Wave polarization in transversely isotropic and orthorhombic media

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ABSTRACT

Elastic waves can, in principle, be classified according to their propagation velocity, e.g., using the values of the slowness or the group velocity surfaces along a given direction. The investigation is restricted to media with a distinct outer (P) sheet of the wave surface. We show that there are media with the same velocity distribution but drastically different polarization behavior. Such media are kinematically identical but dynamically different. Therefore, classification according wave velocity alone is not sufficient, and the identification of the wave type should be based on both velocity and the polarization distribution.

For transversely isotropic symmetry there are at most two and, for orthorhombic symmetry, at most four media that have the same velocity distribution. There is always one with a polarization distribution topologically similar to that of isotropy, which is called "normal polarization." All the other media are said to possess "anomalous polarization." Note that this does not imply that such media can be found, but only that they are not forbidden by the laws of physics.

In orthorhombic media there is a different set of four media closely related to the above-mentioned set. The two sets share all velocities along the axes, but the velocity distribution off the axes is different in the second set. All members of this set possess anomalous polarization.

INTRODUCTION

Wave propagation in orthorhombic media has been discussed, e.g., by Musgrave (1981), Helbig (1994) and Schoenberg and Helbig (1997). Musgrave (1981) was concerned with the kinematic classification of orthorhombic media by means of the intersection of the slowness surface with planes of symmetry. Schoenberg and Helbig (1997) concentrated on the geometry of the entire slowness surface and discuss some aspects of polarization, but mostly restrict the discussion to media that are likely to occur in geological settings. In this chapter, we concentrate on polarization, and extend the discussion to all orthorhombic media that conceivably could occur, i.e., that satisfy the stability conditions.

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Slowness and polarization form together the "eigensystem" of the Kelvin-Christoffel matrix in a given direction; thus, they are closely related. The question arises whether this makes the polarization a simple consequence of the slowness, in the sense that the polarization behavior could be derived uniquely from the slowness surface alone. In other words, are there distinct elastic tensors that give rise to different polarization behavior, but identical propagation behavior?

We consider media for which the fastest wave is quasi-compressional and the slower wave is quasi-transverse. This occurs when the compressional stiffnesses $\{c_{11}, c_{22}, c_{33}\}$ are larger than the shear stiffnesses $\{c_{44}, c_{55}, c_{66}\}$; this is strictly true along the axes of symmetry. With the term "anomalous polarization," we describe the strange feature predicted to exist in some transversely isotropic and orthorhombic media *for one and the same slowness surface*. While the polarization along the axes of symmetry agrees with the normal polarization, there exist two "zones" where the polarization of the fast in-plane wavefront is (quasi-) transverse and that of the slow in-plane wavefront is (quasi-) longitudinal.

Consider a transversely isotropic medium whose symmetry axis is in the x_3-direction and assume that the quantity $A_{13} = C_{13} + C_{55}$ (density normalized stiffnesses) is less than zero. Then, such a medium has the same slowness (and group velocity) surface as the medium obtained by replacing C_{13} with $-(A_{13} + C_{55})$, but its polarization behavior is anomalous in the sense that there are some propagation directions for which shear waves are faster than longitudinal waves. A normally polarized, transversely isotropic medium ($A_{13} > 0$) may have one stable anomalous companion, (i.e., $A^*_{13} = -A_{13} < 0$), with the symmetry axis bisecting the angle between the normal and the anomalous polarization vector corresponding to the same propagation vector.

For media of the orthorhombic symmetry class, anomalous polarization *without change of the slowness surface* is possible if it occurs simultaneously in two of the three planes of symmetry. A "normal" orthorhombic medium may have at most three anomalous companions, in such a way that two of the quantities, $A_{12} = C_{12} + C_{66}$, $A_{13} = C_{13} + C_{55}$ and $A_{23} = C_{23} + C_{44}$, are negative. Then, the stiffness matrices of the members of the quadruplet share the same diagonal stiffnesses, but differ in the off-diagonal stiffnesses. Anomalous polarization in the (x_I, x_J)-plane occurs for $A_{IJ} < 0$. If two of them, say A_{IK} and A_{JK}, become negative, the x_K-axis bisects the angle between the normal and the anomalous polarization. In addition, there exists a second quadruplet, with a common slowness surface, that has identical intersections with the planes of symmetry, but differs for propagation directions outside the planes of symmetry. Three companions of this second quadruplet have anomalous polarizations in one plane, and the other has anomalous polarizations in all three planes.

We take the discussion beyond the realm of ray geometry by calculating finite-band wavefields (displayed as snapshots). This is important since it is not immediately obvious how kinematic and dynamic features that are derived under the ray-geometric approximation show up in observational data. Most of the material in the introductory part of the chapter can be found elsewhere. It is repeated here since it is crucial to the arguments in later parts of the chapter.

Slowness surface and polarization

The propagation of elastic waves in anisotropic media in a given direction is governed by a homogeneous system of three linear equations in the three unknown directions of the polarization (or displacement) vector. The coefficients of these equations are combinations of the parameters of the material—the elements of the elastic stiffness tensor and the density—and the direction cosines of the propagation vector. The matrix of these coefficients is called the Kelvin-Christoffel matrix. Nontrivial solutions require that the determinant of the Kelvin-Christoffel matrix vanish. The corresponding equation is called the Kelvin-Christoffel equation. It can be formulated either in terms of the velocity v or in terms of the slowness $s = 1/v$. Slowness times distance is the time needed to cover that distance, and the projection of the slowness vector onto a line of receivers is the "apparent slowness," the time between two receivers with unit separation. Note that these concepts apply to plane waves ("propagation in a given direction"); thus, v is the velocity of plane waves. In nondispersive media, v is the *phase velocity*.

For a given direction, the slowness is given by (the square root of the inverse of) the three eigenvalues of the Kelvin-Christoffel equation, while the polarization vectors are the corresponding eigenvectors. The Kelvin-Christoffel equation thus establishes a relation between the direction of propagation and the directions of the three corresponding displacement vectors.

The set of end points of the slowness vectors in all directions is called the *slowness surface*, which is a convenient embodiment of the propagation properties of the medium. Since three plane waves can propagate with finite slowness in any direction, the slowness surface is a closed surface that is intersected by a "ray" from the center generally in three distinct points. The slowness surface is a convenient and concise representation of the "propagation properties" of plane waves, and we shall sometimes use the term *propagation properties* as a synonym for "slowness surface" or "field of all propagation vectors."

Directions in which the ray intersects the surface in fewer points—corresponding to multiple roots of the Kelvin-Christoffel equation—are called singular directions, and the corresponding multiple point of the slowness surface, a singularity. With a single exception, occurring in transverse isotropy, singularities are "point singularities." These come in two types: for point singularities, sensu strictu, the set of tangents through the singular point form a double cone. In the so-called kissing singularities, there is at least one common tangent to the two touching sheets. In general, kissing singularities can be seen as the coalescence of two point singularities under continuous variation of the elastic stiffnesses. The common tangent is parallel to the connection between the two point singularities just before coalescence.

In view of reciprocity, the traveltime between two points is independent of direction. Thus, slowness in a given direction is the same as in the opposite direction. Thus, the slowness surface is point symmetric, and, according to Neumann's principle (see, e.g., Nye, 1987), its symmetry is at least as high as that of the tensor of elastic stiffnesses, i.e., the slowness surface of an orthorhombic medium has three mutually perpendicular planes of symmetry. The properties of wave propagation in an orthorhombic medium are completely described by those in one octant. The symmetry is such that any feature on an axis (the line of intersection of two symmetry planes) occurs twice, once for each direction of the axis. Any feature in precisely one symmetry plane occurs four times, once in each quadrant of the plane; and any feature occurring away from the three symmetry planes occurs eight times, once in each octant. Such features can be singularities, but there are several other features discussed below.

Transverse isotropy is a subclass of the orthorhombic symmetry class, characterized by an axis of rotational invariance. In view of the reciprocity of elastic wave propagation, this implies that every plane that contains the axis of rotational invariance is a plane of symmetry, and that the plane perpendicular to the axis is an individual plane of symmetry. Rotational invariance means, for example, that any feature not on the axis is drawn out to a "circle of latitude," and reflected in the individual plane of symmetry. For example, a point singularity not on the axis is drawn out to a line singularity. In the terminology of Crampin (1981), this is called an "intersection singularity." Commonly there is a kissing singularity on the axis. Since it can be seen as the coalescence of a line singularity into itself, there is a common tangent perpendicular to the axis in all sagittal planes (i.e., in all azimuths).

Though the slowness surface is met by a ray from the center in three points, it generally does not have three distinct sheets, but is multiply interconnected in the singularities. There is a wide variety of topologically different types of slowness surfaces constrained only by the symmetry requirements and the fact that every elastic slowness surface is of degree six (the Kelvin-Christoffel equation is of degree six in the components of the slowness vector). One consequence of this was pointed out by Musgrave (1981): A distinct innermost sheet must be convex (see Helbig, 1994). Any line intersects a surface of degree six in at most six points. To intersect a distinct inner sheet, the line must have four intersections with the outer system of sheets (connected or distinct) and thus can meet the inner sheet in at most two points—precisely the definition of convexity. With the stiffness ranking $\{c_{11}, c_{22}, c_{33}\} > \{c_{44}, c_{55}, c_{66}\}$, a medium of orthorhombic symmetry has a distinct inner sheet and two interconnected outer sheets. This ranking is common, but many orthorhombic materials with different rankings are known, e.g., spruce (Musgrave, 1981; figure 10 on page 423 shows the slowness surface for spruce). In this study the standard ranking, i.e., all compressional stiffnesses larger than all shear stiffnesses, is assumed.

For transverse isotropy, stability requires that $c_{11} > c_{66}$, but puts no constraint on c_{55}. Thus, there might be media without a distinct inner sheet of the slowness surface. Isotropic media are a subclass of transversely isotropic media, where the stiffness ranking is required by stability. Thus there is always a distinct inner sheet. In isotropic media, the two outer sheets degenerate to a single sheet. In the terminology introduced above, one could call every direction a singular direction. This is generally avoided because of the apparent self-contradiction.

The three polarization vectors for a given propagation direction are, as eigenvectors of a symmetric matrix, mutually perpendicular. A few general statements about the polarization directions can be made on the basis of the symmetry of the elastic tensor.

- If the propagation direction coincides with an *axis of symmetry*, one of the three polarization directions coincides with the propagation direction (it is purely longitudinal), and the other two are constrained to the transverse plane perpendicular to the axis (and thus the propagation direction), i.e., they are "purely transverse." Any direction for which this arrangement of polarizations holds is called a "longitudinal direction." If a longitudinal direction coincides with an element of symmetry, i.e., if its occurrence is determined by the symmetry of the stiffness tensor, it is called a "generic" longitudinal direction. In addition, there can be "free" longitudinal directions. Their occurrence and direction are determined by the relations between the elements of the stiffness tensor. For the condition of occurrence and the maximum number, see Helbig (1993, 1994).
- If the propagation direction lies in a *plane of symmetry*, one purely transverse polarization direction is perpendicular to this plane, while the other two polarization directions lie in the plane of symmetry. One consequence of this is that, for propagation directions in a plane of symmetry, the system of three equations in three unknown direction cosines decouples into a single equation and a system of two equations in two unknown direction cosines.

From the points listed above follow the well-known "polarization properties" of media of higher symmetry. We use the term "polarization properties" somewhat loosely but unambiguously for the overall polarization behavior embodied in the field of all polarization vectors. One has, for instance,

- In isotropic media any direction is an axis of symmetry, thus all directions are longitudinal directions. Since, also, all planes are planes of symmetry, the polarization direction of the transverse waves in the transverse plane is arbitrary and can be characterized with respect to geometric features extraneous to the medium (direction of gravity, free surface, direction of borehole, etc.).
- Transversely isotropic media have one axis of rotational invariance, which is a longitudinal direction. Any plane containing this axis is a plane of symmetry; thus for any direction of propagation one polarization is always purely transverse, perpendicular to the plane containing the propagation direction and the axis of rotational invariance (the sagittal plane). This polarization is referred to as "cross-plane" or "anti-plane" polarization. It is often called "*SH* polarization," since, for a vertical axis of rotational invariance, all sagittal planes are vertical. The other two polarizations lie in the sagittal plane (in-plane polarization). The plane perpendicular to the axis of rotational in-

variance is a plane of symmetry; thus, there is a second purely transverse polarization (parallel to the axis) if the propagation vector lies in the individual plane of symmetry, and all directions perpendicular to the axis are longitudinal directions. Thus, for all directions other than along the axis, the polarization vectors are fixed. Along the axis of symmetry the two shear displacements are constrained to the transverse plane, but are otherwise indeterminate.

- Orthorhombic media have three mutually perpendicular planes of symmetry and three two-fold axes. Thus there are three (generic longitudinal) directions where all three polarization vectors are determined by the symmetry properties of the stiffness tensor; and three planes where, to each propagation direction, correspond one purely transverse (cross-plane) polarization and two in-plane polarizations.

In isotropic media, the two (or, rather, three) sheets of the slowness surface are identified according to the polarization of P for the inner sheet and S for the outer sheet. The names were originally based on the order of the slownesses (the time of arrival of the corresponding wave) with P for the first (primary) and S for the second. Today, the indicators P and S are often connected with polarization, i.e., P with compression (or longitudinal) and S with shear (or transverse), with the specification SH and SV for waves with transverse displacements in the horizontal and vertical planes, respectively. For isotropic media this shift of meaning causes no difficulty. For transversely isotropic media with a vertical axis, one often uses this terminology, sometimes with the qualifier "quasi-", i.e., qP, qSV, and SH, since the first two of these waves are generally not purely longitudinal and transverse, respectively. For a nonvertical axis, the terminology "in-plane polarization" and "cross-plane polarization" is to be preferred. For orthorhombic media, even this breaks down, because one cannot classify in this way the three arrivals outside the planes of symmetry. Moreover, polarization is connected to the sheets of the slowness surface in a much more complicated way. Thus, one has to go back to the names based on the ranking of the three slownesses. It would have been simple to go back to P, S and T (for tertiary), but the shift of meaning for P and S makes this potentially confusing. One speaks thus of the inner, central, and outer sheet or, if the slowness surface is of the standard topological type, and the connection with the polarization simple, of P-waves, fast S- and slow S-waves.

