^{-1} + \sqrt{1 + Q_0^{-2}} - 1 \right) \approx \frac{c_0}{Q_0}, \tag{44}$$

where the approximation holds for low-loss solids ($Q_0 \gg 1$). This is a simple relation between the maximum velocity dispersion and the minimum Q (higher attenuation). The Kramers-Kronig relations are more general and reflect the fact that if velocity dispersion is known for all frequencies then Q is known for all frequencies and vice versa.

Test of the Kramers-Kronig relations

It is shown in this section the consistency of the Kramers-Kronig relations applied to experimental data of wave propagation obtained for rocks. We consider the data used by Zhou et al. (2009) in their Figs. 12 and 13. These data are consistent with the Kramers-Kronig relationships, where the best fit is given by the Zener model (Fig. 2, black-solid line). Model parameters can be obtained from experimental data by best fitting the formula for the pertinent variable (e.g., complex modulus, attenuation, phase velocity) using a least-square criterion. One may also look for a "compromise solution" by minimizing the sum of the misfits, each one

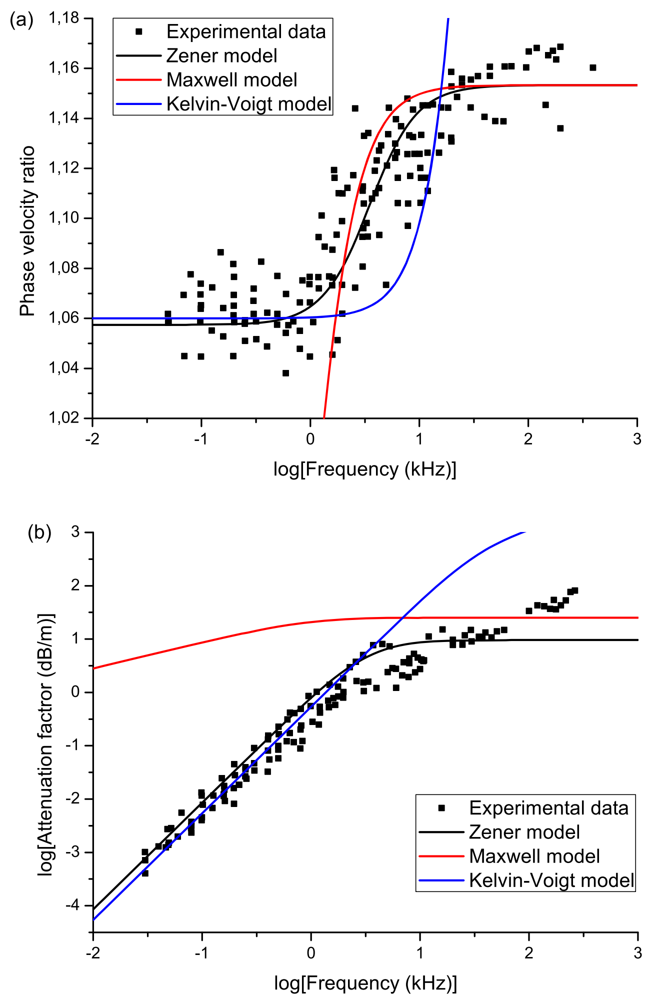

Fig. 2 Experimental phase velocity ratio and attenuation factor (symbols) (Zhou et al. 2009) compared with the mechanical model results (solid lines) (**a** , **b**). The Zener model provides the best fit with $\tau = 4.17 \times 10^{-5}$ s, $Q_0 = 11.5$ and $c_\infty = 1730$ m/s. These values correspond to a relaxation peak centered at 3.5 kHz. The parameters of the Maxwell model are $\tau = 0.0001$ s and $c_\infty = 1730$ m/s, and the Kelvin-Voigt model has $\tau = 5 \times 10^{-6}$ s and $c_0 = 1590$ m/s. The water sound speed for the normalization of the phase velocity is 1500 m/s

normalized by variance. The Maxwell and Kelvin-Voigt model give a good fit of the velocity at the high- and low-frequency limits, respectively, whereas attenuation is best described by the second model. Phase velocity and attenuation have been obtained as follows:

$$c_p = \frac{1}{s_\mathrm{R}} = \left(\sqrt{\frac{\rho}{M}}\right)_\mathrm{R}^{-1} \tag{45}$$

and

$$\alpha = -8.686\,\omega s_\mathrm{I} = -8.686\,\omega \left(\sqrt{\frac{\rho}{M}}\right)_\mathrm{I}, \tag{46}$$

where s is the slowness (27) and M is given by Eq. 38.

