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Time-domain Modeling of Constant-Q Seismic Waves Using Fractional Derivatives

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Abstract — Kjartansson's constant-Q model is solved in the time-domain using a new modeling algorithm based on fractional derivatives. Instead of time derivatives of order 2, Kjartansson's model requires derivatives of order 2γ, with $0 < \gamma < 1/2$, in the dilatation-stress formulation. The derivatives are computed with the Grünwald-Letnikov and central-difference approximations, which are finite-difference extensions of the standard finite-difference operators for derivatives of integer order. The modeling uses the Fourier method to compute the spatial derivatives, and therefore can handle complex geometries. A synthetic cross-well seismic experiment illustrates the capabilities of this novel modeling algorithm.

Key words: Viscoelastic waves, fractional calculus, numerical modeling, seismology.

1. Introduction

Constant-Q models provide a good parameterization of seismic attenuation in rocks, in oil exploration and seismology. By reducing the number of parameters they allow an improvement of seismic inversion. Moreover, there is physical evidence that attenuation is almost linear with frequency (therefore Q is constant) in many frequency bands. BLAND (1960) and KJARTANSSON (1979) discuss a linear attenuation model with the required characteristics, but the idea is much older (SCOTT-BLAIR, 1949). Kjartansson's constant-Q model is based on a creep function of the form $t^{2\gamma}$, where t is time and $\gamma \ll 1$ for seismic applications. This model is completely specified by two parameters, i.e., phase velocity at a reference frequency and Q. Therefore, it is mathematically far simpler than any nearly constant Q, as for instance, a spectrum of Zener models (CARCIONE *et al.*, 1988). Due to its simplicity, Kjartansson's model is used in many seismic applications, mainly in its frequency-domain form. MAINARDI

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and TOMIROTTI (1997) interpreted the constant-Q model in terms of fractional derivatives and obtained its 1-D Green's function.

Seismic modeling in inhomogeneous media can, in principle, be performed in the frequency domain. However, the method is expensive when using differential formulations, since it involves solution of many Helmholtz equations. The alternative is to compute the solution through a time-convolution, although the resulting algorithm is relatively expensive. A purely differential – as opposed to integro-differential – formulation can be obtained by using fractional derivatives (CAPUTO and MAINARDI, 1971). Instead of time derivatives of order 2, Kjartansson's model requires derivatives of order $2 - 2\gamma$ with $0 < \gamma < 1/2$ in the dilatation formulation of the wave equation, and 2γ in the dilatation-stress formulation. The equation becomes parabolic since the phase velocity has no upper bound. Fractional derivatives appear also in Biot theory, related to memory effects in porous rocks at seismic frequencies with $\gamma = 1/4$ (GUREVICH and LOPATNIKOV, 1995) and at low- and high-frequency limits (FELLAH and DEPOLLIER, 2000). Fractional derivatives can be computed with the Grünwald-Letnikov and central-difference approximations, which are finite-difference extensions of the standard finite-difference approximation for derivatives of integer order (GRÜNWALD, 1867; LETNIKOV, 1868, GORENFLO, 1997). Unlike the standard operator of differentiation, the fractional operator increases in length as time increases, since it must keep the memory effects. However, after a given time period the operator can be truncated (short memory principle).

In the first part of this work we review the constant-Q model and calculate the complex modulus, phase velocity, and attenuation factor versus frequency. We then recast the acoustic wave equation in the time-domain in terms of fractional derivatives, and obtain the Grünwald-Letnikov and central-difference approximations. Then, we investigate the accuracy of the time discretization by comparing the exact and the finite-difference (FD) phase velocities and attenuation factors. The model is discretized on a mesh, and the spatial derivatives are calculated with the Fourier method by using the Fast Fourier Transform. This approximation is infinitely accurate for band-limited periodic functions with cutoff spatial wavenumbers smaller than the cutoff wavenumbers of the mesh. Finally, we test the modeling algorithms with an analytical solution for a 2-D homogeneous medium, and illustrate the method with seismic applications in inhomogeneous media.

2. Constant-Q Model

2.1. Stress-strain Relation

Stress σ and strain ϵ in a 1-D linear anelastic medium are related by a convolutional relation (BLAND, 1980),

$$\sigma(t) = \psi(t) * \dot{\epsilon}(t) \tag{1}$$

where ψ is the relaxation function, t is the time variable, the symbol "$*$" denotes time convolution, and a dot above a variable indicates time differentiation.

Let us define the relaxation function (KJARTANSSON, 1979)

$$\psi(t) = \frac{M_0}{\Gamma(1-2\gamma)} \left(\frac{t}{t_0}\right)^{-2\gamma} H(t) \ , \tag{2}$$

where M_0 is a bulk modulus, Γ is Euler's Gamma function, t_0 is a reference time, γ is a dimensionless parameter, and H is the Heaviside step function. The parameters M_0, t_0 and γ have precise physical meanings, which will become clear in the following analysis. Let us take the Fourier transform of equation (1). We obtain

$$\tilde{\sigma}(\omega) = \mathscr{F}(\dot{\psi}(t))\tilde{\epsilon}(\omega) \equiv M(\omega)\tilde{\epsilon}(\omega) \ , \tag{3}$$

where $\mathscr{F}$ is the Fourier transform operator, $M(\omega)$ is the complex modulus, and a tilde denotes the Fourier transform. After some calculations we gain

$$M(\omega) = M_0 \left(\frac{\mathrm{i}\omega}{\omega_0}\right)^{2\gamma} \ , \tag{4}$$

where $\omega_0 = 1/t_0$ is the reference frequency.