Wave surface, rays and synthetic wave fields

The wavefront at time t, due to a point source, can be regarded as the envelope of plane waves in all directions that leave the origin at $t = 0$. A vector connecting the origin with a point on the envelope is called a ray of length gt, with g the "wave velocity," and a vector of the same direction and length g is called the "wave velocity vector." The set of end points of all wave velocity vectors is called the "wave surface." It is geometrically identical to the wavefront at $t = 1$, but the coordinates of the wave surface are velocities, while those of the wavefront are of dimension length from origin.

The slowness surface and wave surface are related to each other as polar reciprocals, i.e., the normal of one surface is parallel to the corresponding radius vector of the other surface. An element of inflection on one surface, where the sense of rotation of the normal changes with respect to that of the radius vector, thus corresponds to a cuspoidal element in the other surface. The slowness surface has no cuspoidal elements, but it can have elements of inflection of different types. The wave surface can have no elements of inflection, but can have cuspoidal elements and can be much more complicated than the slowness surface.

The elastodynamic equations are solved by a direct grid method. In this approach, the model space is discretized and the three components of the displacement vector are computed at each grid point of the numerical mesh.

SLOWNESS, POLARIZATION, AND RAY VELOCITY OF AN ORTHORHOMBIC MEDIUM

The Kelvin-Christoffel equations for orthorhombic media can be written in terms of the slowness *s* as (e.g., Helbig, 1994)

$$T_{11}\alpha_1 + A_{12}\beta_1\beta_2\alpha_2 + A_{13}\beta_1\beta_3\alpha_3 = 0, \tag{1}$$

$$A_{12}\beta_1\beta_2\alpha_1 + T_{22}\alpha_2 + A_{23}\beta_2\beta_3\alpha_3 = 0, \tag{2}$$

$$A_{13}\beta_1\beta_3\alpha_1 + A_{23}\beta_2\beta_3\alpha_2 + T_{33}\alpha_3 = 0, \tag{3}$$

where β_i are the direction cosines of the propagation vector, α_i are the direction cosines of the displacement vector,

$$A_{23} = C_{23} + C_{44} \qquad A_{13} = C_{13} + C_{55}, \qquad A_{12} = C_{12} + C_{66}, \tag{4}$$

$$\begin{aligned} T_{11} &= C_{11}\beta_1^2 + C_{66}\beta_2^2 + C_{55}\beta_3^2 - \frac{1}{s^2} \\ T_{22} &= C_{66}\beta_1^2 + C_{22}\beta_2^2 + C_{44}\beta_3^2 - \frac{1}{s^2} \\ T_{33} &= C_{55}\beta_1^2 + C_{44}\beta_2^2 + C_{33}\beta_3^2 - \frac{1}{s^2} \end{aligned} \tag{5}$$

with C_{IJ}, $I, J = 1,\ldots, 6$ the density normalized stiffnesses, the square roots of which are wave velocities. The six diagonal elements are related to the velocities along the coordinate axes.

The Kelvin-Christoffel equation is obtained by setting the determinant of the system (1)–(3) equal to zero. This yields an equation of third degree in $v^2 = 1/s^2$ (or sixth degree in *s*) (e.g., see Helbig, 1994, p. 172):

$$-\left(\frac{1}{s^2}\right)^3 + I_1\left(\frac{1}{s^2}\right)^2 - I_2\left(\frac{1}{s^2}\right) + I_3 = 0, \tag{6}$$

with the coefficients

$$I_1 = \beta_1^2(C_{11} + C_{66} + C_{55}) + \beta_2^2(C_{66} + C_{22} + C_{44}) + \beta_3^2(C_{55} + C_{44} + C_{33}), \tag{7}$$

$$\begin{aligned} I_2 &= \beta_1^4[C_{11}(C_{66} + C_{55}) + C_{66}C_{55}] \\ &+ \beta_2^4[C_{22}(C_{44} + C_{66}) + C_{44}C_{66}] \\ &+ \beta_3^4[C_{33}(C_{55} + C_{44}) + C_{55}C_{44}] \\ &+ \beta_2^2\beta_3^2[C_{22}C_{33} + C_{22}C_{55} + C_{33}C_{66} + C_{44}(C_{44} + C_{55} + C_{66}) - A_{23}^2] \\ &+ \beta_1^2\beta_3^2[C_{11}C_{33} + C_{33}C_{66} + C_{11}C_{44} + C_{55}(C_{44} + C_{55} + C_{66}) - A_{13}^2] \\ &+ \beta_1^2\beta_2^2[C_{11}C_{22} + C_{11}C_{44} + C_{22}C_{55} + C_{66}(C_{44} + C_{55} + C_{66}) - A_{12}^2] \end{aligned} \tag{8}$$

$$
\begin{aligned}
I_3 &= \beta_1^6 C_{11}C_{66}C_{55} + \beta_2^6 C_{22}C_{44}C_{66} + \beta_3^6 C_{33}C_{55}C_{44} \\
&+ \beta_2^2\beta_3^2[C_{44}(\beta_2^2 C_{22}C_{55} + \beta_3^2 C_{33}C_{66}) \\
&+ (\beta_2^2 C_{66} + \beta_3^2 C_{55})(C_{22}C_{33} + C_{44}^2 - A_{23}^2)] \\
&+ \beta_1^2\beta_3^2[C_{55}(\beta_3^2 C_{33}C_{66} + \beta_1^2 C_{11}C_{44}) \\
&+ (\beta_3^2 C_{44} + \beta_1^2 C_{66})(C_{11}C_{33} + C_{55} - A_{13}^2)] \\
&+ \beta_1^2\beta_2^2[C_{66}(\beta_1^2 C_{11}C_{44} + \beta_2^2 C_{22}C_{55}) \\
&+ (\beta_1^2 C_{55} + \beta_2^2 C_{44})(C_{11}C_{22} + C_{66}^2 - A_{12}^2)] \\
&+ \beta_1^2\beta_2^2\beta_3^2[C_{11}C_{22}C_{33} + 2C_{44}C_{55}C_{66} \\
&+ 2A_{23}A_{13}A_{12} + C_{11}(C_{24}^4 - A_{23}^2) + C_{22}(C_{55}^2 - A_{13}^2) + C_{33}(C_{66}^2 - A_{12}^2)].
\end{aligned} \tag{9}
$$

Note that multiplication of equation (6) by s^6 changes the equation of order three in one unknown s^2 into an equation of order three for the squares of the components of the of the slowness vector.

There are several (equivalent) expressions for the polarization. For instance, the three direction cosines of the polarization vector corresponding to the solution s and a given propagation direction β_i are proportional to the corresponding minors of the matrix defined by the linear system (1)–(3). For instance, in terms of the cofactors of the last row, we have

$$
\alpha_1/\alpha_2/\alpha_3 =
$$

$$
(A_{12}A_{23}\beta_2^2 - A_{13}T_{22})\beta_1\beta_3/(A_{13}A_{12}\beta_1^2 - A_{23}T_{11})\beta_2\beta_3/(T_{11}T_{22}\beta_1^2 - A_{12}\beta_1^2\beta_2^2), \tag{10}
$$

which can be brought into the form

$$
\alpha_1/\alpha_2/\alpha_3 =
$$

$$
\frac{\beta_1}{(A_{12}A_{13}\beta_1^2 - A_{23}T_{11})}\Big/\frac{\beta_1}{(A_{12}A_{23}\beta_2^2 - A_{13}T_{22})}\Big/\frac{\beta_3}{(A_{13}A_{23}\beta_3^2 - A_{12}T_{33})}. \tag{11}
$$

We assume the normalization criterion $\alpha_i\alpha_i = 1$. Again, multiplication by s^6 changes these expressions into a relation between the coordinates of the slowness surface and the polarization vector.

Let us define $F = F(\beta_1, \beta_2, \beta_3)$ to be the determinant of the system (1)–(3). Then, $F = 0$ is the equation of the slowness surface, and the group ray or wave velocity vector is given by (Helbig, 1994)

$$
\underset{\sim}{\mathbf{v}}_g = \frac{\nabla F}{(\nabla F \cdot \underset{\sim}{\beta})}, \tag{12}
$$

where $\boldsymbol{\beta}$ is the propagation unit vector and the dot denotes scalar product. Alternatively, the group velocity can be obtained from the energy velocity, since in elastic media they coincide. Its evaluation involves the calculation of the direction cosines of the displacement vector also (Helbig, 1994, p.163).

ANOMALOUS POLARIZATIONS IN TRANSVERSELY ISOTROPIC MEDIA

The analysis will be carried out for the sagittal (s_1, s_3)-plane of transversely isotropic media, but is equally valid for the three planes of symmetry of orthorhombic media. Moreover, the results apply to any plane containing the symmetry axis of a transversely isotropic medium.

A useful expression for studying the relationship between slowness surface and polarization behavior in the planes of symmetry is obtained by multiplying the ith row of equations (1)–(3) by α_i and then taking the difference. For instance, (row 1) $\times\, \alpha_1$ − (row 3) $\times\, \alpha_3$ yields

$$T_{11}\alpha_1^2 - T_{33}\alpha_3^2 = (A_{23}\beta_3\alpha_3 - A_{12}\beta_1\alpha_1)\beta_2\alpha_2. \tag{13}$$

The right-hand side vanishes for propagation in the (x_1, x_3)-plane ($\beta_2 = 0$) and one simply has

$$\frac{\alpha_1^2}{\alpha_3^2} = \frac{T_{33}}{T_{11}} = \frac{C_{55}s_1^2 + C_{33}s_3^2 - 1}{C_{11}s_1^2 + C_{55}s_3^2 - 1}, \tag{14}$$

where s_1 and s_3 are the components of the slowness vector. The system (1)-(3) reduces in the (x_1, x_3)-plane to

$$(C_{11}s_1^2 + C_{55}s_3^2 - 1)\alpha_1 + A_{13}s_1s_3\alpha_3 = 0, \tag{15}$$

$$(C_{66}s_1^2 + C_{44}s_3^2 - 1)\alpha_2 = 0, \tag{16}$$

$$A_{13}s_1s_3\alpha_1 + (C_{55}s_1^2 + C_{33}s_3^2 - 1)\alpha_3 = 0, \tag{17}$$

and the Kelvin-Christoffel equation decouples into

$$C_{66}s_1^2 + C_{44}s_3^2 - 1 = 0,\ \ \alpha_2 = 1, \tag{18}$$

and

$$(C_{11}s_1^2 + C_{55}s_3^2 - 1)(C_{55}s_1^2 + C_{33}s_3^2 - 1) - A_{13}^2s_1^2s_3^2 = 0,\ \ \alpha_2 = 0, \tag{19}$$

with α_1 and α_3 determined from equations (15) or (17).

In the (s_1, s_3)-plane the two equations (18) and (19) describe an ellipse and a quartic, respectively. However, in the (S_1, S_3)-plane (the "squared slowness" plane) with $S_i = s_i^2$, they describe a straight line and a quadric, respectively. The set of all quadrics goes through the "anchor points" ($1/C_{11}$, 0), (0,$1/C_{55}$), (0, $1/C_{33}$), and ($1/C_{55}$, 0), with the shape controlled by A_{13}. If we regard A_{13} as a set parameter, then a limiting member of the set is obtained for $A_{13} = 0$, i.e., the two straight lines

$$C_{11}S_1 + C_{55}S_3 = 1\ \text{ and } C_{55}S_1 + C_{33}S_3 = 1. \tag{20}$$

For $A_{13}^2 = (C_{11} - C_{55})\,(C_{33} - C_{55})$ the quadric degenerates into the two straight lines through ($1/C_{11}$, 0), (0,$1/C_{33}$), and (0, $1/C_{55}$), ($1/C_{55}$, 0), respectively, and corresponding to an ellipse and a circle, respectively, in the slowness plane. The fact that for $A_{13}^2 = (C_{11} - C_{55})\,(C_{33} - C_{55})$ the slowness surface (and the wave surface) of P and S waves in transversely isotropic media are ellipsoids and spheres, respectively, was mentioned by Rudzki (1911). For $A_{13}^2 < (C_{11} - C_{55})\,(C_{33} - C_{55})$ the quadric is a hyperbola passing through the anchor points, and the two lines lie in the first quadrant "outside" the hyperbola. For $A_{13}^2 > (C_{11} - C_{55})(C_{33} - C_{55})$ the quadric is an ellipse through the anchor points, and the two lines lie in the first quadrant "inside" the squared slowness ellipse.

The numerator and denominator of equation (14) are expressions of the scaled distance of the point (S_1, S_3) from the straight lines in equation (20), respectively. This establishes a graphical relationship between the (squared) tangent of the propagation angle, the squared slowness surface, and the (squared) tangent of the in-plane polarization angle (see Helbig, 1994). An example is shown in Figure 1. Lines of constant $\tan^2\beta$ pass through the origin, and lines of constant $\tan^2\alpha$ through the point P_{13}, the intersection of the two limiting members [equation (20)].We can read off $\tan^2\beta$ and $\cot^2\beta$ on the sides of the square in the lower left. For instance, $\beta = \pi/6$ (= 30°) and $\beta = \pi/3$ (= 60°) correspond to $\tan^2\beta = 1/3$ and $\tan^2\beta = 3$, respectively, $\beta = \pi/4$ (= 45°) to $\tan^2\beta = 1$; $\tan^2\alpha$ (and $\cot^2\alpha$) can be read off similarly on the parallelogram between the origin and P_{13}.