The Zener model does not capture the high-frequency behavior of the attenuation. A porous medium model, such one based on the Biot theory, that includes a high-frequency viscodynamic operator to describe deviations of the fluid flow from the Poiseuille regime, might correct this deficiency (e.g., Carcione 2014).

The Kelvin-Voigt model asymptotically approximates Zener model at low frequencies, likewise for Maxwell model at high frequencies. This is evident in Fig. 2. The higher flexibility of Zener model is related to the fact that it has three free parameters, whereas the other two models depend on two parameters only.

Conclusions

We have provided alternative mathematical demonstrations of the Kramers-Kronig relations between the real and imaginary parts of the complex stiffness modulus, as well as between the phase velocity dispersion and the attenuation and quality factors. The conditions to satisfy the relations have been clearly analyzed, i.e., causality (analyticity), linearity, regularity, and square-integrability of the stiffness modulus, $M = M_\mathrm{R} + \mathrm{i}M_\mathrm{I}$; basically $M_\mathrm{R} - M(\infty)$ and M_I are the correct Hilbert transform pairs, where the infinity refers to time frequency. The examples are illustrative: the Zener and Maxwell models satisfy the relations while the Kelvin-Voigt and constant-Q models do not. The Zener model is shown to fit experimental data (wave velocity and attenuation) obtained for ocean bottom sediments.

The classical derivation of the relations and its variants are thoroughly discussed in the referenced paper by Labuda and Labuda. These approaches are based on the theory of complex holomorphic functions and requires quite involved computations and arguments. Our derivation uses the modern formulation of the Sokhotski-Plemelj formula and of harmonic analysis, which in turn relies on the theory of distributions (also called "generalized functions") developed a few decades after the publication of the original papers by Kramers and Kronig.

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Appendix: the Zener and Maxwell models verify the Kramers-Kronig relations

In this Appendix we derive, first, a very compact unified formulation of Kramers-Kronig relations, using Hilbert transform. Then we show, through detailed *ad hoc* computations, that the reduced complex modulus of the Zener model verifies this condition.

A.1 Compact formulation of Kramers-Kronig relations

As we have seen in the main text, Kramers-Kronig relations for a complex-valued function f of a real variable ω may be expressed as follows:

$$\begin{cases} \mathrm{Re} f &= \mathcal{H}(\mathrm{Im} f) \\ \mathrm{Im} f &= -\mathcal{H}(\mathrm{Re} f) \end{cases} \tag{47}$$

where symbol $\mathcal{H}$ denotes Hilbert transform.

Summing $(1)_1$ and $(1)_2$ multiplied by the imaginary unit i, and rearranging, we obtain the following:

$$f = -\mathrm{i}\,\mathcal{H}(f) \tag{48}$$

where we have used the linearity of Hilbert transform.

Conversely, taking real and imaginary parts of Eq. 48, we obtain (47). Therefore, the single condition (48) is equivalent to Kramers-Kronig relations.

A.2 Zener model

We recall that the complex modulus of the Zener model may be expressed as follows:

$$M(\omega) = M_\infty - \frac{M_\infty - M_0}{1 + \mathrm{i}\,\tau\,\omega} \tag{49}$$

Now, we show that the reduced complex modulus

$$\overline{M}(\omega) = M(\omega) - M_\infty = -\frac{M_\infty - M_0}{1 + \mathrm{i}\,\tau\,\omega} \tag{50}$$

verifies formulation (48) of Kramers-Kronig relations, i.e.,

$$\overline{M}(\omega) = (M_\infty - M_0)\,\mathrm{i}\,\mathcal{H}\left(\frac{1}{1 + \mathrm{i}\,\tau\,\omega}\right) \tag{51}$$

The Hilbert transform of a function $f(\omega)$ is given by the following:

$$\begin{aligned}\mathcal{H}f(\omega) &= \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} g(\omega,\omega')\,\mathrm{d}\omega' = \frac{1}{\pi}\lim_{L\to\infty}\mathcal{P}\int_{-L}^{L} g(\omega,\omega')\,\mathrm{d}\omega' \\ &= \frac{1}{\pi}\lim_{L\to\infty}\lim_{\epsilon\to 0^+}\left(\int_{-L}^{\omega-\epsilon} g(\omega,\omega')\,\mathrm{d}\omega' + \int_{\omega+\epsilon}^{L} g(\omega,\omega')\,\mathrm{d}\omega'\right) \\ &= \frac{1}{\pi}\lim_{L\to\infty}\lim_{\epsilon\to 0^+}[G(\omega,\omega-\epsilon) - G(\omega,-L) \\ &\quad + G(\omega,L) - G(\omega,\omega+\epsilon)]\end{aligned} \tag{52}$$