2.2. *Phase Velocity and Attenuation Factor*

The complex velocity is

$$V = \sqrt{\frac{M}{\rho}} \ , \tag{5}$$

where ρ is the density. The phase velocity c is the frequency ω divided by the real part of the complex wavenumber. Then,

$$c = \left[\mathrm{Re}\left(\frac{1}{V}\right)\right]^{-1} \ . \tag{6}$$

Substituting equations (4) and (5) in (6) yields

$$c = c_0 \left|\frac{\omega}{\omega_0}\right|^{\gamma} \tag{7}$$

with

$$c_0 = \sqrt{\frac{M_0}{\rho}} \left[\cos\left(\frac{\pi\gamma}{2}\right)\right]^{-1} \ . \tag{8}$$

The attenuation factor is given by

$$\alpha = -\omega \ \mathrm{Im}\left(\frac{1}{V}\right) = \tan\left(\frac{\pi\gamma}{2}\right)\mathrm{sgn}(\omega)\frac{\omega}{c} \quad . \tag{9}$$

The quality factor is defined as the peak energy stored during a cycle divided by the energy loss during the cycle. It is given by (e.g., CARCIONE and CAVALLINI, 1994)

$$Q = \frac{\mathrm{Re}(V^2)}{\mathrm{Im}(V^2)} = \frac{1}{\tan(\pi\gamma)} \quad . \tag{10}$$

Firstly, we derive from equation (7) that c_0 is the phase velocity at $\omega = \omega_0$, the reference frequency, and that

$$M_0 = \rho c_0^2 \cos^2\left(\frac{\pi\gamma}{2}\right) \quad . \tag{11}$$

Secondly, it follows from equation (10) that Q is independent of frequency, so that

$$\gamma = \frac{1}{\pi}\tan^{-1}\left(\frac{1}{Q}\right) \tag{12}$$

parameterizes the attenuation level. Hence we see that $Q > 0$ is equivalent to $0 < \gamma < 1/2$. Moreover, $c \to 0$ when $\omega \to 0$, and $c \to \infty$ when $\omega \to \infty$. It follows that very high frequencies of the signal propagate at almost infinite velocity, and the differential equation describing the wave motion is parabolic (e.g., PRÜSS, 1993).

2.3. *Wave Equation in Differential Form*

Let us consider a 2-D wave equation of the form

$$\frac{\partial^\beta w}{\partial t^\beta} = D\Delta w + f \quad , \tag{13}$$

where $w(x, z, t)$ is a field variable, β is the order of the time derivative, D is a positive parameter, Δ is the 2-D Laplacian operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \quad , \tag{14}$$

and f is a forcing term. Consider a plane wave

$$\exp[\mathrm{i}(\omega t - k_x x - k_z z)] \quad , \tag{15}$$

with ω real and (k_x, k_z) the complex wavevector. Substituting the ansatz (15) in the wave equation (13) with $f = 0$ yields the dispersion equation

$$(\mathrm{i}\omega)^\beta + Dk^2 = 0 \quad , \tag{16}$$

where $k = (k_x^2 + k_z^2)^{1/2}$ is the complex wavenumber. Equation (16) is the Fourier transform of equation (13). The properties of the Fourier transform when it acts on fractional derivatives are well established, and a rigorous treatment is available in the literature (e.g., DATTOLI *et al.*, 1998). Since $k^2 = \rho\omega^2/M$, comparison of equations (16) and (4) yields

$$\beta = 2 - 2\gamma, \qquad \text{and} \qquad D = \frac{M_0}{\rho}\omega_0^{-2\gamma} \ . \tag{17}$$

Equation (13), together with (17), is the wave equation corresponding to Kjartansson's stress-strain relation (KJARTANSSON, 1979). In order to obtain realistic values of the quality factor, corresponding to wave propagation in rocks, $\gamma \ll 1$ and the time derivative in equation (13) has a fractional order.

Kjartansson's wave equation (13) is a particular version of a more general wave equation for variable material properties. The convolutional constitutive equation (3) can be written in terms of fractional derivatives. In fact, it is easy to show, using equations (4) and (17), that it is equivalent to

$$\sigma = \rho D \frac{\partial^{2-\beta}\epsilon}{\partial t^{2-\beta}} \ . \tag{18}$$

Coupled with the constitutive equation (18) are the momentum equations

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u_x}{\partial t^2} \ , \tag{19}$$

$$\frac{\partial \sigma}{\partial z} = \rho \frac{\partial^2 u_z}{\partial t^2} \ , \tag{20}$$

where u_x and u_z are the displacement components. Redefining

$$\epsilon = \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \tag{21}$$

as the dilatation field, differentiating and adding equations (19) and (20), the substitution of equation (18) yields

$$\Delta_\rho\left(\rho D \frac{\partial^{2-\beta}\epsilon}{\partial t^{2-\beta}}\right) = \frac{\partial^2 \epsilon}{\partial t^2} \ , \tag{22}$$

where

$$\Delta_\rho = \frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\frac{1}{\rho}\frac{\partial}{\partial z} \ . \tag{23}$$

Multiplying by $(\mathrm{i}\omega)^{\beta-2}$ the Fourier transform of equation (22) produces, after an inverse Fourier transform, the inhomogeneous wave equation

$$\frac{\partial^\beta \epsilon}{\partial t^\beta} = \Delta_\rho(\rho D \epsilon) + s \ , \tag{24}$$

where we included the seismic source s. This equation is of type (13) if the medium is homogeneous.