A direct connection between polarization and propagation directions for the planes of symmetry can be established in the following way: rearrange the coupled equations (15) and (17) to read

$$C_{11}\frac{s_1}{s_3} + C_{55}\frac{s_3}{s_1} - \frac{1}{s_1 s_3} = -A_{13}\frac{\alpha_3}{\alpha_1}, \tag{21}$$

and

$$C_{55}\frac{s_1}{s_3} + C_{33}\frac{s_3}{s_1} - \frac{1}{s_1 s_3} = -A_{13}\frac{\alpha_1}{\alpha_3}. \tag{22}$$

Subtraction of the two equation gives

$$(C_{11} - C_{55})\frac{s_1}{s_3} - (C_{33} - C_{55})\frac{s_3}{s_1} = A_{13}\left(\frac{\alpha_1}{\alpha_3} - \frac{\alpha_3}{\alpha_1}\right). \tag{23}$$

This can be written as a quadratic equation in $\alpha_1/\alpha_3 = \tan\alpha$ in terms of $s_1/s_3 = \tan\beta$:

$$\tan^2\alpha - \frac{1}{A_{13}}\left[(C_{11} - C_{55})\tan\beta - \frac{C_{33} - C_{55}}{\tan\beta}\right]\tan\alpha - 1 = 0. \tag{24}$$

Note that since the last term is −1, the two solutions differ by π/2 (i.e., the two directions are orthogonal). Note further that with a change in the sign of A_{13} the polarization angle changes sign, but not magnitude.

The fact that there can be two different polarization distributions and the same transversely isotropic slowness surface was first pointed out by Helbig and Schoenberg (1987), who called this phenomenon "anomalous polarization". Two transversely isotropic stiffness tensors $\underset{\approx}{\mathbf{C}}$ and $\underset{\approx}{\mathbf{C}}^*$, that have all elements identical except

$$C_{13}^* = -(C_{13} + 2C_{55}), \text{ i.e., } A_{13}^* = -A_{13}, \tag{25}$$

give rise to the same slowness surface.

For an "anomalous companion medium" to a transversely isotropic medium to exist, it must be stable. For an anomalous transversely isotropic medium, stability is guaranteed if

$$C_{33}(C_{11} - C_{66}) > C_{13}^{*2} \tag{26}$$

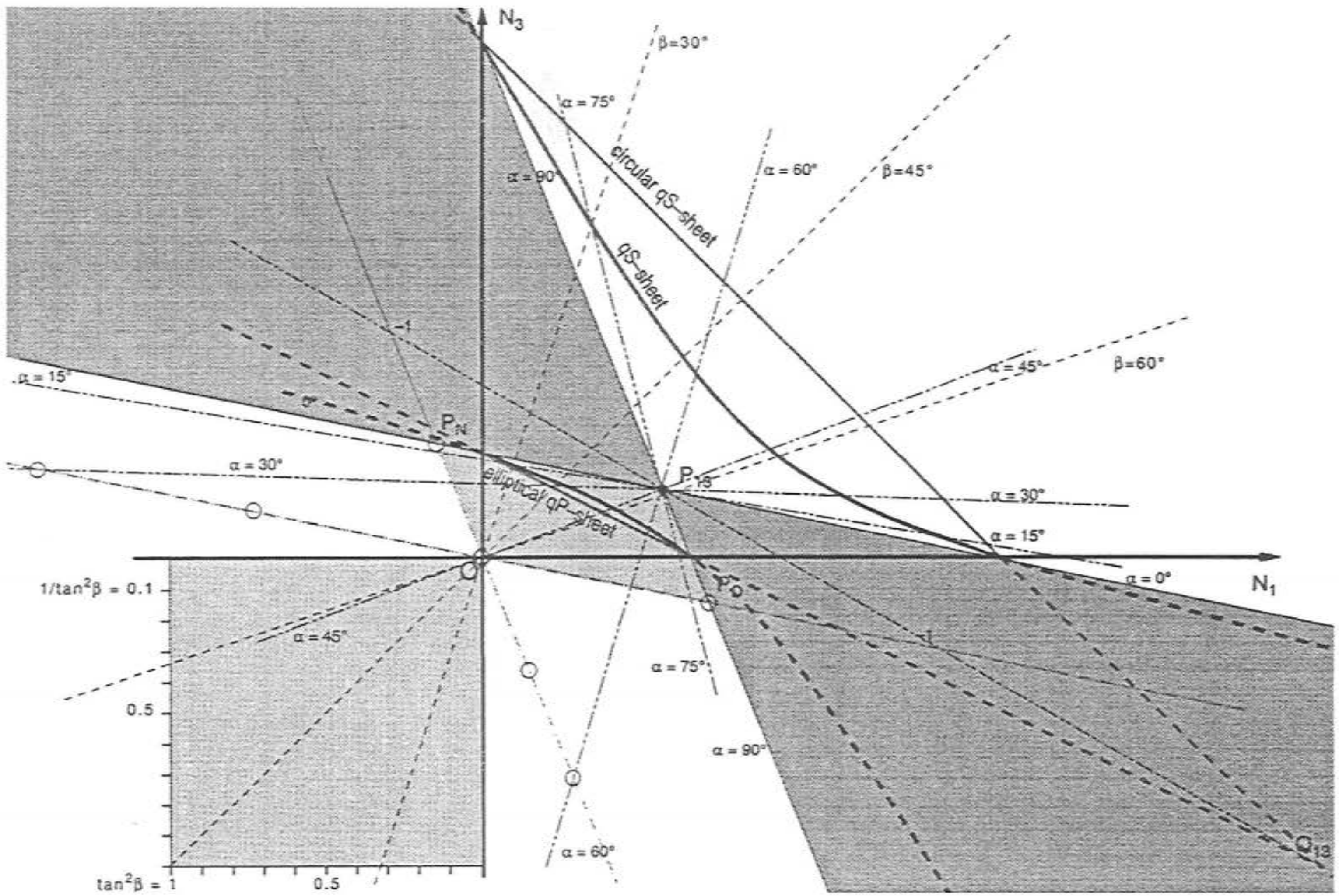


FIG. 1. An example of the graphical relationship between the (squared) tangent of the propagation angle β, the squared slowness surface, and the (squared) tangent of the in-plane polarization angle α as a means to explain the relationship between expressions in equations (14) and (20). For additional examples and further details, see Helbig (1994).

(Helbig, 1994). Substituting equation (25) into equation (26) gives the constraint on C_{13} corresponding to the "normal" medium:

$$-C_{55} < C_{13} < \sqrt{C_{33}(C_{11} - C_{66})} - 2C_{55}, \tag{27}$$

since $C_{55} > 0$ and $C_{13} + C_{55} > 0$.

With the definition in equation (24), the polarization angle α is measured against the x_3-axis, and a change in sign can be interpreted as a change in sign in the x_1-component. The x_3-axis bisects the angle between the two polarization directions.

Alternatively, equation (24) can be written as a quadratic equation in $\tan\beta$ in terms of $\tan\alpha$:

$$\tan^2\beta + \frac{2A_{13}\cot 2\alpha}{C_{11} - C_{55}}\tan\beta - \frac{C_{33} - C_{55}}{C_{11} - C_{55}} = 0, \tag{28}$$

where the property $\tan\alpha - (\tan\alpha)^{-1} = -2(\tan 2\alpha)^{-1}$ has been used.

ANOMALOUS POLARIZATION IN ORTHORHOMBIC MEDIA

Anomalous polarization exists both inside and outside the planes of symmetry of orthorhombic media. The formal existence is discussed in the following section entitled Existence of Anomalous Media. The relations between the polarization direction and the propagation direction are more complicated than in the planes of symmetry, since to each propagation direction there must be three mutually perpendicular polarization directions. However, anomalous polarization outside the planes of symmetry operates in the same manner as in the planes of symmetry.

Existence of anomalous media

The slowness surface, and thus the wave surface, is completely determined by the Kelvin-Christoffel equation (6), i.e., by its coefficients. The dependence of the three coefficients I_1, I_2, and I_3 on the nine elastic stiffnesses of an orthorhombic medium can be described as follows:

1. The "diagonal" stiffnesses C_{11}, C_{22}, C_{33}, C_{44}, C_{55}, and C_{66} determine via equation (7) the three intersections of the slowness surface with each of the three axes of symmetry. The slowness surfaces of all media with the same diagonal stiffnesses form a set of bi-cubic surfaces that pass through these nine points.
2. The absolute value of the combinations $A_{23} = C_{23} + C_{44}$, $A_{13} = C_{13} + C_{55}$, and $A_{12} = C_{12} + C_{66}$ determine the shape of the curves of intersection of the slowness surface with the planes of symmetry. The slowness surfaces of all media with the same diagonal stiffnesses and the same A_{IJ}^2 form a set of bi-cubic surfaces that have identical curves of intersection with the planes of symmetry. This follows directly from the observation that the coefficients contain A_{IJ} in squared form only if one of the three direction cosines β_i of the propagation vectors vanishes, i.e., if it is one of the planes of symmetry [see equation (9)].
3. The slowness surfaces of all media with the same diagonal stiffnesses and the same A_{IJ}, taking the algebraic sign into account, form two sets of bi-cubic surfaces. Within each set there are four identical slowness surfaces. This follows directly from the observation that only the third coefficient in equation (9) contains a term $A_{23}A_{13}A_{12}$ that can change its sign with a change in sign of a single A_{IJ}. The members of the first set are those where an even number (including 0) of A_{IJ} are negative, and the members of the other set are those with an odd number of minus signs.

Polarization behavior

It follows from equations (8) and (9) that the slowness surface (and the wave surface) are not affected by a simultaneous sign change of precisely two of the three off-diagonal factors A_{IJ}. However, this change affects the polarization behavior. It follows from the symmetric form of equation (10) that a change in the algebraic sign of precisely two of the A_{IJ} results in a change in algebraic sign of precisely the two corresponding components $\alpha_K (I \neq J \neq K \neq I)$ of the polarization vector. If A_{IK} and A_{JK} change sign simultaneously, the x_K-axis bisects the two polarization directions. The different polarization arrangements that are possible without a change in the slowness surface are

A_{23}	+	−	+	−
A_{13}	+	−	−	+
A_{12}	+	+	−	−
α_1	+	−	+	−
α_2	+	−	−	+
α_3	+	+	−	−

(29)

For transverse isotropy, anomalous polarization means that the slowness surface is divided into zones: In the normal zones near the x_3-axes (the poles) and near the (x_1, x_2)- plane (the equator), the polarization of the inner (fastest) sheet is close to longitudinal, while for intermediate angles (the anomalous zones), the polarization of this sheet is close to "in-plane transverse." For orthorhombic media the same global distribution of polarizations exists, except that the anomalous polarization is superimposed on the nonrotationally symmetric topology of the orthorhombic slowness surface. Moreover, each of the three symmetry planes can take the role of "equator," so that a given orthorhombic slowness surface may be associated with four different polarizations: A "normal" polarization and three types of "anomalous" polarizations, in the above scheme with the equator the (x_1, x_2)-plane, the (x_2, x_3)-plane, and the (x_1, x_3)-plane and the corresponding anomalous zones, respectively.

The four polarization distributions corresponding to the "normal" slowness surface, with the sign combinations from equation (29), are shown in Figure 2. It shows the intersections of the slowness surface with the three planes of symmetry, and the polarization vector for the fastest (innermost) sheet wherever it makes an angle $\delta > \pi/4$ with the propagation vector. The "zones" of anomalous polarization are clearly visible in Figure 2.

The four sign combinations not listed in equation (29) correspond to another quadruplet of polarization distributions with a common slowness surface. This slowness surface has the same intersections with the planes of symmetry as the quadruplet equation (29). In the region centered about the spatial diagonal the two slowness surfaces differ (see Figure 3). Polarization distributions for the second type of slowness surface are shown in Figure 4. Note that the stability of all four versions of the first slowness surface does not guarantee the stability of the media shown in Figure 4.

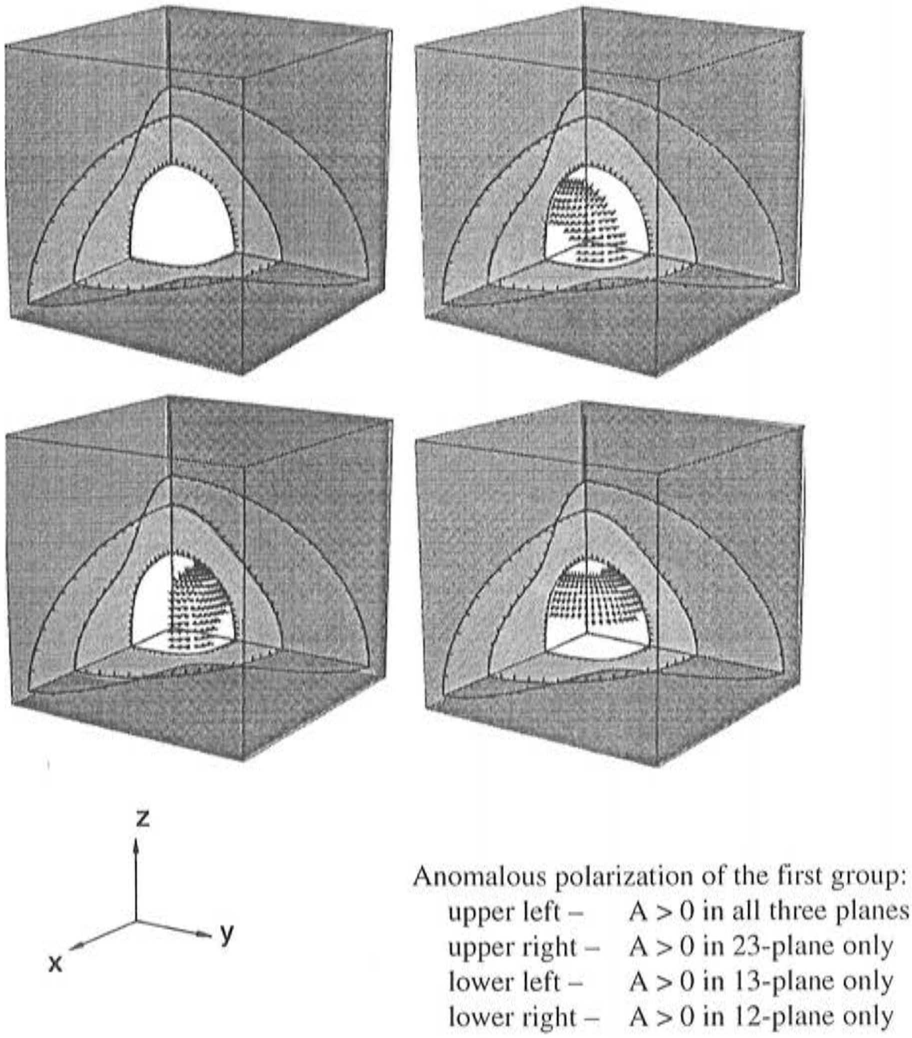


FIG. 2. Distribution of anomalous polarization of the fastest sheet of the slowness surface for $A_{12}A_{23}A_{13} > 0$. Polarization vectors are plotted if they make an angle $\delta > \pi/4$ with the propagation direction. Top left: all three $A_{IJ} > 0$; top right: only $A_{23} > 0$, bottom left: only $A_{13} > 0$; bottom right: only $A_{12} > 0$.

