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SHORT NOTE

Two equivalent expressions of the Kramers-Kronig relations

J.M. Carcione[1,2], B. Gurevich[3] and J. Ba[1]

[1] *School of Earth Sciences and Engineering, Hohai University, Nanjing, China*
[2] *National Institute of Oceanography and Applied Geophysics - OGS, Trieste, Italy*
[3] *Curtin University, Perth, Australia*

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1. Introduction

Natural and man-made materials show viscoelastic behaviour, by which stress and strain are related by a relaxation function or a complex stiffness modulus in the time and frequency domains, respectively. Viscoelasticity led to significant research in material science and seismology (Christensen, 1982). Real-world systems should satisfy the Kramers-Kronig relations (KKRs), known from the beginning of the 20th century from the works of de Laer Kronig (1926) and Kramers (1927) on electromagnetism, showing the interrelation between the real and imaginary parts of the complex susceptibility. Electrical and mechanical representations include the Debye model, used to describe the behaviour of dielectric materials, and the Zener viscoelastic model, respectively, both being mathematically equivalent (Carcione, 2014).

In viscoelasticity, the KKRs connect the real and imaginary parts of the stiffness modulus. Carcione *et al*. (2019) provide a complete derivation of the relations using the Sokhotski-Plemelj equation, showing explicitly what are the conditions for the relations to hold. There are many forms of the relations. In geophysics, the book of Mavko *et al*. (2009) provides the most popular expression, based on the relaxed modulus. In this note, we show that this expression is equivalent to a simpler one involving the unrelaxed modulus. Two different demonstrations are given, illustrating the eclectic mathematical apparatus available to obtain the relations. Moreover, we develop the KKRs relations for the creep function (creep compliance) and derive them for seismological applications, i.e. based on the seismic velocity and attenuation factor.

2. The Kramers-Kronig relations

2.1. Relaxation function

The stress (σ)-strain (ϵ) relation of a viscoelastic solid is (e.g. Carcione, 2014):

$$\sigma = \psi * \dot{\epsilon} = \dot{\psi} * \epsilon \tag{1}$$

where ψ is the relaxation function, '$*$' denotes time convolution and a dot above a variable time differentiation. Let us define

$$\hat{\psi}(t) = \psi(t) - M_R \geq 0, \tag{2}$$

such that

$$M_R = \psi(\infty) \text{ and } \hat{\psi}(t) \to 0 \text{ when } t \to \infty, \tag{3}$$

where M_R is the relaxed modulus.

The Fourier transform of Eq. 1 gives

$$\mathcal{F}[\sigma(t)] = M(\omega)\mathcal{F}[\epsilon(t)], \tag{4}$$

where ω is the angular frequency, $\mathcal{F}$ is the Fourier-transform operator and

$$M(\omega) = \mathcal{F}[\dot{\psi}(t)] = \int_{-\infty}^{\infty} \dot{\psi}(t) \exp(-\mathrm{i}\omega t) dt \tag{5}$$

is the complex modulus, with i = $\sqrt{-1}$, such that

$$M(\omega = 0) = \psi(t = \infty) = M_R \qquad M(\omega = \infty) = \psi(t = 0) = M_U, \tag{6}$$

where M_U is the unrelaxed modulus.

Since physically-admitted relaxation functions are causal, we have $\psi(t) \equiv \breve{\psi} H(t)$, where ψ has no restriction and H is the Heaviside function. Then, $\dot{\psi} = \delta(t)\breve{\psi} + \dot{\breve{\psi}} H(t)$, where δ is Dirac's delta, and

$$M(\omega) = \psi(0^+) + \int_0^{\infty} \dot{\breve{\psi}}(t) \exp(-\mathrm{i}\omega t) dt, \tag{7}$$

because $\breve{\psi}(0) = \psi(0^+)$. Since $\breve{\psi}(t) = \psi(t)$ for $t > 0$, Eq. 7 becomes

$$M(\omega) = \psi(\infty) + \mathrm{i}\omega \int_0^{\infty} [\psi(t) - \psi(\infty)] \exp(-\mathrm{i}\omega t) dt, \tag{8}$$