3. Numerical Algorithm

3.1. Grünwald-Letnikov and Central-difference Approximations to the Fractional Derivative

The Grünwald-Letnikov and central-difference approximations to the Riemann-Liouville fractional derivative of a function f are

$$\frac{\partial^\nu f(t)}{\partial t^\nu} \sim \frac{1}{h^\nu} \sum_{j=0}^{J} (-1)^j \binom{\nu}{j} f(t - jh) \tag{25}$$

and

$$\frac{\partial^\nu f(t)}{\partial t^\nu} \approx \frac{1}{h^\nu} \sum_{j=0}^{J} (-1)^j \binom{\nu}{j} f\left[t + \left(\frac{\nu}{2} - j\right)h\right] \ , \tag{26}$$

respectively, where h is the time step, and $J = t/h - 1$. These expressions are derived in Appendix A. They are first- and second-order accurate, respectively. The fractional derivative of f at time t depends on all the previous values of f. This is the memory property of the fractional derivative, related to field attenuation in our particular example. However, the binomial coefficients $\binom{\nu}{j}$ are negligible for j exceeding an integer J. This allows us to use the short-memory principle and hence to replace $\sum_{j=0}^{J}$ with $\sum_{j=0}^{L}$, where $L < J$ is the effective memory length (a small constant integer).

3.2. Dilatation-stress Formulation

3.2.1. Time discretization

Use of the Grünwald-Letnikov approximation (25) in equation (24) results in an implicit time-integration scheme, which can be expensive in terms of computer time and storage. An explicit scheme can be obtained if the wave equation is written in the dilatation-stress formulation. Using equations (18) and (22), and including the source term yields

$$\frac{\partial^2 \epsilon}{\partial t^2} = \Delta_\rho \sigma + s \ . \tag{27}$$

On the other hand, using (17), equation (18) becomes

$$\sigma = \rho D \frac{\partial^{2\gamma} \epsilon}{\partial t^{2\gamma}} \ . \tag{28}$$

Equations (27) and (28) are discretized at $t = (n-1)h$ and $t = nh$, respectively, by using the central-difference and Grünwald-Letnikov approximations. We obtain

$$\epsilon^n = h^2(\Delta_\rho \sigma + s)^{n-1} + 2\epsilon^{n-1} - \epsilon^{n-2} \tag{29}$$

from (26)–(27), and

$$\sigma^n = \rho D h^{-2\gamma} \sum_{j=0}^{J} (-1)^j \binom{2\gamma}{j} \epsilon^{n-j} \tag{30}$$

from (25) and (28).

3.2.2. FD complex velocity

The dispersion relation relates the frequency with the wavenumber and allows the calculation of the phase velocity corresponding to each Fourier component. Time discretization implies an approximation of the dispersion relation.

Assuming constant material properties and substituting the ansatz (15) with $t = nh$ in equations (29) and (30) gives the following dispersion relation:

$$\sin\left(\frac{\omega h}{2}\right) = \frac{1}{2}\sqrt{D} k h^{1-\gamma} \left[\sum_{j=0}^{n-2} (-1)^j \binom{2\gamma}{j} \exp(-\mathrm{i}\omega j h)\right]^{1/2} , \tag{31}$$

where k is the complex wavenumber. The FD approximation to the complex velocity is $\bar{V} = \omega/k$ where ω and k satisfy equation (31). If $\gamma = 0$, this velocity is real and we obtain the FD phase velocity

$$\bar{c} = \frac{c_0}{\mathrm{sinc}(\theta)} , \tag{32}$$

where $\mathrm{sinc}(\theta) = \sin(\theta)/\theta$ and $\theta = \omega h/2$. Equation (32) indicates that the FD velocity is greater than the true phase velocity. If $\gamma \neq 0$, the FD complex velocity can be written as

$$\bar{V} = \frac{\sqrt{D} h^{-\gamma}}{\mathrm{sinc}(\theta)} \left[\sum_{j=0}^{n-2} (-1)^j \binom{2\gamma}{j} \exp(-2\mathrm{i}\theta j)\right]^{1/2} , \tag{33}$$

where equation (31) has been used.

3.3. Dilatation Formulation

3.3.1. Time discretization

An explicit scheme can be obtained with the central-difference approximation (26). In this case, equation (24) is discretized at $t = nh$. We obtain

$$\epsilon^{n+\beta/2} = h^\beta \Delta_\rho(\rho D \epsilon^n) - \sum_{j=1}^{J} (-1)^j \binom{\beta}{j} \epsilon^{n-j+\beta/2} + s^n \quad . \tag{34}$$

In order to compute the spatial derivatives at nh we require an approximation for ϵ^n. Since $\beta \approx 2$, the simplest approximation is $\epsilon^n \approx \epsilon^{n-1+\beta/2}$. Similarly, the source can be introduced at times $t = (n - 1 + \beta/2)h$.