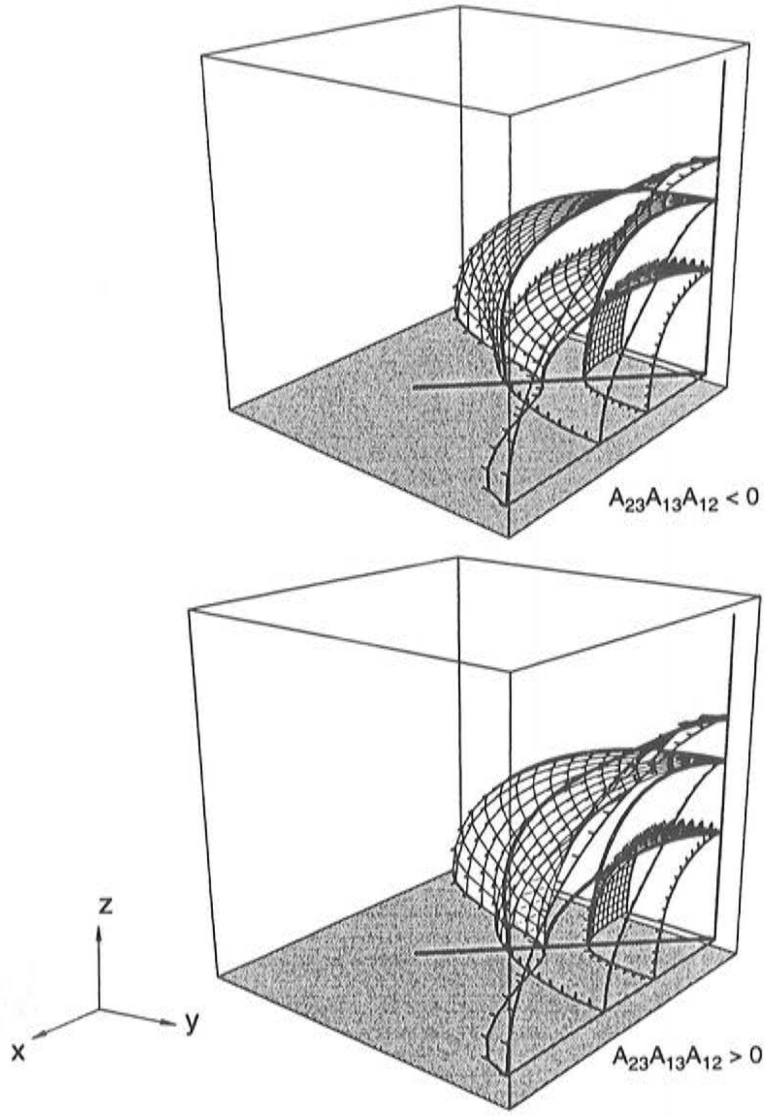


FIG. 3. Two slowness surfaces with the same curves of intersection with the planes of symmetry. Top: with an odd number of negative A_{IJ}; bottom: with an even number of negative A_{IJ}, including the "normal" medium. To show the deviation of the two slowness surfaces in the region surrounding the spatial (1,1,1) diagonal, the curve of intersection with the plane bisecting the (x_1, x_3)-plane and (x_2, x_3)-plane of the upper figure is copied into the lower figure (heavy curves).

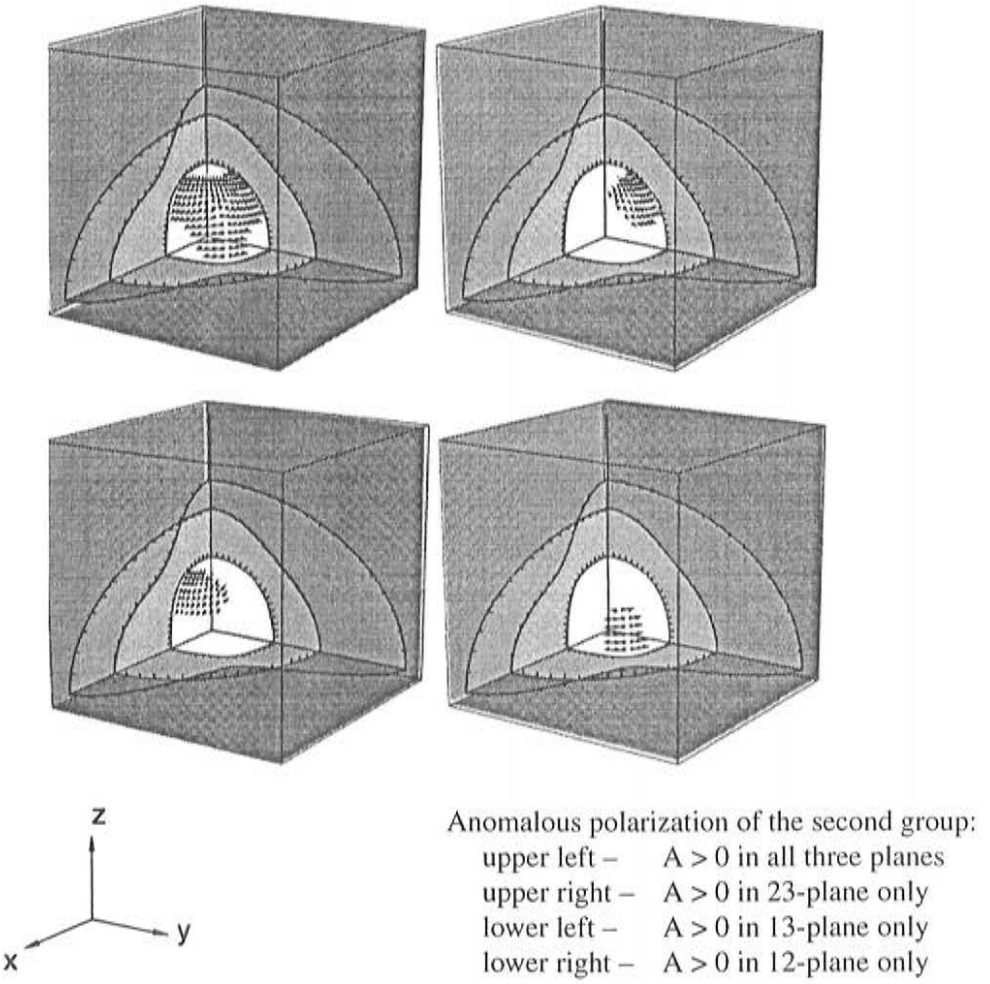


FIG. 4. Distribution of anomalous polarization of the fastest sheet of the slowness surface for $A_{12}A_{23}A_{13} < 0$. Polarization vectors are plotted if they make an angle $\delta > \pi/4$ with the propagation direction. Top left: all three $A_{IJ} < 0$; top right: only $A_{23} < 0$, bottom left: only $A_{13} < 0$; bottom right: only $A_{12} < 0$.

Stability constraints on the elastic stiffnesses

An orthorhombic medium is stable if

$$C_{11} > 0,\ C_{22} > 0,\ C_{33} > 0,\ C_{44} > 0,\ C_{55} > 0,\ C_{66} > 0,$$
$$C_{22}C_{33} - C_{23}^2 > 0,\ C_{11}C_{33} - C_{13}^2 > 0,\ C_{11}C_{22} - C_{12}^2 > 0, \tag{30}$$
$$C_{11}C_{22}C_{33} + 2C_{23}C_{13}C_{12} - C_{23}^2C_{11} - C_{13}^2C_{22} - C_{12}^2C_{33} > 0$$

(see, e.g., Helbig, 1994, p. 192).

In the terms of the dimensionless parameters

$$\xi^2 = \frac{C_{23}^2}{C_{22}C_{33}},\quad \eta^2 = \frac{C_{13}^2}{C_{11}C_{33}},\quad \zeta^2 = \frac{C_{12}^2}{C_{11}C_{22}},$$
$$\Xi^2 = \frac{C_{44}^2}{C_{22}C_{33}},\quad H^2 = \frac{C_{55}^2}{C_{11}C_{33}},\quad Z^2 = \frac{C_{66}^2}{C_{11}C_{22}}, \tag{31}$$

the last four inequalities take the form

$$\xi^2 < 1,\quad \eta^2 < 1,\quad \zeta^2 < 1,$$
$$\xi^2 + \eta^2 + \zeta^2 - 2\xi\eta\zeta < 1. \tag{32}$$

The three inequalities in the first row of equation (32) constrain the point (ξ, η, ζ) to lie inside the cube with corners at $(\pm 1, \pm 1, \pm 1)$. The second inequality further constrains the point to lie inside the closed part of the surface one obtains by replacing the "less than" sign by an equality sign (Musgrave, 1990; Helbig, 1994). This is a surface of third degree with four singularities at $(+1, +1, +1)$, $(-1, -1, +1)$, $(-1, +1, -1)$, and $(+1, -1, -1)$. The six lines through these four points lie entirely on the surface [e.g., equation (33)].

The equation

$$\xi^2 + \eta^2 + \zeta^2 - 2\xi\eta\zeta - 1 = 0 \tag{33}$$

is satisfied, for example, by all points on the pair of straight lines $\eta = \xi$, $\zeta = 1$, and $\eta = -\xi$, $\zeta = -1$ and the two other pairs obtained through exchange of ξ, η, and ζ.

Rewriting equation (33) and setting $\zeta = \zeta_1$ yields an ellipse for $|\zeta_1| < 1$ and a hyperbola for $|\zeta_1| > 1$:

$$\frac{x^2}{1+\zeta_1} + \frac{y^2}{1-\zeta_1} = 1,$$
$$\text{with}\quad x = \frac{1}{\sqrt{2}}(\xi + \eta),\quad y = \frac{1}{\sqrt{2}}(\xi - \eta). \tag{34}$$

A plane containing the ζ-axis and bisecting the other two axes intersects in a parabola:

$$\eta = \xi \Rightarrow x = \sqrt{2}\xi, y = 0 \Rightarrow x^2 = 1 + \zeta. \tag{35}$$

A plane parallel to a face of the cube intersects the surface in an ellipse if the plane passes through the cube, otherwise in a hyperbola; and the axes of the intersection quadrics are parallel to the diagonal of the face of the cube [e.g., equation (34)]. A plane perpendicular to an edge of the tetrahedron passing through the opposite edge, i.e., through two parallel face diagonals of the cube, intersect the surface in a parabola [e.g., equation (35)]. It follows that the surface consists of four open parts extending to infinity, and one closed part connected with the open parts in the four singularities. The closed part of the surface is entirely contained inside the cube; thus it is the "constraint surface" (32), the hull of the constraint volume (see Figure 5).

The resulting relationship between (ξ, η, ζ) and (Ξ, H, Z) is shown in Figures 6 and 7. At least one anomalous companion medium, with two negative A_{IJ}, exists if one of the points $P_\zeta = (\Xi, H, \zeta)$, $P_\eta = (\Xi, \eta, Z)$ or $P_\xi = (\xi, H, Z)$ lies inside the constraint volume. In the example in Figure 6, both points $P_\zeta = (0.5, -0.032, 0.7)$ and $P_\zeta^* = (-0.5, -0.032, 0.7)$ lie inside the constraint ellipse (intersection of the plane $\zeta = 0.7$ with the constraint volume). Anomalous polarization exists if the point $P = (\xi, \eta, \zeta)$ lies inside the intersection of the ellipse with its own image shifted by $-2(\Xi, H, 0)$. Since the intersection has center P_ζ^*, a second point P^* inside the intersection can be found such that $P_\zeta^* = (P + P^*) / 2$. The example shows that media with the same $P_\zeta = (\Xi, H, \zeta)$, only with either a negative A_{23}^* or a negative A_{13}^*, would not be stable, since the points $(-\Xi, H, \zeta)$ and $(\Xi, -H, \zeta)$ both lie outside the constraint ellipse. Thus, no pairs of stable points exist such that these points are their average. Figure 7 shows the situation for simultaneous sign changes in A_{23}, A_{13}, and A_{12}: Now the point $(-\Xi, -H, -Z)$ must lie in the intersection of the constraint volume with the constraint volume shifted by $2(-\Xi, -H, -Z)$. Again this puts a stronger constraint on the admissible points (Ξ, H, Z) —the constraint volume becomes smaller—than the sign change in only two of the A_{IJ}.

Note that the condition

$$\{C_{11}, C_{22}, C_{33}\} > \{C_{44}, C_{55}, C_{66}\} > 0 \text{ implies } 1 > \{\Xi, H, Z\} > 0, \tag{36}$$

i.e., for the media we consider here, the point (Ξ, H, Z) lies in the first octant of the constraint cube [equation (32)]. Figure 8 shows that the constraint surface occupies a sizable part of this unit cube in the first octant. Thus, the possible existence of companion media with anomalous polarization should not be rare. This does, of course, not mean that such companion media should be observed frequently, only that they are rarely ruled out by stability considerations.