where

$$g(\omega,\omega') = \frac{f(\omega')}{\omega-\omega'} \tag{53}$$

and G is a primitive of the integrand in Eq. 52, i.e.,

$$G(\omega,\omega') = \int g(\omega,\omega')\,\mathrm{d}\omega' \tag{54}$$

Therefore, the Hilbert transform in Eq. 51 is given by the following:

$$\mathcal{H}\left(\frac{1}{1+\mathrm{i}\,\tau\,\omega}\right) = \frac{1}{\tau\,\omega-\mathrm{i}} \tag{55}$$

as shown in the Lemma below.

Substituting (55) into (51) and rearranging, one obtains an identity, which proves that the reduced complex modulus $\overline{M}(\omega)$ verifies Kramers-Kronig relations.

Lemma *The Hilbert transform of function $f(\omega) = 1/(1+\mathrm{i}\,\tau\,\omega)$ is given by the following:*

$$\mathcal{H}\left(\frac{1}{1+\mathrm{i}\,\tau\,\omega}\right) = \frac{1}{\tau\,\omega-i} \tag{56}$$

Proof We rely on identity (52), where now

$$g(\omega,\omega') = \frac{1}{(\omega-\omega')\,(1+\mathrm{i}\,\tau\,\omega')} \tag{57}$$

and hence its primitive is

$$G(\omega,\omega') = \frac{2\,\tan^{-1}(\tau\,\omega') + 2\,\mathrm{i}\,\ln(\omega'-\omega) - \mathrm{i}\,\ln[1+(\tau\,\omega')^2]}{2\,(\tau\,\omega-\mathrm{i})} \tag{58}$$

as can be checked by differentiating with respect to ω'. Substituting (58) into (52), one obtains (56). Computations are cumbersome, but straightforward: indeed two divergent terms appear, $\ln(\epsilon)$ and $\ln(-\epsilon)$, but they are present only in the combination $\ln(\epsilon) - \ln(-\epsilon) = \ln(-1) = \mathrm{i}\,\pi$.

A.3 Maxwell model

The Maxwell model is obtained from Fig. 1 by removing the parallel spring. Its reduced complex modulus is as follows:

$$\overline{M}(\omega) = M(\omega) - M_\infty = -\frac{M_\infty}{1+\mathrm{i}\,\tau\,\omega} \tag{59}$$

(e.g., Carcione 2014). From Eqs. 48 and 56, we have the following:

$$\overline{M}(\omega) = -\mathrm{i}\,\mathcal{H}(\overline{M}) = \mathrm{i}M_\infty\mathcal{H}\left(\frac{1}{1+\mathrm{i}\,\tau\,\omega}\right) = \frac{\mathrm{i}M_\infty}{\tau\,\omega-\mathrm{i}} = \overline{M}(\omega), \tag{60}$$

so we have shown that the Maxwell model satisfies the relations.

A.4 Kelvin-Voigt model

The Kelvin-Voigt model is obtained from Fig. 1 by removing the series spring. Its complex modulus is as follows:

$$M(\omega) = M_0(1+\mathrm{i}\,\tau\,\omega) \tag{61}$$

(e.g., Carcione 2014). $M_\infty = \infty$ in this case. We may consider $\overline{M} = \mathrm{i}\,\tau\,\omega\,M_0$, since the Hilbert transform of a constant (M_0) is zero. This is the complex modulus of the dashpot.

The primitive function is as follows:

$$G(\omega,\omega') = -\mathrm{i}\,\tau\,M_0[\omega' + \omega\ln(\omega'-\omega)]. \tag{62}$$

It can be easily seen that the calculation (52) diverges, and therefore the Kelvin-Voigt solid does not satisfy the relations. Another explanation may be found by considering the relaxation rate $\dot{\psi} = M_0\delta + M_0\tau\delta$, which is the inverse Fourier transform of the complex modulus. One immeditely sees that it is causal, but both of its terms are singular distributions. Thus, there is no way, because of the lack of regularity, to introduce a "reduced complex modulus" as in the previous cases. Equivalently, one may notice that both terms in the complex modulus $M = M_0 + \mathrm{i}\omega\tau M_0$ are not square-integrable.

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