3.3.2. FD complex velocity

Substituting the ansatz (15) with $t = nh$ in equation (34) gives the FD complex velocity

$$\bar{V} = 2\mathrm{i}\theta\sqrt{D}h^{\beta/2-1}\exp(-\mathrm{i}\theta)\left[1 + \sum_{j=1}^{n-1}(-1)^j \binom{\beta}{j}\exp(-2\mathrm{i}\theta j)\right]^{-1/2} \quad . \tag{35}$$

In the above formula we have written $\exp(-\mathrm{i}\theta)$ in place of $\exp(-\mathrm{i}\beta\theta/2)$ because of the approximation $\epsilon^n \approx \epsilon^{n-1+\beta/2}$. Here lies, essentially, the difference between the Grünwald-Letnikov and central-difference approximations.

3.4. Accuracy. FD Phase Velocity and Attenuation Factor

The FD phase velocity is given by

$$\bar{c} = \left[\mathrm{Re}\left(\frac{1}{\bar{V}}\right)\right]^{-1} \quad , \tag{36}$$

and the FD attenuation factor is

$$\bar{\alpha} = -\omega \,\mathrm{Im}\left(\frac{1}{\bar{V}}\right) \quad . \tag{37}$$

If $h \to 0$ (i.e., $n \to \infty$), equation (33) becomes

$$\bar{V} \;\to\; \sqrt{D}\left[\frac{1 - \exp(-\mathrm{i}\omega h)}{h}\right]^\gamma \quad , \tag{38}$$

where we used the property

$$(1 - z)^{2\gamma} = \sum_{j=0}^{\infty} (-1)^j \binom{2\gamma}{j} z^j, \qquad z = \exp(-\mathrm{i}\omega h) \quad , \tag{39}$$

which is convergent if $|z| < 1$ (e.g., Itô, 1987). Using L'Hôpital rule as $h \to 0$ in equation (38) yields

$$\bar{V} = \sqrt{D}(\mathrm{i}\omega)^\gamma \quad , \tag{40}$$

which by virtue of equations (4), (5) and (17) gives the complex velocity V.

Similarly, when $h \to 0$, equation (35) becomes

$$\bar{V} = \sqrt{D}(\mathrm{i}\omega)^{1-\beta/2} \ , \tag{41}$$

which is equivalent to (40).

Using the same arguments, the attenuation factor (9) is obtained from equation (37) if $h \to 0$.

4. Examples

Attenuation measurements in a relatively homogeneous medium (Pierre shale) were made by McDONAL *et al.* (1958) near Limon, Colorado. They reported a constant-Q behavior with attenuation $\alpha = 0.12f$, where α is given in dB per 1000 ft and the frequency f in Hz. Conversion of units implies α (dB/1000 ft) = 8.686 α (nepers/1000 ft) = 2.6475 α (nepers/km). For low-loss solids, the quality factor is

$$Q = \frac{\pi f}{\alpha c} \ ,$$

with α given in nepers per unit length (TOKSÖZ and JOHNSTON, 1981). Since c is approximately 7000 ft/s (2133.6 m/s), the quality factor is $Q \simeq 32.5$. We consider a reference frequency $f_0 = \omega_0/(2\pi) = 250$ Hz, corresponding to the dominant frequency of the seismic source used in the experiments. Then, $\gamma = 0.0097955$, $\beta = 1.980409$, and $c_0 = \sqrt{M_0/\rho} = 2133.347$ m/s. The phase velocity (6) and attenuation factor (9) versus frequency $f = \omega/2\pi$ are shown in Figures 1a and 1b, respectively, where the open circles are the experimental points, the broken and dotted lines are the FD approximations (36) and (37), using the dilatation-stress and dilatation formulations, respectively. The memory lengths are 120 and 60, respectively. The curves correspond to a time step $h = 0.05$ ms. The dilatation formulation is more accurate because it is based on the central-difference approximation (26), and because the decay of the binomial coefficients in the series expansion is faster than in the dilatation-stress formulation. The latter fact is illustrated in Figure 2, which shows the logarithm of the absolute value of the binomial coefficients versus the summation index for the dilatation-stress formulation ($\nu = 2\gamma$, continuous line) and for the dilatation formulation ($\nu = \beta$, broken line).

The medium is discretized on a numerical mesh, with uniform vertical and horizontal grid spacings of 2 m, and 77 × 77 grid points. The spatial derivatives are calculated with the Fourier method by using the fast Fourier transform (FFT) (KOSLOFF and BAYSAL, 1982). The source, applied at the center of the mesh, is a Ricker-type wavelet, whose amplitude spectrum is a Gaussian function centered at 250 Hz. A band-limited source, such as a Butterworth filter with a low cut-off