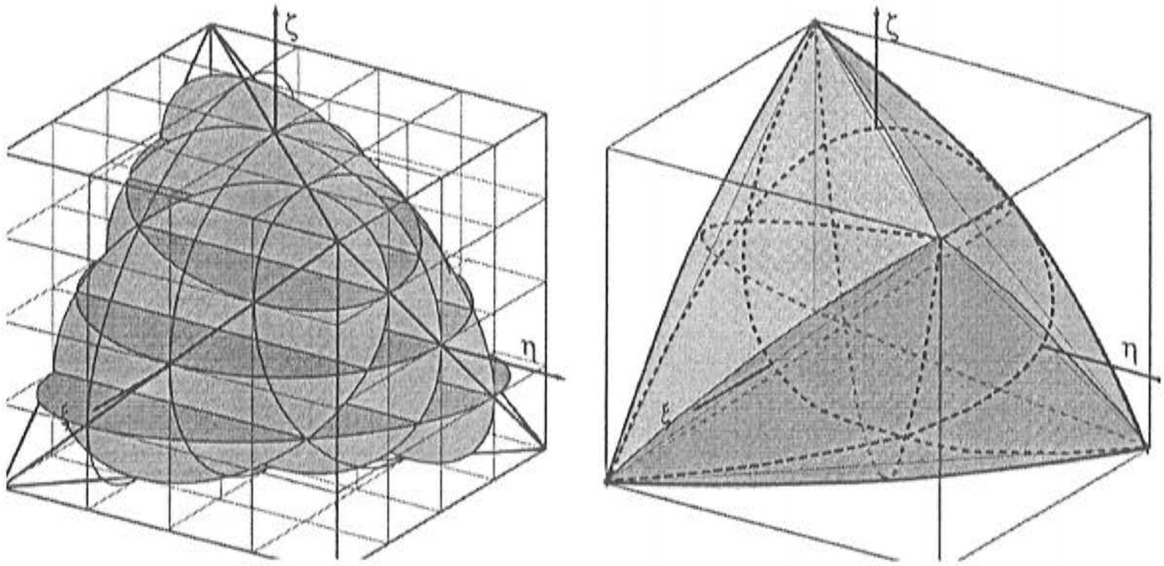


FIG. 5. Constraint volume of orthorhombic media. Stability holds if the diagonal stiffnesses are positive and the point ξ, η, ζ lies inside the constraint volume. This volume is contained in the cube and comprises the tetrahedron. Left: planes parallel to the sides of the cube intersect in ellipses (outside the cube in hyperbolae). Right: planes through opposite face diagonals and opposite edges intersect in parabolas.

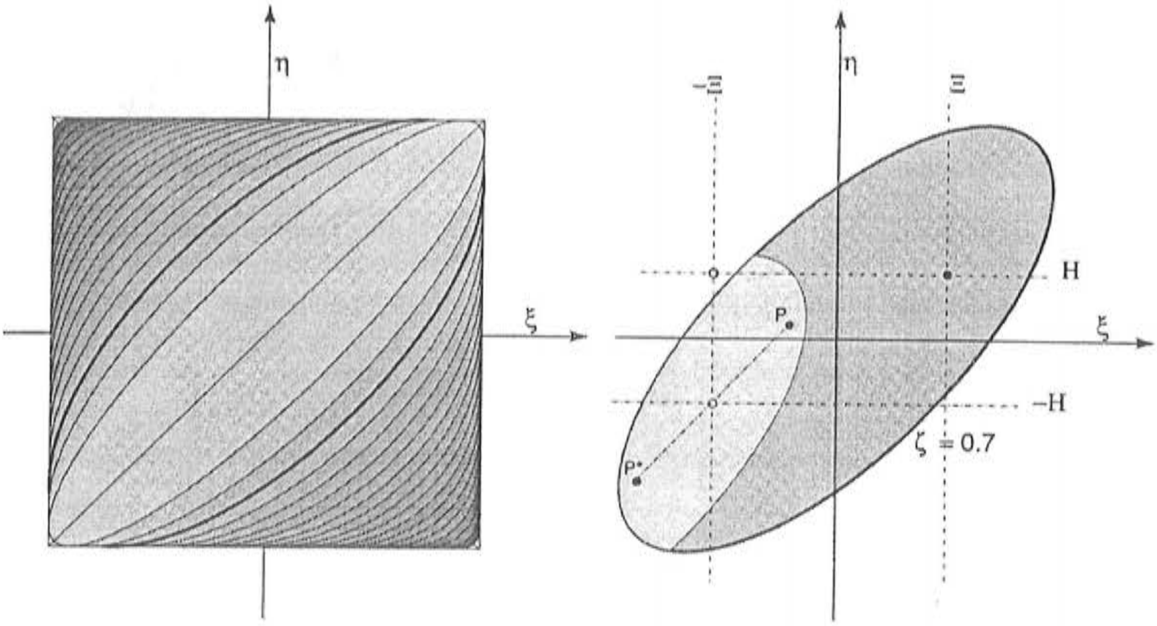
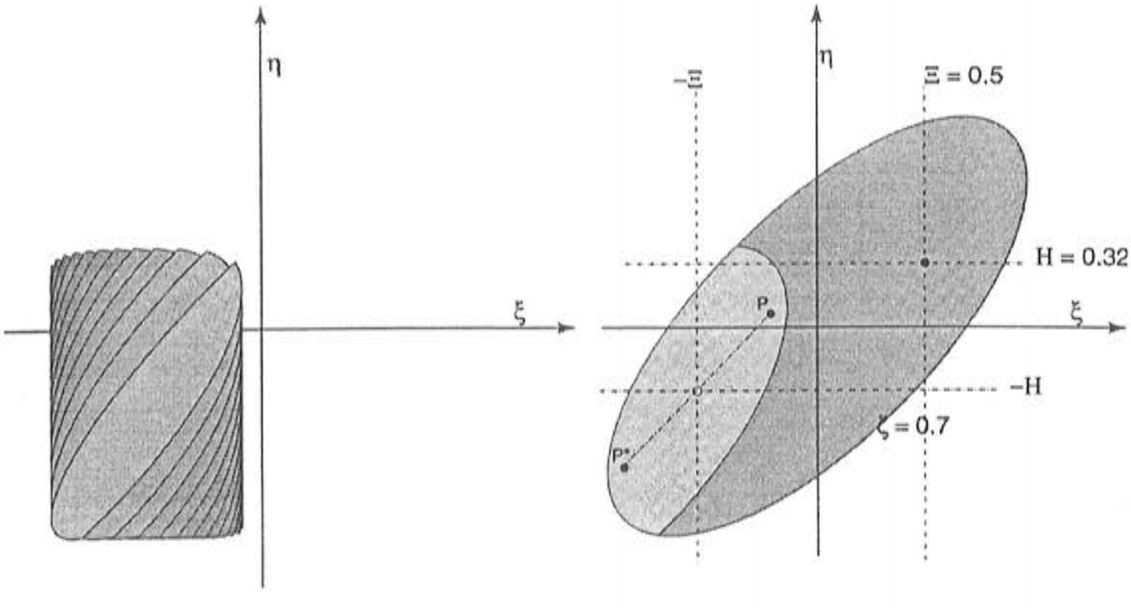


FIG. 6. Left: constraint volume viewed along the ζ-axis. Right: condition for the existence of anomalous polarization in the (x_2, x_3)- and (x_1, x_3)-planes for a given $\zeta = 0.7$. The points $P = (\xi, \eta, 0.7)$ and $P = (\xi^*, \eta^*, 0.7)$ must both lie inside the constraint ellipse for $\zeta = 0.7$ and $(-\Xi, -H, 0.7)$ must bisect PP^*. This is possible with stable stiffnesses if both P and P^* lie inside the intersection of the ellipse and its image after a shift by $-2(\Xi, H, 0)$. The intersection is nonempty if $(\Xi, H, 0.7)$ satisfies the constraint condition in equation (32), i.e., it lies inside the constraint ellipse.



A B

FIG. 7. The constraint volume for the existence of anomalous polarization in all three planes of symmetry is the intersection of the constraint body of Figure 6 with its own image shifted by −2(Ξ, *H*, Z). Each level of this body is the intersection of the two ellipses (indicated at right). A. Intersection of two constraint bodies for Ξ = 0.5, *H* = 0.32, and Z = 0.1. B. Intersection (lighter shade) of two ellipses (Z = 0). Each "level" of the intersection of the two bodies at right corresponds to an intersection of two ellipses.

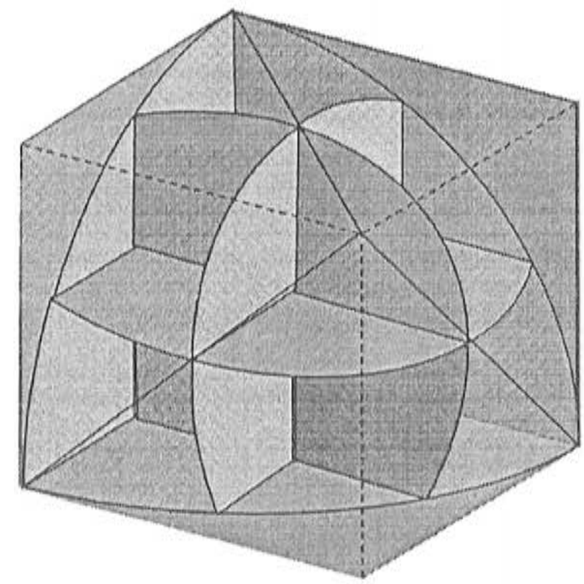


FIG. 8. The first octant of the constraint body (indicated by partial ellipses) occupies a sizable part of the unit cube to which (Ξ, *H*, *Z*) is constrained. Thus, many of the media considered here might have anomalously polarized companions.

SIMULATIONS

In this section, the theoretical results are further verified by means of full wave numerical modeling. First, we consider the examples given by Helbig and Schoenberg (1987) in a plane containing the symmetry axis of a transversely isotropic medium. Since this is an axis of rotational invariance, a 2-D simulation is enough to analyze the polarization behavior. On the other hand, for orthorhombic media we use a 3-D modeling technique in order to analyze the polarization features out of the symmetry planes.

The forward modeling codes are based on the Fourier pseudospectral method for computing the spatial derivatives and Chebychev expansions of the evolution operator as the time-integration technique. The details on the 2-D and 3-D algorithms can be found in Carcione et al. (1988, 1992), respectively. These algorithms possess spectral accuracy for band-limited signals and are not affected by temporal or spatial numerical dispersion.

2-D wave modeling in transversely isotropic media

Equation (6) gives the slowness equation for a transversely isotropic medium if $C_{11} = C_{22}$, $C_{13} = C_{23}$, $C_{44} = C_{55}$, and $C_{12} = C_{11} - 2C_{66}$. Then, the polarization behavior is governed by A_{13}, with equation (24) giving the polarization angle. For the simulation of the examples illustrated by Helbig and Schoenberg (1987) in their Figures 2 and 3 (case 1), 4 (case 2), 5 (case 3) and 6 (case 4), respectively, we adopt their convention for defining the material properties:

$$A = \frac{C_{13}}{C_{55}} + 1, \quad B = \frac{C_{11}}{C_{55}} - 1, \quad \text{and} \quad C = \frac{C_{33}}{C_{55}} - 1. \tag{37}$$

Table 1 shows the values of these constants for normal polarization behavior. Case 3 does not satisfy our basic assumption $\{C_{11}, C_{22}, C_{33}\} > \{C_{44}, C_{55}, C_{55}\}$, and is included only for the sake of comparison. The parameters of the medium with anomalous polarization are the same except for a change in sign for A.

Although irrelevant for the present analysis, we choose $C_{55} = 2.25 \times 10^6$ Pa in mks units. The size of the numerical mesh is $N_X = N_Z = 225$, and the motion is initiated by a directional force making an angle $\pi/4$ with the symmetry axis. In all cases, the central frequency f_c of the source function is the same, and is related to the grid size $D_X = D_Z$ by the Nyquist criterion $4D_X f_c = \sqrt{C_{55}}$.

The snapshots corresponding to the four cases are represented in Figures 9,10,11 and 12, respectively, where only one quadrant of the 2-D model is represented. Figures (a) and (b) represent the ray velocity section for normal and anomalous polarizations, respectively, with the "tadpoles" indicating the polarization directions. Figures (c) and (d) are the respective snapshots with the segments representing the displacement vector every third grid point of the mesh. The ray velocities are normalized with respect to the shear velocity $\sqrt{C_{55}}$.

As predicted by the theory [e.g., equation (15) or (17)], the x_3-axis bisects the angle between the normal and the anomalous polarizations. In general, the anomaly is more pronounced around $\beta = \pi/4$, where the faster wave is transversely polarized and the slower wave is longitudinally polarized. This is particularly evident in case 2, which shows an example of a nearly isotropic wavefront with nonisotropic particle motion directions. Moreover, note that the fastest branch of the triplication event in case 1 (see Figures 9b and 9d) has longitudinal polarization. In case 3, both media display anomalous polarizations. This is caused by the choice $C_{33} < 1 < C_{11}$, i.e., the shear velocity in the x_3-direction is faster than the compressional velocity in that direction. The fast sheet is purely transverse in the x_3-direction and becomes purely longitudinal in the x_1-direction, while the opposite behavior occurs for the inner sheet (see Figures 11a and 11c). The medium with $A < 0$ (Figures 11b and 11d) follows the same rules but is even more anomalous. When A is close to zero in case 4 (Figure 12), the fastest branch of the cusp combines with the outer qP wavefront, and the two sheets degenerate into two intersecting ellipses, one with fixed particle motion in the x_1-direction and the other with fixed particle motion along the x_3-direction. In this case, the anomalous polarized medium ($A < 0$) does not differ greatly from the "normally" polarized medium ($A > 0$), as can be appreciated in Figure 12.