[Equivalence of Eqs. 7 and 8 follows by integrating Eq. 8 by parts[1]]

[1] Proof: $\mathrm{i}\omega \int_0^{\infty} [\psi(t) - \psi(\infty)] \exp(-\mathrm{i}\omega t) dt = -\int_0^{\infty} [\psi(t) - \psi(\infty)] d \exp(-\mathrm{i}\omega t) = -|[\psi(t) - \psi(\infty)] \exp(-\mathrm{i}\omega t)|_{t=0}^{t=\infty} + \int_0^{\infty} \exp(-\mathrm{i}\omega t) d\psi = -[\psi(\infty) - \psi(\infty)] \exp(-\mathrm{i}\omega\infty) + [\psi(0) - \psi(\infty)] \exp(-\mathrm{i}\omega 0) + \int_0^{\infty} \dot{\psi}(t) \exp(-\mathrm{i}\omega t) dt = \psi(0) - \psi(\infty) + \int_0^{\infty} \dot{\psi}(t) \exp(-\mathrm{i}\omega t) dt.$

We decompose the complex modulus into real and imaginary parts

$$M(\omega) = M_1(\omega) + \mathrm{i}M_2(\omega), \tag{9}$$

where

$$M_1(\omega) = \psi(\infty) + \omega \int_0^\infty [\psi(t) - \psi(\infty)] \sin(\omega t) dt \tag{10}$$

is the storage modulus, or

$$M_1(\omega) = \omega \int_0^\infty \psi(t) \sin(\omega t) dt, \tag{11}$$

and

$$M_2(\omega) = -\int_0^\infty \dot{\psi}(t) \sin(\omega t) dt = \omega \int_0^\infty [\psi(t) - \psi(\infty)] \cos(\omega t) dt \tag{12}$$

is the loss modulus. To obtain Eq. 11, we have used the property (Golden and Graham, 1988):

$$\omega \int_0^\infty \sin(\omega t) dt = 1 \tag{13}$$

Then, using Eqs. 2 and 3, Eqs. 11 and 12 become

$$M_1(\omega) = M_R + \omega \int_0^\infty \hat{\psi}(t) \sin(\omega t) dt \tag{14}$$

and

$$M_2(\omega) = \omega \int_0^\infty \hat{\psi}(t) \cos(\omega t) dt. \tag{15}$$

Eq. 15 is a cosine transform, whose reverse transformation is

$$\hat{\psi}(t) = \frac{2}{\pi} \int_0^\infty \frac{M_2(\omega')}{\omega'} \cos(\omega' t) d\omega'. \tag{16}$$

Substituting Eq. 16 into Eq. 14 and re-ordering terms yields

$$M_1(\omega) = M_R + \frac{2\omega}{\pi} \int_0^\infty \frac{M_2(\omega')}{\omega'} \left[\int_0^\infty \sin(\omega t) \cos(\omega' t) dt \right] d\omega'. \tag{17}$$

The integral in square brakets in the right-hand side of Eq. 17 is:

$$\int_0^\infty \sin(\omega t) \cos(\omega' t) dt = \frac{1}{2} \int_0^\infty \left\{ \sin[(\omega + \omega')t] + \sin[(\omega - \omega')t] \right\} dt$$

$$= -\frac{1}{2} \left| \frac{\cos[(\omega + \omega')t]}{\omega + \omega'} + \frac{\cos[(\omega - \omega')t]}{\omega - \omega'} \right|_0^\infty \tag{18}$$

$$= \frac{1}{2} \left[\frac{1}{\omega + \omega'} + \frac{1}{\omega - \omega'} \right] = \frac{\omega}{\omega^2 - \omega'^2},$$

where the integrals have been handled as

$$\lim_{\beta \to 0} \int_0^\infty \exp(-\beta t) \sin[(\omega \pm \omega')t] dt \tag{19}$$

in the second equality, to warrant convergence at the upper limit [a more rigorous proof, based on complex-variable theory, is given in Landau and Lifschit (1958)].