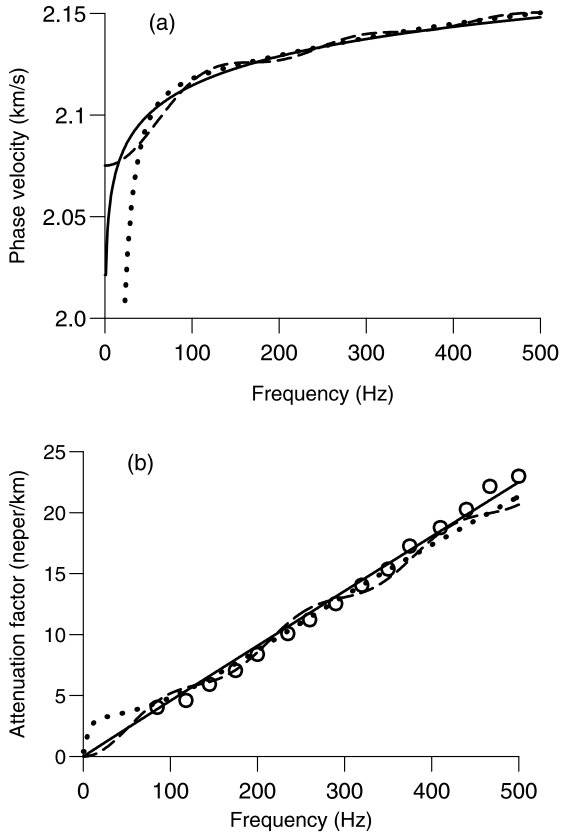

Figure 1
Phase velocity (a) and attenuation factor (b) versus frequency in Pierre shale (continuous line) as given by Eqs. (6) and (9), respectively. The broken and dotted lines are the FD approximations using the dilatation-stress (33) and the dilatation formulations (35), respectively. The memory lengths are, respectively, 120 and 60. The open circles are the experimental data reported by McDonal *et al.* (1958).

frequency, should be used to avoid aliasing problems, since the phase velocity approaches zero at zero frequency. The maximum allowed frequency is $f_{\max} = c_{\min}/(2d)$, where d is the grid spacing and $c_{\min}$ is the phase velocity at the low cut-off frequency. However, this effect can be neglected, in virtue of the form of the phase-velocity curve (Fig. 1a) and because the energy of the Ricker wavelet is concentrated around its central frequency. The time step used in this simulation is 0.05 ms. Figure 3 compares two snapshots of the dilatation field computed at 36 ms, where (a) corresponds to the lossless case ($\gamma = 0$), and (b) to a lossy model of Pierre shale. The attenuation is evident in the latter case.

A 2-D analytical solution of equation (24) in a homogeneous medium can easily be obtained. The solution to the acoustic (lossless) equation is the zero-order Hankel

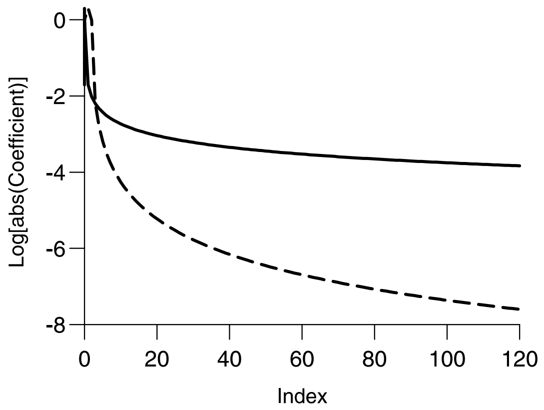

Figure 2
Decimal logarithm of the absolute value of the binomial coefficients versus the summation index for the dilatation-stress formulation ($\nu = 2\gamma = 2 \times 0.0097955$) (continuous line) and for the dilatation formulation ($\nu = \beta = 1.980409$) (broken line).

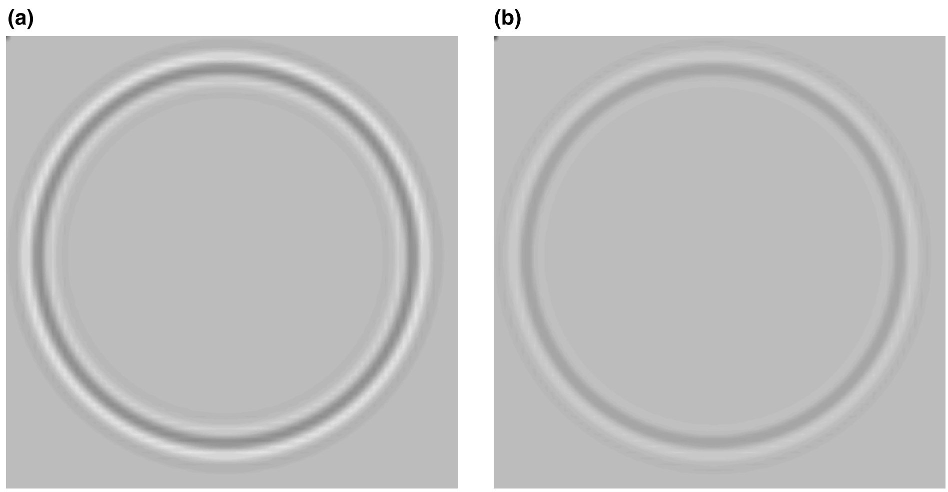

Figure 3
Snapshots of the dilatation field in a lossless medium equivalent to Pierre shale (a), and in a dissipative model of Pierre shale (b).

function of the second kind (MORSE and FESHBACH, 1953, sec. 11.2; CARCIONE *et al.*, 1988a),

$$G(x, z, x_0, z_0, \omega) = -\mathrm{i}\pi H_0^{(2)}\left(\frac{\omega r}{c_0}\right) \tag{42}$$

where (x_0, z_0) is the source location, and

$$r = \left[(x - x_0)^2 + (z - z_0)^2 \right]^{1/2} . \tag{43}$$

The viscoacoustic solution is obtained by invoking the correspondence principle (BLAND, 1960), i.e., by substituting the acoustic velocity c_0 with the complex velocity (5). We set $G(-\omega) = G^*(\omega)$, where the superscript $*$ denotes complex conjugation. This equation ensures that the inverse Fourier transform of the Green's function is real. The frequency-domain solution is then given by $w(\omega) = G(\omega)F(\omega)$, where F is the Fourier transform of the source. Because the Hankel function has a singularity at $\omega = 0$, we assume $G = 0$ for $\omega = 0$, an approximation that has no significant effect on the solution (note, moreover, that $F(0)$ is small). The time-domain solution $w(t)$ is obtained by a discrete inverse Fourier transform. We have tacitly assumed that w and dw/dt are zero at time $t = 0$.