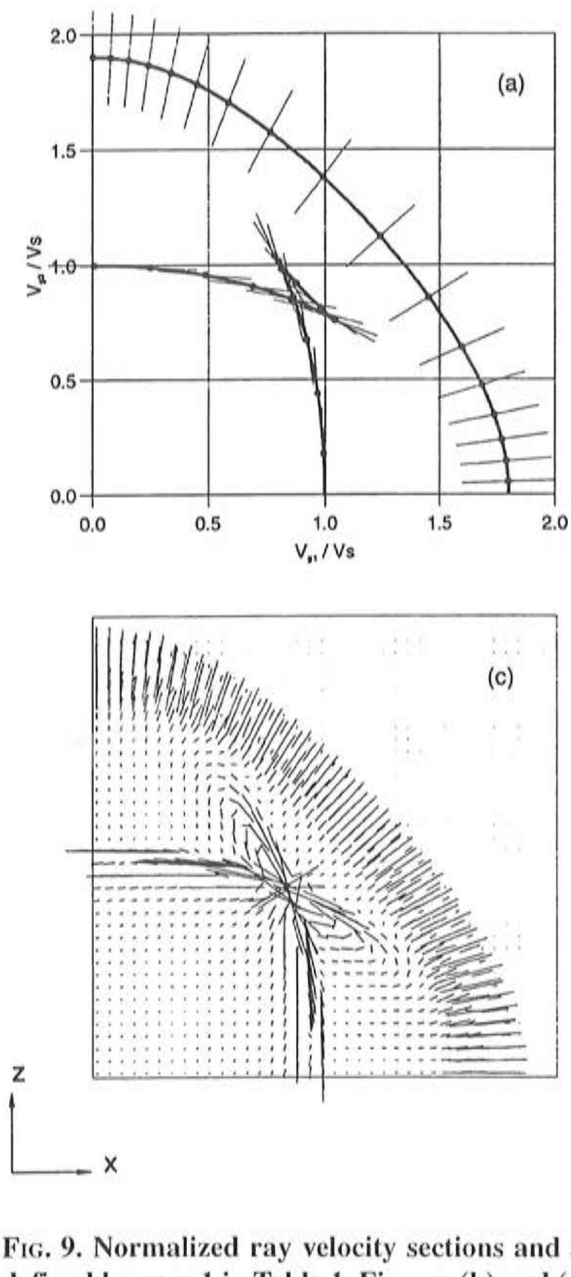


Fig. 9. Normalized ray velocity sections and snapshots for the transversely isotropic media defined by case 1 in Table 1. Figures (b) and (d) correspond to the anomalous medium.

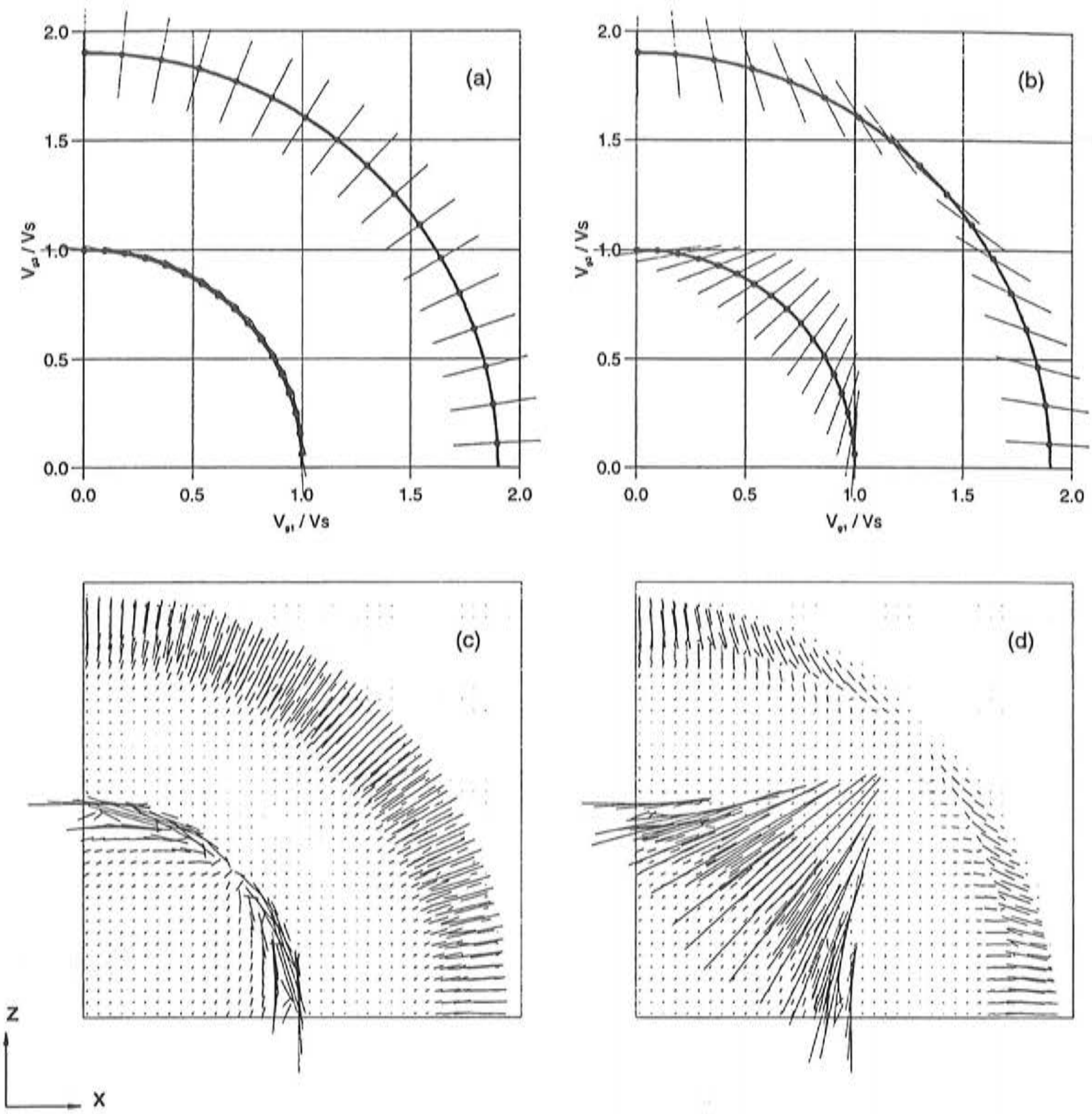


Fig. 10. Normalized ray velocity sections and snapshots for the transversely isotropic media defined by case 2 in Table 1. Figures (b) and (d) correspond to the anomalous medium.

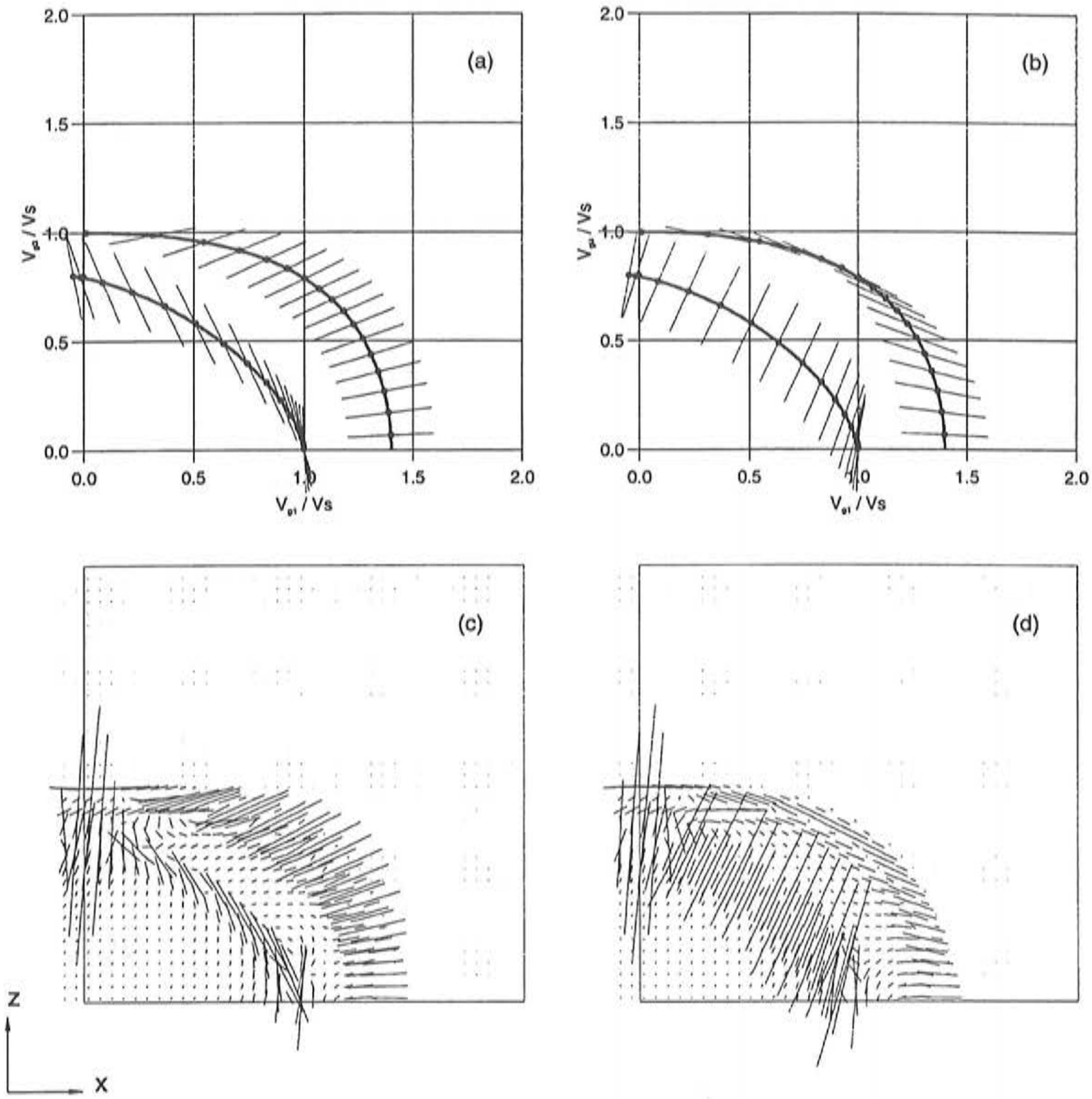


FIG. 11. Normalized ray velocity sections and snapshots for the transversely isotropic media defined by case 3 in Table 1. Figures (b) and (d) correspond to the anomalous medium.

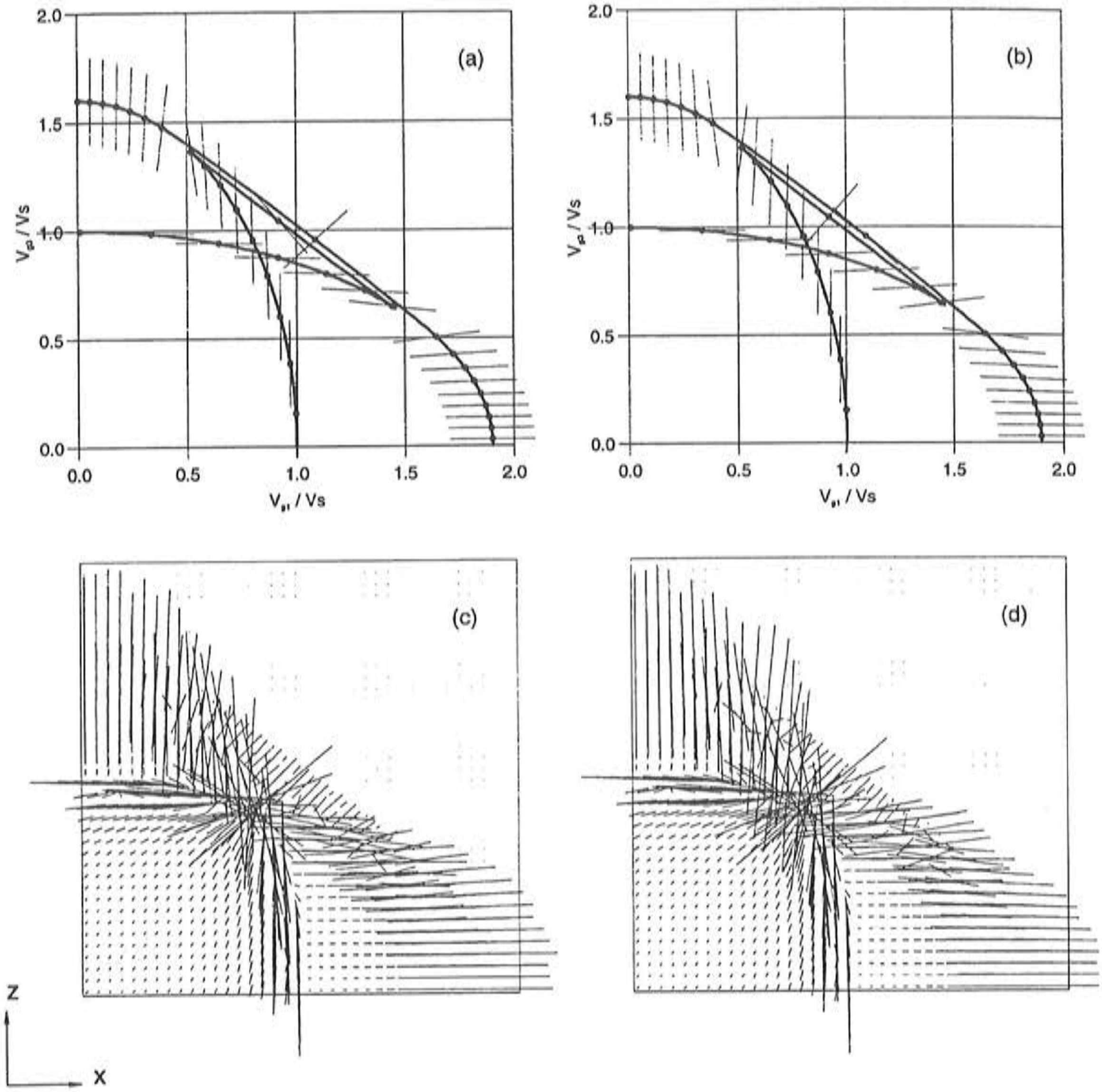


FIG. 12. Normalized ray velocity sections and snapshots for the transversely isotropic media defined by case 4 in Table 1. Figures (b) and (d) correspond to the anomalous medium.