Then, substituting Eq. 18 into Eq. 17 gives

$$M_1(\omega) = M_R + \frac{2}{\pi} \int_0^\infty \frac{\omega^2 M_2(\omega')}{\omega'(\omega^2 - \omega'^2)} d\omega'. \tag{20}$$

Since in a real-world system, the stress and strain must be real-valued, the relaxation function should be real-valued, and hence $M(\omega)$ must be Hermitian (Bracewell, 2000), i.e.

$$M_1(\omega) = M_1(-\omega), \quad M_2(\omega) = -M_2(-\omega) \tag{21}$$

(Carcione 2014) indicating that M_1 and M_2 are even and odd functions of ω, respectively.

Then, the integrand in Eq. 20 is an even function and we have:

$$M_1(\omega) = M_R + \frac{1}{\pi} \int_{-\infty}^\infty \frac{\omega^2 M_2(\omega')}{\omega'(\omega^2 - \omega'^2)} d\omega'. \tag{22}$$

Since

$$\frac{\omega^2}{\omega'(\omega^2 - \omega'^2)} = \frac{\omega}{\omega'(\omega - \omega')} - \frac{\omega}{\omega^2 - \omega'^2}, \tag{23}$$

we have

$$M_1(\omega) = M_R + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\omega M_2(\omega')}{\omega'(\omega-\omega')} d\omega' - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\omega M_2(\omega')}{\omega^2-\omega'^2} d\omega'. \quad (24)$$

The second integral is zero, because M_2 is an odd function. Then:

$$M_1(\omega) = M_R + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\omega M_2(\omega')}{\omega'(\omega-\omega')} d\omega'. \quad (25)$$

This is Mavko *et al*. (2009) first equation.

We can further simplify Eq. 25. Since

$$\frac{\omega}{\omega'(\omega-\omega')} = \frac{1}{\omega'} + \frac{1}{\omega-\omega'}, \quad (26)$$

we have

$$M_1(\omega) = M_R + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{M_2(\omega')}{\omega'} d\omega' + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{M_2(\omega')}{\omega-\omega'} d\omega'. \quad (27)$$

From Eq. 16, the first integral is $\hat{\psi}(0)$ since $M_2(\omega')/\omega'$ is an even function. Using Eq. 2

$$\hat{\psi}(0) = \psi(0) - M_R = M_U - M_R. \quad (28)$$

Then,

$$M_1(\omega) = M_U + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{M_2(\omega')}{\omega-\omega'} d\omega' \quad (29)$$

is another expression of the Kramers-Kronig relation, involving the unrelaxed modulus.

Following the same procedure for the imaginary part of the stiffness, we obtain

$$M_2(\omega) = -\frac{\omega}{\pi}\int_{-\infty}^{\infty} \frac{[M_1(\omega') - M_R]}{\omega'(\omega-\omega')} d\omega'. \quad (30)$$

This is Mavko *et al.* (2009) second equation. Similarly, the companion equation to Eq. 29 is

$$M_2(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{M_1(\omega')}{\omega-\omega'} d\omega', \quad (31)$$

since $M_2(\infty) = 0$.

Eqs. 25 to 30 and 29 to 31 are two equivalent Hilbert transform pairs (Bracewell, 2000), defining two different expressions of the KKRs. The Cauchy principal value of the improper integrals is implied in these calculations.