Figure 4 compares numerical (dotted and broken lines) and analytical (solid line) solutions in the lossy case, at 40 m from the source. In this case, we used the dilatational formulation with memory lengths 20 (broken line) and 40 (solid line). The agreement is excellent for $L = 40$, while $L = 20$ yields a degraded numerical solution.

Finally, we provide an example of seismic wave propagation in inhomogeneous media. The geological model is shown in Figure 5, and the material properties are indicated in Table 1, with the same reference frequency $f_0 = 80$ Hz for all the media. The low velocity and low quality factor of medium 4 simulate an unconsolidated sandstone. Absorbing strips, of width 18 grid points, are implemented at the four boundaries of the mesh (CARCIONE *et al.*, 1988). The source is a Ricker wavelet with central frequency of 80 Hz, and the wavefield is computed by using a time step of 0.2 ms. The synthetic seismograms recorded in the receiver well, corresponding to the lossless case (a) and lossy case (b), are shown in Figure 6. The simulation based on the dilatation formulation is 4.5 times faster than the simulation based on the dilatation-stress formulation.

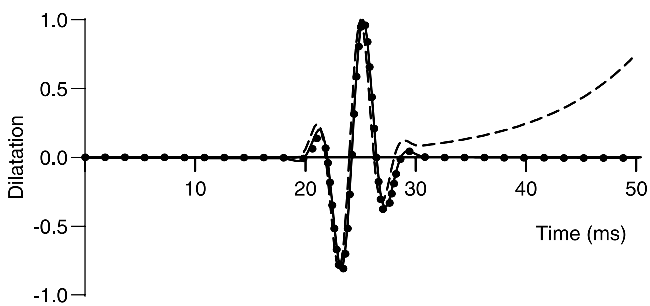

Figure 4
Comparison between numerical and analytical solutions at 400 m from the source. The dotted and broken lines correspond to memory lengths 40 and 20, respectively, and the solid line is the analytical solution. The medium is Pierre shale.

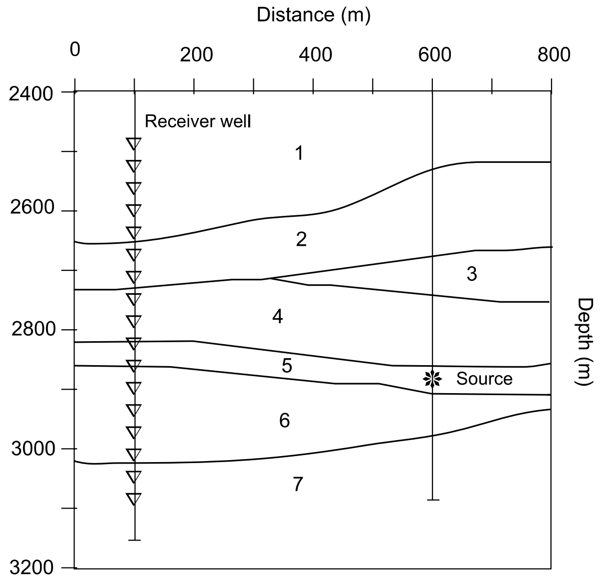

Figure 5
Geological model.

Table 1

Material properties

Medium	c_0 (km/s)	ρ (g/cm^3)	Q
1	3.2	2.5	100
2	3.3	2.52	110
3	3.6	2.58	120
4	2.9	2.4	30
5	3.6	2.7	140
6	3.7	2.71	150
7	3.85	2.72	165

5. *Conclusions*

The concept of fractional derivative has been used to simulate constant-Q wave propagation (Pierre shale). The equations were expressed in the dilatation-stress and dilatation formulations. The second approach is more accurate and efficient, however our numerical experiments indicate that the absorbing-boundary algorithm performs better with the first formulation. The validity and accuracy of the algorithms are verified by comparison with a novel 2-D analytical solution. The modeling is illustrated with a cross-well seismic experiment, using a Kjartansson's attenuation model, but this approach can provide important applications for porous media as