Figure 13 represents the u_x-component for a spherical explosive source, where (a) corresponds to the normal medium, and (b) to the anomalous medium (Case 1 in Table 1). As can be seen, the phase behavior of the compressional wave in the horizontal direction is the same in both cases (the same happens for the vertical displacements). The change in sign when crossing the diagonal directions (Figure 13b), and the fact that the displacement vanishes along those directions, indicate that, at the vertical and horizontal symmetry axes, the wave has the same phase as in the normally polarized medium, i.e., the displacement is directed outwards (or inwards) along the whole wavefront.

Finally, Figure 14 illustrates two u_x-component seismograms corresponding to a single interface. The upper layer is the quasi-isotropic medium indicated with case 2 in Table 1 [(a) normal polarization and (b) anomalous polarization] and the lower medium is isotropic with compressional and shear velocities of 3500 m/s and 2000 m/s, respectively. The source (a vertical force) is located at 1040 m above the interface and has a central frequency of 19 Hz. As expected (see Figure 10), the differences between the two seismograms are important. Note that the events in (b) cannot be rigorously identified as qP or qS, as in Figure 14a.

Table 1. Material properties of the transversely isotropic media.

Case	A	B	C
1	1.2	2.24	2.61
2	2.6	2.61	2.61
3	0.75	0.96	−0.36
4	0.1	2.61	1.56

3-D wave modeling in orthorhombic media

The normalized stiffness matrix of the chosen orthorhombic medium with normal polarization is

$$\underset{\approx}{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} = \begin{bmatrix} 10 & 2 & 1.5 & 0 & 0 & 0 \\ 2 & 9 & 1 & 0 & 0 & 0 \\ 1.5 & 1 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (38)$$

where, in order to obtain "realistic" mks units, the numbers should be multiplied by 10^6. In the following example we assume a simultaneous change in sign of A_{12} and A_{13}. Then, $C^*_{12} = -4$ and $C^*_{13} = -5.5$. Figure 15 shows the slowness surfaces of the normal and anomalous media with the corresponding polarization directions. Note that the anomalous surface is identical to the normal one because both A_{12} and A_{13} have changed sign.

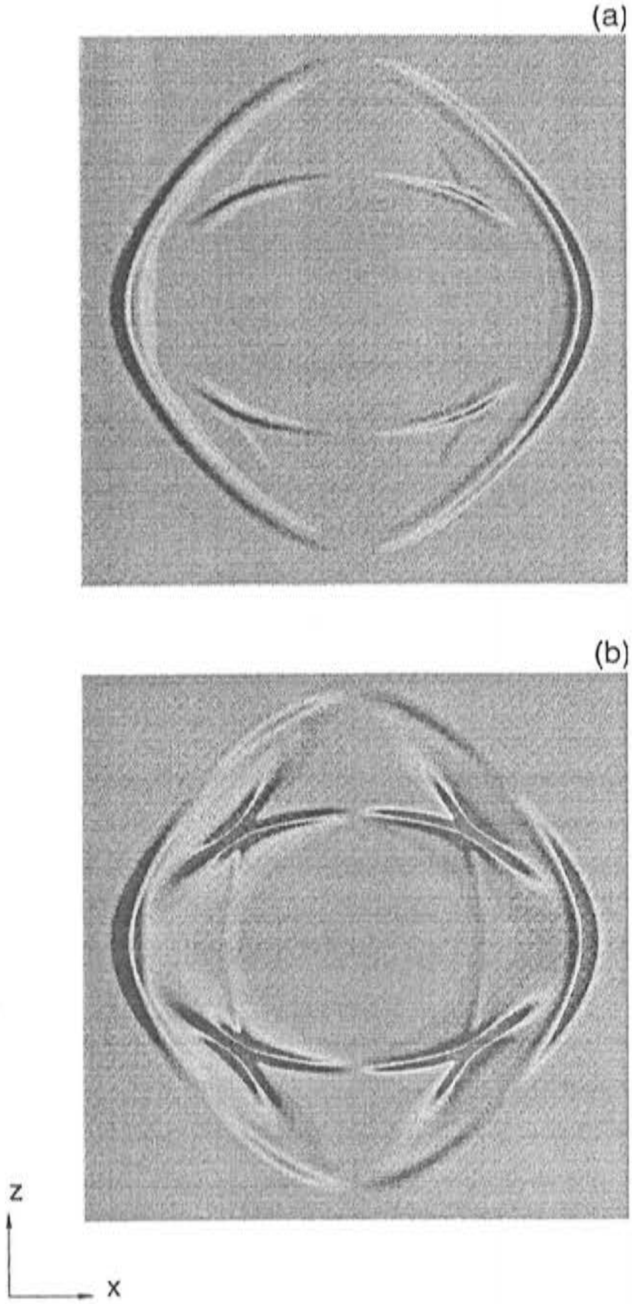
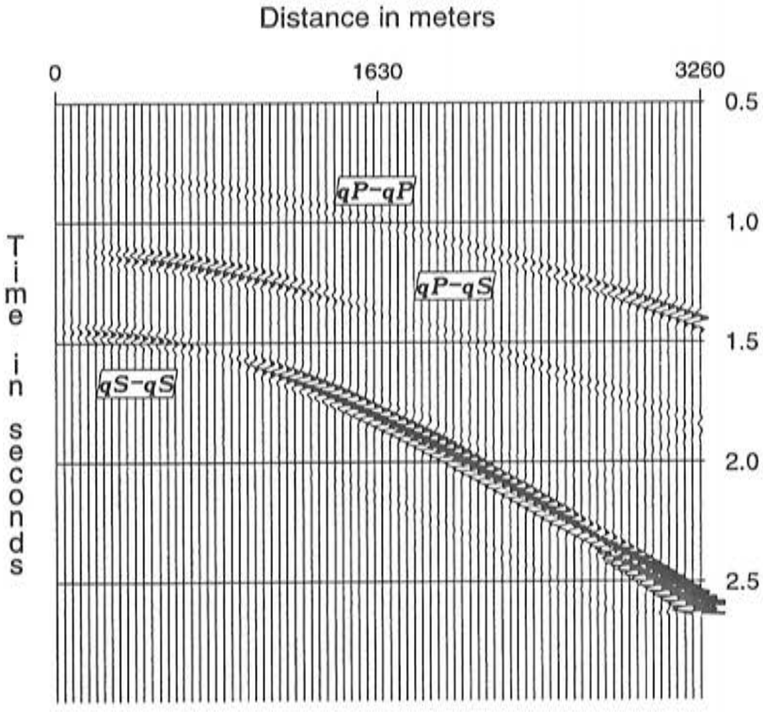
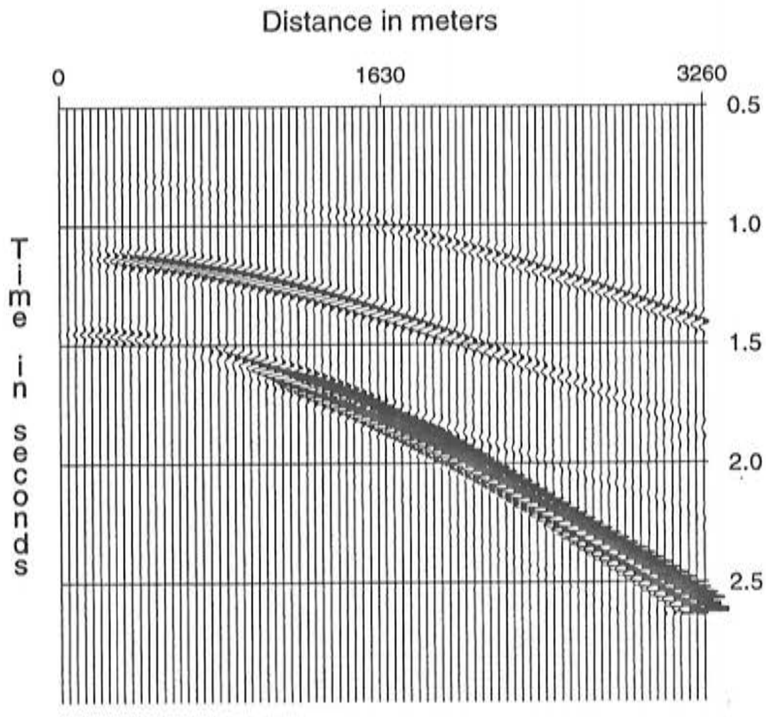


Fig. 13. Snapshots of the u_x-component for (a) normally polarized transversely isotropic medium and (b) anomalously polarized transversely isotropic medium. The source is purely dilatational and the medium corresponds to case 1 of Table 1.



(a)



(b)

FIG. 14. Seismograms of the u_x-component corresponding to a single interface. The upper layer is the quasi-isotropic medium indicated with case 2 in Table 1 [(a) normal polarization and (b) anomalous polarization] and the lower medium is isotropic with compressional and shear velocities of 3500 m/s and 2000 m/s, respectively.

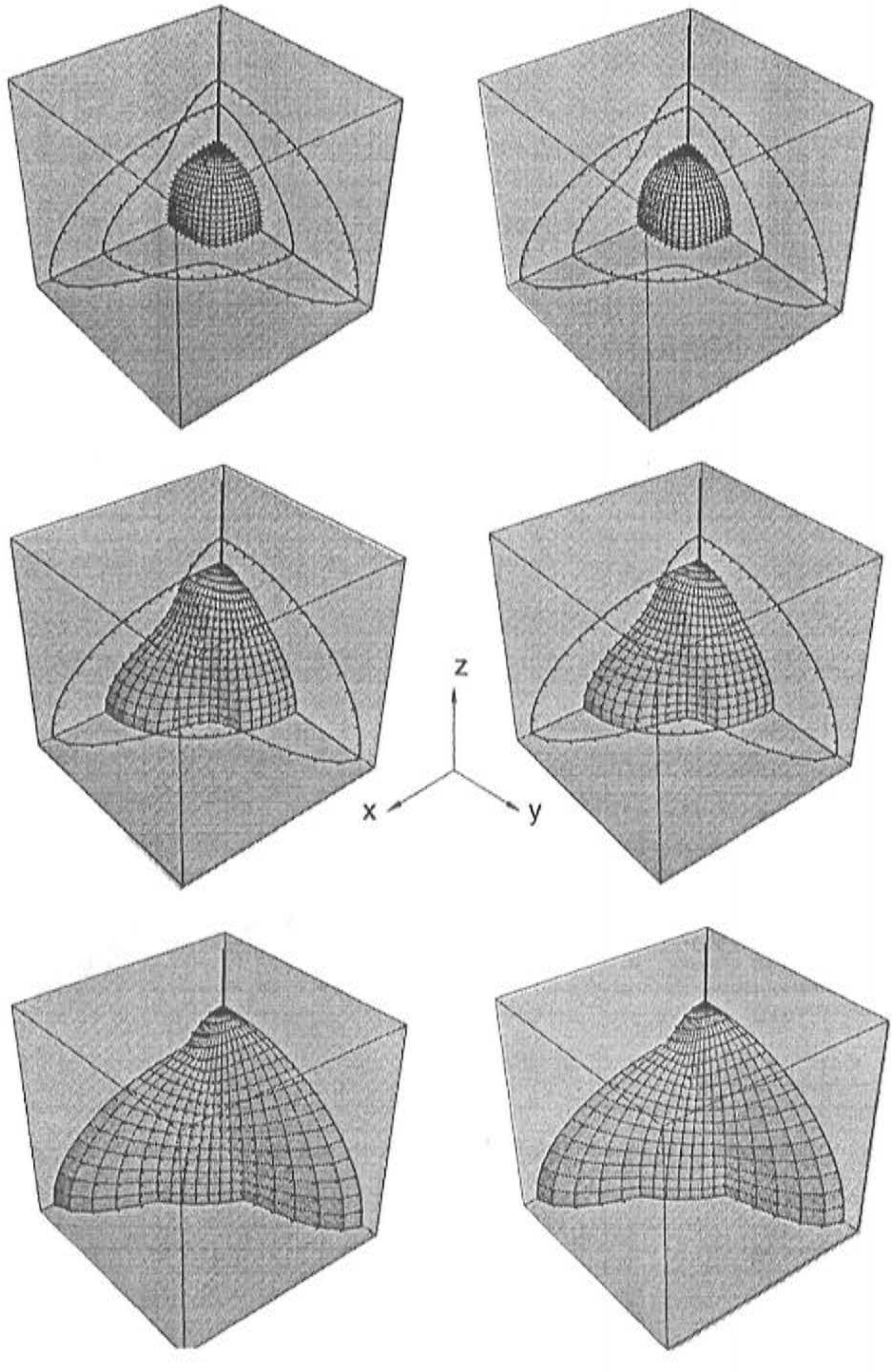


Fig. 15. The three sheets of the slowness surface with normal (left) and anomalous (right) polarization. Axis of anomalous polarization is the x_3-direction. The off-plane singular direction is shown by a short line.

The size of the numerical mesh is $N_X = N_Y = N_Z = 165$, the grid spacing is uniform and the criterion $4D_X f_c = \sqrt{C_{66}}$ is used to avoid wavefield aliasing. The source is a directional force making an angle of $\pi/4$ with the coordinate axes.

Figures 16 shows the ray velocities and corresponding snapshots in the three symmetry planes of the normal and anomalous media. The polarization is indicated on the curves; when it is not plotted, the particle motion is perpendicular to the respective plane (anti-plane shear waves). Only one octant is shown from symmetry considerations. As can be seen, a sign change in A_{12} and A_{13} only affects the (x_1, x_2)- and (x_1, x_3)-planes, leaving unaltered the polarizations in the (x_2, x_3)-plane. As in the transversely isotropic case, the anomaly is more pronounced at about 45° where the polarization of the fastest wave is quasi-transverse and the cusp lid is essentially longitudinal. Moreover, the cross-plane shear wave with polarization perpendicular to the respective symmetry plane can be clearly seen in Figures 16c and d.