2.2. Alternative demonstration

Let us define (Carcione *et al.*, 2019):

$$\nu(t) = \dot{\psi}(t) - M_U \delta(t) \tag{32}$$

such that $M_U = M(\omega = \infty)$ and set $N = \mathcal{F}[\nu]$. We have:

$$N = M - M_U, \tag{33}$$

where $N(\omega = \infty) = 0$. Since $\nu(t)$ is real, $N(\omega)$ is Hermitian; that is

$$N(\omega) = N^*(-\omega), \tag{34}$$

or

$$N_1(\omega) = N_1(-\omega), \qquad N_2(\omega) = -N_2(-\omega). \tag{35}$$

Furthermore, ν can split into even and odd functions of time, ν_e and ν_o, respectively, as

$$\nu(t) = \frac{1}{2}[\nu(t) + \nu(-t)] + \frac{1}{2}[\nu(t) - \nu(-t)] \equiv \nu_e + \nu_o. \tag{36}$$

Since ν is causal, $\nu_o(t) = \mathrm{sgn}(t)\nu_e(t)$, and

$$\nu(t) = [1 + \mathrm{sgn}(t)]\nu_e(t), \tag{37}$$

whose Fourier transform is

$$\mathcal{F}[\nu(t)] = N(\omega) = M_1(\omega) - M_U - \left(\frac{\mathrm{i}}{\pi\omega}\right) * [M_1(\omega) - M_U], \tag{38}$$

because $\mathcal{F}[\nu_e] = M_1 - M_U$ and $\mathcal{F}[\mathrm{sgn}(t)] = 2/(\mathrm{i}\omega)$ (Bracewell, 2000). Eqs. 33 and 38 and the fact that $(\omega - \omega')^{-1}$ is ω-symmetric and odd imply

$$M_2 = -\left(\frac{1}{\pi\omega}\right) * (M_1 - M_U) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{M_1(\omega')d\omega'}{\omega - \omega'}, \tag{39}$$

i.e. Eq. 31 (alternatively, one could also use the property that the Hilbert transform of a constant is zero).

Similarly, since $v_e(t) = \text{sgn}(t)v_o(t)$, it is $v(t) = [\text{sgn}(t) + 1]v_o(t)$, and because $\mathcal{F}[v_o] = \text{i}M_2$, we obtain:

$$M_1 - M_U = \left(\frac{1}{\pi\omega}\right) * M_2 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{M_2(\omega')d\omega'}{\omega - \omega'}, \tag{40}$$

i.e. Eq. 29.

In mathematical terms, $M_1 - M_U$ and M_2 are Hilbert transform pairs. Causality also implies that M has no poles (or is analytic) in the lower half complex ω-plane (Golden and Graham, 1988).

2.3. Creep function

The strain-stress relation is

$$\epsilon = \chi * \dot{\sigma} = \dot{\chi} * \sigma, \tag{41}$$

where χ is the creep function (e.g. Carcione, 2014). Let us define

$$\hat{\chi}(t) = \chi(t) - J_U \geq 0, \tag{42}$$

such that

$$J_U = \chi(0) \quad \text{and} \quad \hat{\chi}(t) \to 0 \quad \text{when} \quad t \to 0, \tag{43}$$

where J_U is the unrelaxed compliance.

From Eq. 41, we may write

$$\epsilon(t) = \int_{-\infty}^{t} \chi(t - t')\dot{\sigma}(t')dt' = -\int_{0}^{\infty} \chi(\tau)\dot{\sigma}(t - \tau)d\tau, \tag{44}$$

because χ is causal and we defined $\tau = t - t'$ (the time derivatives are calculated with respect to the arguments). Evidencing explicitly the instantaneous response

$$\epsilon(t) = J_U\sigma(t) - \int_{0}^{\infty} \hat{\chi}(\tau)\dot{\sigma}(t - \tau)d\tau, \tag{45}$$

where we used Eq. 42.

Now, substitute a Fourier component for the stress, $\sigma(t) = \sigma_0 \exp(\text{i}\omega t)$, to obtain

$$\epsilon(t) = J_U\sigma(t) + \text{i}\omega\sigma_0 \int_{0}^{\infty} \hat{\chi}(\tau)\exp[\text{i}\omega(t - \tau)]d\tau. \tag{46}$$