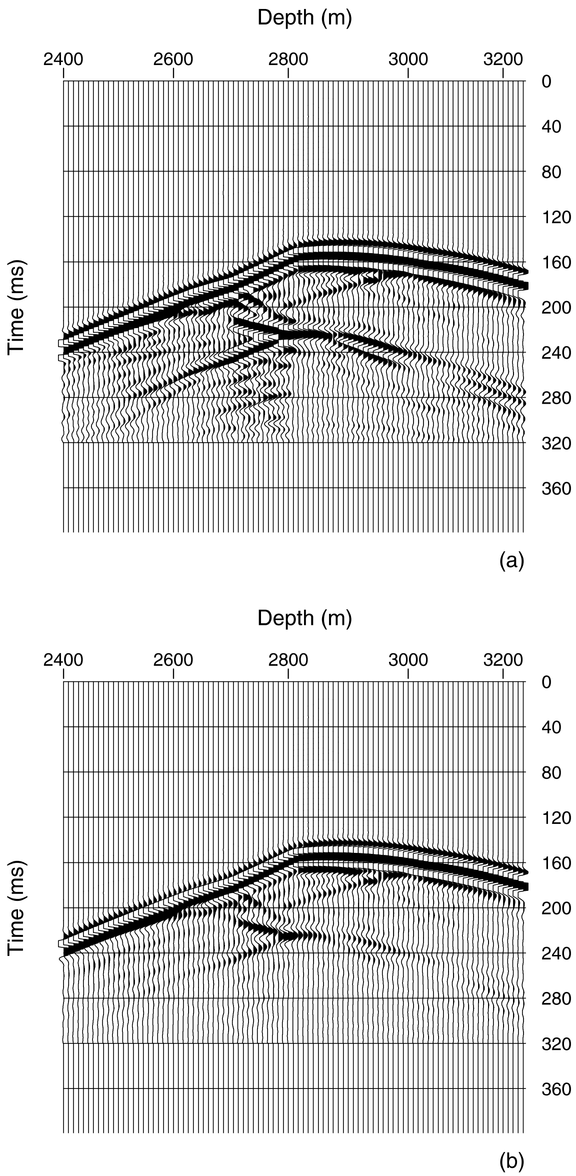

Figure 6
Acoustic (a) and viscoacoustic (b) synthetic seismograms of the dilatation field, corresponding to the model illustrated in Figure 5.

well, since fractional derivatives appear in Biot theory, related to viscodynamic effects at seismic frequencies.

Further research goals on this subject include: (i) an optimal finite-difference approximation of the fractional-derivative operator to reduce numerical dispersion

$[O(h^n), n \geq 2]$ and memory storage. The latter is closely related to the memory length, which in the present case exceeds the number of memory variables used in nearly constant-Q modeling algorithms based on mechanical models; (ii) to improve the absorbing-boundary algorithm in the dilatation formulation; (iii) the generalization to the elastic P-SV case; and (iv) applications of the method to wave propagation in porous media.

Appendix A. The Fractional Derivative

The notion of fractional derivative here adopted can be easily introduced via Fourier transform, since it is intended to generalize the rule of the Fourier transform for the common derivative of integer order of a well-behaved function of time by allowing noninteger powers of the frequency. If the time Fourier transform is defined as

$$[\mathscr{F}\phi](\omega) = \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\omega t}\phi(t)\mathrm{d}t \ , \tag{44}$$

it is well known that

$$\left[\mathscr{F}\phi^{(n)}\right](\omega) = (\mathrm{i}\omega)^n[\mathscr{F}\phi](\omega) \ , \tag{45}$$

where n is any positive *integer* number, and $\phi^{(n)}$ is the n-th derivative of ϕ. Our fractional derivative is defined in such a way that

$$\left[\mathscr{F}\phi^{(\alpha)}\right](\omega) = (\mathrm{i}\omega)^\alpha[\mathscr{F}\phi](\omega) \ , \tag{46}$$

where now α is any positive *real* number. For α not integer, one can show that such a derivative is a special pseudo-differential operator that can be properly defined by introducing the integer m such that $m - 1 < \alpha < m$ and putting

$$\phi^{(\alpha)}(t) = \frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\int_{-\infty}^{t} \phi(\tau)\frac{1}{(t-\tau)^{\alpha+1-m}}\,\mathrm{d}\tau \tag{47}$$

or, equivalently,

$$\phi^{(\alpha)}(t) = \frac{1}{\Gamma(m-\alpha)}\int_{-\infty}^{t} \frac{\mathrm{d}^m\phi(\tau)}{\mathrm{d}\tau^m}\frac{1}{(t-\tau)^{\alpha+1-m}}\,\mathrm{d}\tau \ . \tag{48}$$

We note the possibility of interchanging the integer-order derivative with the integral, since the function $\phi(t)$ is assumed to decay sufficiently fast to zero for $t \rightarrow -\infty$ together with its (relevant) derivatives.

Appendix B. Grünwald-Letnikov and Central-difference Approximations to the Fractional Derivative

Consider the backward first-order approximation of the first derivative,

$$\frac{\partial f(t)}{\partial t} \sim \frac{f(t) - f(t-h)}{h} \quad . \tag{49}$$

This leads to the second derivative

$$\frac{\partial^2 f(t)}{\partial t^2} \sim \frac{1}{h}\left(\frac{\partial f(t)}{\partial t} - \frac{\partial f(t-h)}{\partial t}\right) \sim \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2} \tag{50}$$

and to the third derivative

$$\frac{\partial^3 f(t)}{\partial t^3} \sim \frac{f(t) - 3f(t-h) + 3f(t-2h) - f(t-3h)}{h^3} \quad . \tag{51}$$

The generalization is straightforward. The m-th derivative is

$$\frac{\partial^m f(t)}{\partial t^m} \sim \frac{1}{h^m}\sum_{j=0}^{m}(-1)^j \binom{m}{j} f(t - jh), \qquad m = 0, 1, 2, 3, \ldots . \tag{52}$$

A more accurate (second-order) approximation for the first derivative is

$$\frac{\partial f(t)}{\partial t} \approx \frac{1}{h}\left[f\left(t + \frac{h}{2}\right) - f\left(t - \frac{h}{2}\right)\right] \quad . \tag{53}$$

This leads to the second-order accurate m-th derivative

$$\frac{\partial^m f(t)}{\partial t^m} \approx \frac{1}{h^m}\sum_{j=0}^{m}(-1)^j \binom{m}{j} f\left[t + h\left(\frac{m}{2} - j\right)h\right], \qquad m = 0, 1, 2, 3, \ldots . \tag{54}$$

The upper summation limit may be replaced by any integer larger than m, for example by $t/h - 1$, since

$$\binom{m}{j} = 0 \qquad \text{for } j > m.$$

There are no restrictions in the r.h.s. of equations (52) and (54) that require m to be an integer. Replacing m by any positive real number ν in equations (52) and (54) gives the Grünwald-Letnikov approximation (25) and the central-difference approximation (26), respectively (GORENFLO, 1997). The fractional binomial coefficients can be defined in terms of Euler's Gamma function as

$$\binom{v}{j} = \frac{\Gamma(v+1)}{\Gamma(j+1)\Gamma(v-j+1)}$$

and can be calculated by a simple recursion formula

$$\binom{v}{j} = \frac{v-j+1}{j}\binom{v}{j-1}, \qquad \binom{v}{0} = 1 \ .$$

The extension of the upper limit from v to $t/h - 1$ has an important consequence. While in equations (52) and (54) the series has vanishing terms beyond $j = m$, in equations (25) and (26) these terms are different from zero. The approximations (25) and (26) are actually "differintegration" operators, since it can be shown that for negative v they represent the generalized Riemann sums.

For more details on the theory and applications of fractional calculus, the reader is referred to OLDHAM and SPANIER (1974), GORENFLO and MAINARDI (1997), and PODLUBNY (1999).

REFERENCES

BLAND, D. R., *The Theory of Linear Viscoelasticity* (Pergamon, New York 1960).

CAPUTO, M. and MAINARDI, F. (1971), *Linear Models of Dissipation in Anelastic Solids*, Riv. Nuovo Cimento (Ser. II) *1*, 161–198.

CARCIONE, J. M. and CAVALLINI, F. (1994), *A Rheological Model for Anelastic Anisotropic Media with Applications to Seismic Wave Propagation*, Geophys. J. Int. *119*, 338–348.

CARCIONE, J. M., KOSLOFF, D., and KOSLOFF, R. (1988a), *Wave Propagation Simulation in a Linear Viscoacoustic Medium*, Geophys. J. Roy. Astr. Soc. *93*, 393–407.

CARCIONE, J. M., KOSLOFF, D., and KOSLOFF, R. (1988b), *Wave Propagation Simulation in a Linear Viscoelastic Medium*, Geophys. J. Roy. Astr. Soc. *95*, 597–611.

DATTOLI, G., TORRE, A., and MAZZACURATI, G. (1998), *An Alternative Point of View to the Theory of Fractional Fourier Transform*, J. Appl. Math. *60*, 215–224.

FELLAH, Z. E. A. and DEPOLLIER, C. (2000), *Transient Acoustic Wave Propagation in Rigid Porous Media: A Time-domain Approach*, J. Acoust. Soc. Am. *107*, 683–688.

GORENFLO, R., *Fractional calculus, some numerical methods*. In *Fractals and Fractional Calculus in Continuum Mechanics* (eds. Carpinteri, A. and Mainardi, F.). (Springer Verlag, Wien 1997) pp. 277–290.

GORENFLO, R. and MAINARDI, F. (1997), *Fractional calculus: Integral and differential equations of fractional order*. In *Fractals and Fractional Calculus in Continuum Mechanics* (eds. Carpinteri, A. and Mainardi, F.). (Springer Verlag, Wien 1997) pp. 223–276.

GRÜNWALD, A. K. (1867), *Über "begrenzte" Derivationen und deren Anwendung*, Zeit. angew. Math. Physik *12*, 441–480.

GUREVICH, B. and LOPATNIKOV, S. L. (1995), *Velocity and Attenuation of Elastic Waves in Finely Layered Porous Rocks*, Geophys. J. Int. *121*, 933–947.

ITÔ, K., *Encyclopedic Dictionary of Mathematics*, (The M.I.T. Press, Cambridge, 1987).

KJARTANSSON, E. (1979), *Constant Q-wave Propagation and Attenuation*, J. Geophys. Res. *84*, 4737–4748.

KOSLOFF, D. and BAYSAL, E. (1982), *Forward Modeling by the Fourier Method*, Geophysics *47*, 1402–1412.

LETNIKOV, A. V. (1868), *Theory of Differentiation of Fractional Order*, Math. Sb. *3*, 1–68 (in Russian).

MAINARDI, F. and TOMIROTTI, M. (1997), *Seismic Pulse Propagation with Constant Q and Stable Probability Distributions*, Annali di Geofisica *40*, 1311–1328.

MCDONAL, F. J., ANGONA, F. A., MILSS, R. L., SENGBUSH, R. L., VAN NOSTRAND, R. G., and WHITE, J. E. (1958), *Attenuation of Shear and Compressional Waves in Pierre Shale*, Geophysics *23*, 421–439.

MORSE, P. M. and FESHBACH, H., *Methods of Theoretical Physics*, Vol. II (McGraw-Hill, New York 1953).

OLDHAM, K. B. and SPANIER, J., *The Fractional Calculus* (Academic Press, San Diego 1976).

PODLUBNY, I., *Fractional Differential Equations* (Academic Press, San Diego 1999).

PRÜSS, J., *Evolutionary Integral Equations and Applications* (Birkhäuser, Basel 1993).

SCOTT-BLAIR, G. W., *Survey of General and Applied Rheology* (Pitman, London 1949).

TOKSÖZ, M. N. and JOHNSTON, D. H., *Seismic wave attenuation*. In Geophysics, reprint series No. 2 (SEG, Tulsa 1981).

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