The Fourier transform of Eq. 41 gives

$$\mathcal{F}[\varepsilon(t)] = J(\omega)\mathcal{F}[\sigma(t)] \tag{47}$$

where

$$J(\omega) = \mathcal{F}[\dot{\chi}(t)] = \int_{-\infty}^{\infty} \dot{\chi}(t) \exp(-\mathrm{i}\omega t) dt \tag{48}$$

is the complex compliance. Since

$$J(\omega) = J_1 + \mathrm{i}J_2 = \epsilon/\sigma = J_U + \mathrm{i}\omega \int_0^{\infty} \hat{\chi}(\tau) \exp(-\mathrm{i}\omega\tau) d\tau, \tag{49}$$

we have

$$J_1(\omega) = J_U + \omega \int_0^{\infty} \hat{\chi}(t) \sin(\omega t) dt \tag{50}$$

and

$$J_2(\omega) = \omega \int_0^{\infty} \hat{\chi}(t) \cos(\omega t) dt, \tag{51}$$

which are Eqs. 14 and 15 for the creep function.

Recalling the Hermitian property holds

$$J_1(\omega) = J_1(-\omega), \qquad J_2(\omega) = -J_2(-\omega) \tag{52}$$

and repeating previous mathematical calculations, an equation equivalent to Eq. 22 is obtained

$$J_1(\omega) = J_U + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\omega^2 J_2(\omega')}{\omega'(\omega^2 - \omega'^2)} d\omega', \tag{53}$$

which leads to equations equivalent to Eqs. 25 and 30:

$$J_1(\omega) = J_U + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\omega J_2(\omega')}{\omega'(\omega - \omega')} d\omega' \tag{54}$$

and

$$J_2(\omega) = -\frac{\omega}{\pi} \int_{-\infty}^{\infty} \frac{[J_1(\omega') - J_U]}{\omega'(\omega - \omega')} d\omega', \tag{55}$$

respectively.

Then, the KKRs corresponding to Eqs. 29 and 31 are

$$J_1(\omega) = J_U + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{J_2(\omega')}{\omega - \omega'} d\omega' \tag{56}$$

and

$$J_2(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{J_1(\omega')}{\omega - \omega'} d\omega', \tag{57}$$

respectively, where we have used that $\hat{\chi}(0) = 0$. These two equations can also be deduced from Eqs. 2.4-3 and 2.4-4 of Nowick and Berry (1972) after some calculations, using the property $\omega'/({\omega'}^2 - \omega^2) = 1/(\omega' - \omega) - \omega/({\omega'}^2 - \omega^2)$ and the fact that $J_2(\omega)$ is an odd function.

2.4. Wave velocity and attenuation factor

The KKRs can be applied to wave velocity and attenuation, which is useful in seismology. Let us define the complex wave velocity as

$$v_c = \sqrt{\frac{M}{\rho}}, \tag{58}$$

such that the complex slowness is

$$\frac{1}{v_c} = \frac{1}{v_p} - \frac{\mathrm{i}\alpha}{\omega}, \tag{59}$$

where ρ is the mass density, v_p is the phase velocity and α is the attenuation factor (e.g. Carcione, 2014). Let us identify $1/v_c$ with M, $1/v_p - 1/v_\infty$ with $M_1 - M_U$ and $-\alpha/\omega$ with M_2, where $v_\infty = v_p(\omega = \infty)$ is the unrelaxed velocity. Performing the same mathematical developments to obtain Eqs. 40 and 39, we get

$$\frac{1}{v_p(\omega)} - \frac{1}{v_\infty} = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\alpha(\omega')d\omega'}{\omega'(\omega - \omega')} \tag{60}$$

and

$$\alpha(\omega) = \frac{\omega}{\pi}\int_{-\infty}^{\infty} \frac{d\omega'}{v_p(\omega')(\omega - \omega')}, \tag{61}$$

which are Eqs. 13 in Box 5.8 in Aki and Richards (2009), where we used that the Hilbert transform of a constant is zero. Similar relations between velocity and quality factor Q can easily be obtained by considering that $\alpha = \omega/(2v_pQ)$ (Carcione *et al.*, 2019).

In the case of dispersive lossless media, $M_1(\omega) - M_U$ can depend on ω through functions of ω whose Hilbert transform involves delta functions, which do not represent damping due to their zero bandwidth. For instance, in electromagnetism, a Lorenz model (Carcione *et al*., 2010) can be defined that satisfies the relations, exhibiting zero damping in its electronic resonances, where the real part of the permittivity can take positive and negative values for certain frequencies even though the imaginary part is zero (Poon and Francis, 2009; Orfanidis, 2016). Moreover, lossless dispersion occurs when the working frequency is far away from the resonance frequency, where the energy and group velocities coincide (Carcione *et al*., 2010; Orfanidis, 2016).

On the other hand, an example of dispersion less lossy medium is given by a complex velocity $v_c = \omega c/(\omega - \mathrm{i}\gamma)$, where c is a constant velocity and γ a damping factor (Carcione *et al*., 2016). It is $v_p = c$ and $\alpha = \gamma/c$. However, it can be seen from Eq. 61 that this medium does not satisfy the KKRs.

REFERENCES

Aki K. and Richards P.G.; 2009: *Quantitative seismology: theory and methods, 2nd ed.* W.H. Freeman and Company, San Francisco, CA, USA, 557 pp.

Bracewell R.; 2000: *The Fourier transform and its applications, 3rd ed.* McGraw-Hill Book Co., New York, NY, USA, 630 pp.

Carcione J.M.; 2014: *Wave fields in real media. Theory and numerical simulation of wave propagation in anisotropic, anelastic, porous and electromagnetic media, 3rd ed.* Elsevier, Academic Press, London, UK, 690 pp.

Carcione J.M., Gei D. and Treitel S.; 2010: *The velocity of energy through a dissipative medium*. Geophys., 75, T37-T47.

Carcione J.M., Zhu T., Picotti S. and Gei D.; 2016: *Imaging septaria with Q compensation by reverse-time migration*. Netherlands J. Geosci., 95, 283-291.

Carcione J.M., Cavallini F., Ba J., Cheng W. and Qadrouh A.; 2019: *On the Kramers- Kronig relations*. Rheol. Acta, 58, 21-28.

Christensen R.M.; 1982: *Theory of viscoelasticity. An introduction, 2nd ed*. Elsevier, Academic Press, London, UK, 378 pp., doi: 10.1016/B978-0-12-174252-2.X5001-7.

de Laer Kronig R.; 1926: *On the theory of the dispersion of X-rays.* J. Opt. Soc. Am., 12, 547-557, doi: 10.1364/JOSA.12.000547.

Golden J.M. and Graham G.A.C.; 1988: *Boundary value problems in linear viscoelasticity.* Springer-Verlag, Berlin, Germany, 280 pp.

Kramers H.A.; 1927: *La diffusion de la lumiere par les atomes.* In: Atti Congresso Internazionale dei Fisici, Como, Italy, Vol. 2, pp. 545-557.

Landau L.D. and Lifschitz E.M.; 1958: *Statistical physics.* Addison-Wesley, Boston, MA, USA, 494 pp.

Mavko G., Mukerji T. and Dvorkin J.; 2009: *The rock physics handbook. Tools for seismic analysis in porous media, 2ed*. Cambridge University Press, Cambridge, UK, 525 pp., doi: 10.1017/CBO9780511626753.

Nowick A.S. and Berry B.S.; 1972: *Anelastic relaxation in crystalline solids*. Elsevier, Academic Press, London, UK, 692 pp.

Orfanidis S.; 2016: *Electromagnetic waves and antennas.* Rutgers University, New Brunswick/ Piscataway, Newark and Camden, NJ, USA, 850 pp., <www.ece.rutgers.edu/~orfanidi/ewa>.

Poon J. and Francis B.; 2009: *Kramers-Kronig relations for lossless media*. Department of Electrical Engineering, University of Toronto, Toronto, Canada, Internal Report, 4 pp.

Corresponding author: Jing Ba
School of Earth Sciences and Engineering, Hohai University
8 West Focheng Road, 211100 Nanjing, China
Phone: 86-2583787234; e-mail: jba@hhu.edu.cn