The response of the medium to a purely dilatational source is shown in Figure 17. In Figure 17a, the medium is characterized by a change in sign of A_{23} only. This anomalous medium has the same slowness surface as the normal orthorhombic medium along the planes of symmetry. The results are in agreement with Figure 4, indicating anomalous polarization in the (x_2, x_3)-plane only. In Figure 17b, the medium is characterized by a simultaneous change in sign of A_{12} and A_{13}. It can be shown that the phase behavior of the compressional wave along the x_1-axis is the same in both cases. As in the two-dimensional case, the wave has the same phase as the normally polarized medium, i.e., the displacement is directed outwards (or inwards) along the whole wavefront.

Figure 18 shows a snapshot of the polarization vector in a plane tilted 45° ($x_2 = x_3$) and perpendicular to the (x_2, x_3)-plane. The snapshot corresponds to the normal medium. That for the anomalous medium is represented in Figure 19. Several features can be clearly observed despite the difficulty in interpreting the 3-D polarizations. Referring to Figure 17, the outer wavefront is the qP wave, and the qS_1 and qS_2 can be identified from the vertical and horizontal polarization vectors. The polarization directions are, in general, out of the plane (in particular the displacements in the cusps).

In Figure 19, the zone of anomalous polarization can be clearly distinguished in the outer wavefront. According to the theory [see the polarization arrangement diagram in equatoin (29)], a simultaneous sign change of A_{13} and A_{12} implies that the x_2- and x_3-components of the polarization vector change sign. As mentioned before, this is equivalent to saying that the x_1-axis bisects the angle between the normal and the anomalous polarization vectors. This behavior can be observed by the outer wavefront in Figure 18 and 19.

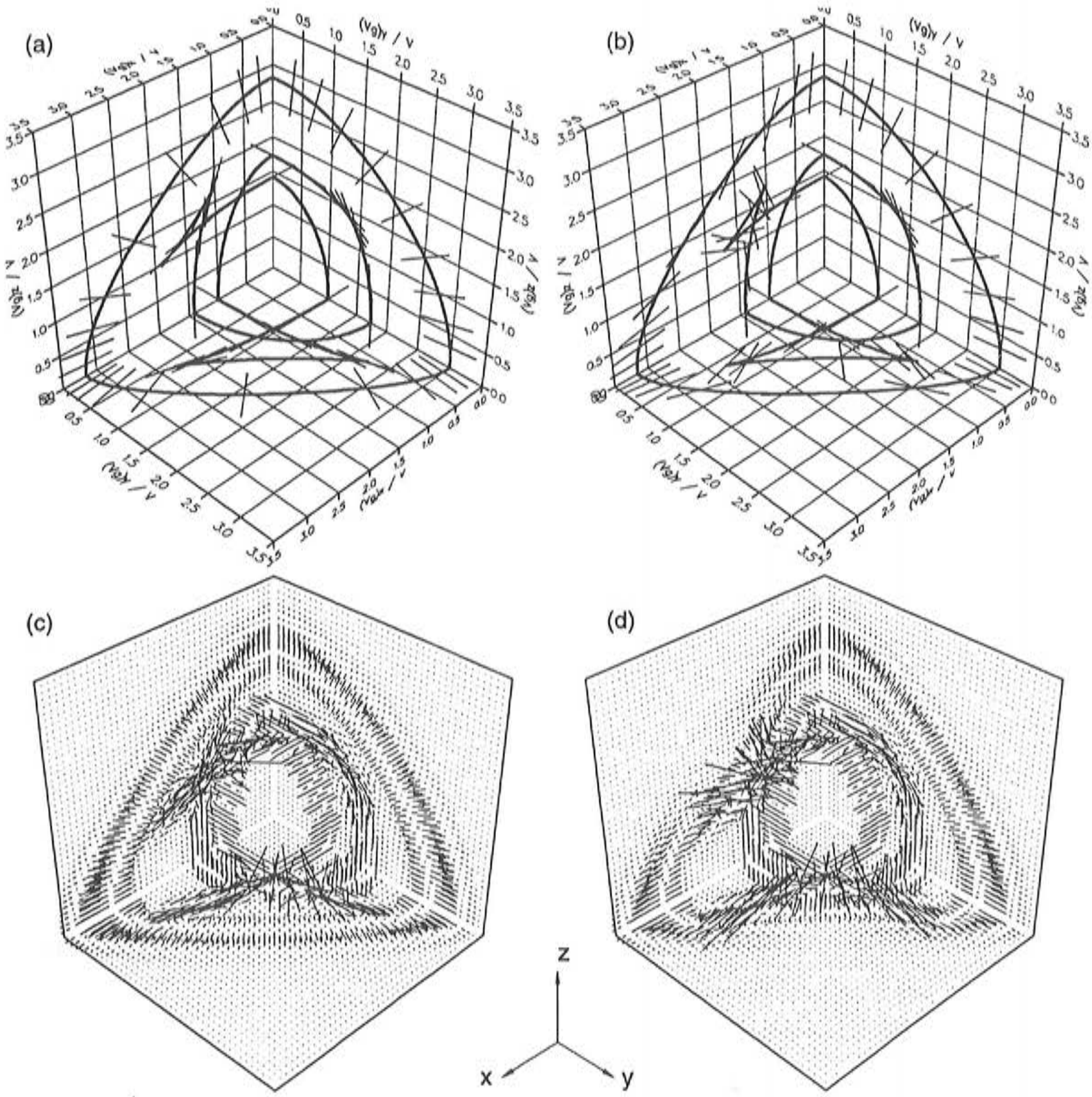


Fig. 16. Ray velocity sections and snapshots at the symmetry planes of the orthorhombic medium. Figures (b) and (d) correspond to the anomalous medium. Only one octant of the model space is displayed for symmetry considerations.

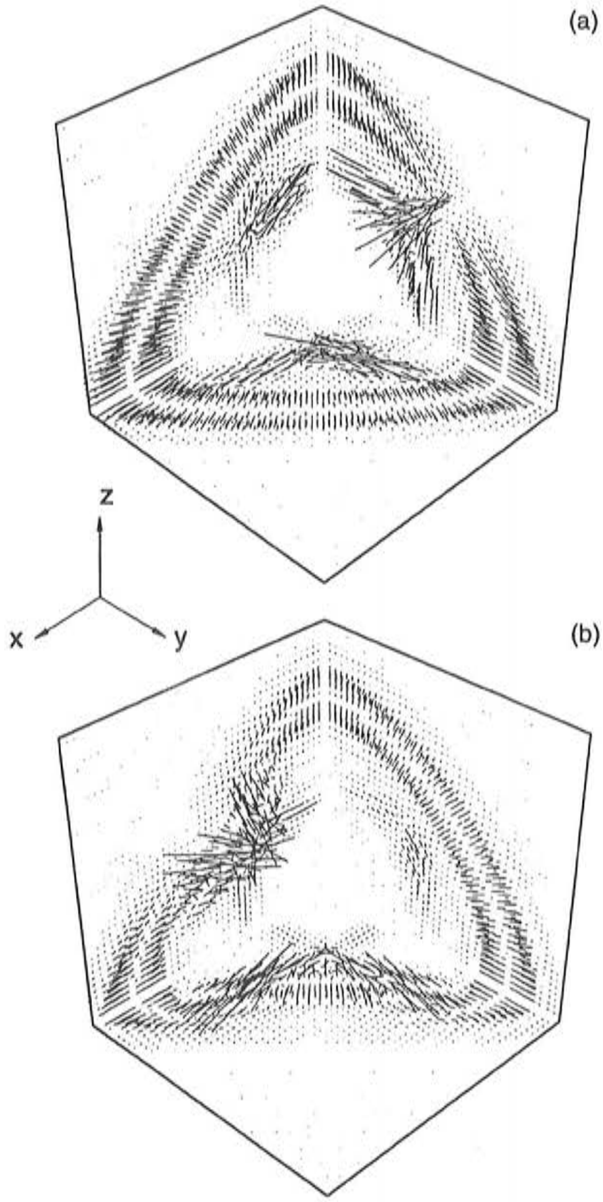


Fig. 17. Snapshots for an explosive source at the symmetry planes of the orthorhombic medium. In (a) the medium corresponds to a change in sign of A_{23} only, and in (b) the medium corresponds to a simultaneous change in sign of A_{12} and A_{13}. The axis orientations are the same as those of Figure 15.

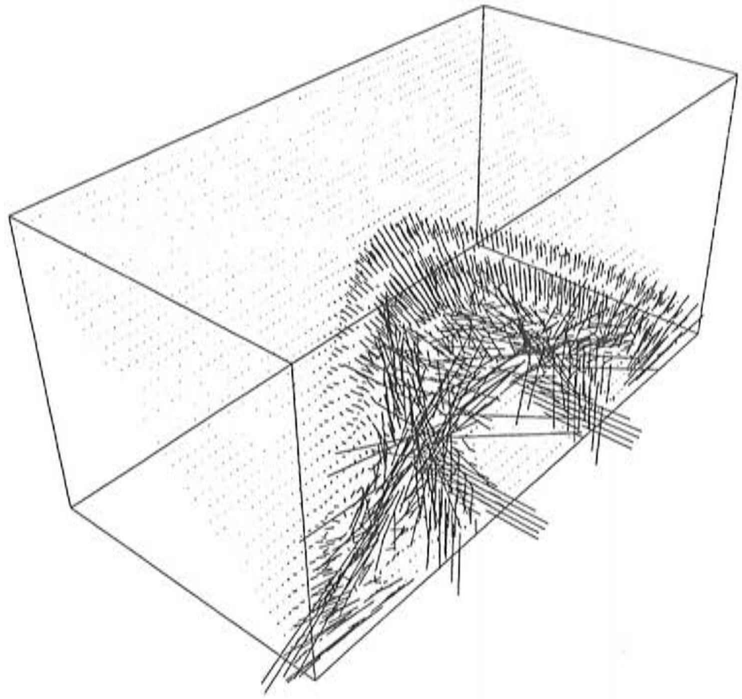


FIG. 18. Snapshot corresponding to the first quadrant of the model space for the normally polarized medium. The plane is tilted 45° ($x_2 = x_3$) and is perpendicular to the (x_2, x_3)-plane. The orientations of the axes are similar to those of Figure 10.

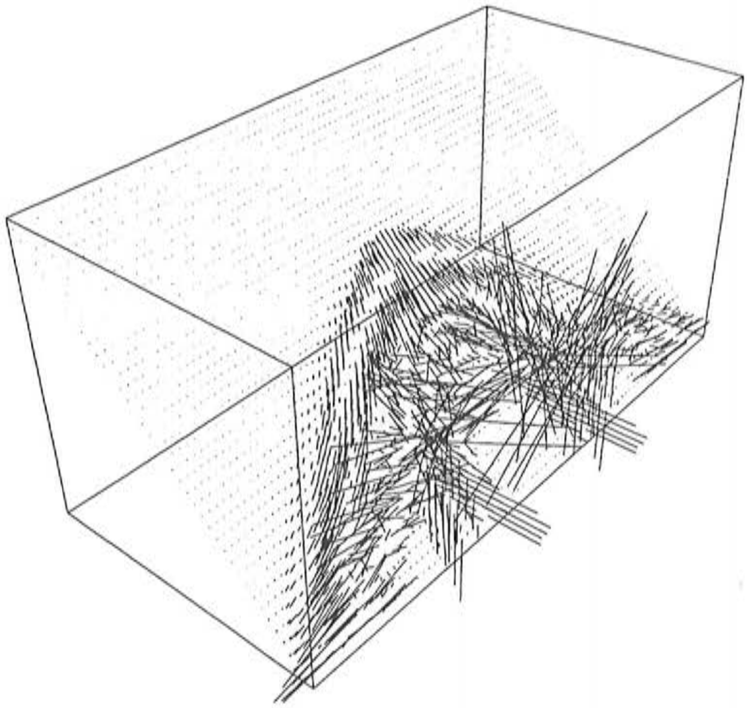


FIG. 19. Snapshot corresponding to the first quadrant of the model space for the anomalous medium. The plane is tilted 45° ($x_2 = x_3$) and is perpendicular to the (x_2, x_3)-plane. The orientations of the axes are similar to those of Figure 10.

CONCLUSIONS

The Kelvin-Christoffel matrix establishes a link between every stable matrix $c_{IJ} \in \Theta_s$ and a slowness-end polarization distribution, i.e., a slowness surface together with the attached polarization. For the inverse problem, the derivation of the elastic matrix from slowness and polarization distribution, it is important to know whether the polarization distribution follows uniquely from the slowness surface. We have shown that this is not the case, even in the subset of Θ_s with members c_{IJ} where $\{c_{11}, c_{22}, c_{33}\} > \{c_{44}, c_{55}, c_{66}\}$.

Consider the set Θ_s of all stable orthorhombic stiffness tensors (for the sake of simplicity referred to as a "natural" coordinate system) with all compressional stiffnesses exceeding all shear stiffnesses, i.e., the set c_{IJ} with $I, J \in \{1, ..., 6\}$, $c_{IJ} = c_{JI}$, $c_{IJ} = 0 \; \forall \; (I > 3 \wedge J \neq I) \vee (J > 3 \wedge J \neq I)$, $c_{IJ} > 0 \; \forall \; I = J$, $c_{II}c_{JJ} - c_{IJ}^2 > 0$ and $c_{11}c_{22}c_{33} + 2c_{23}c_{13}c_{12} - c_{11}c_{23}^2 - c_{22}c_{13}^2 - c_{33}c_{12}^2 > 0$, $c_{II} > c_{JJ}$ for $I \in \{1, 2, 3\}$, $J \in \{4, 5, 6\}$. The entire slowness and displacement distribution in a member of this set is determined completely and uniquely by the corresponding distributions in (a quadrant of each of) the three planes of symmetry, without further explicit reference to the stiffnesses.

Knowledge of the slowness distributions only in the symmetry planes is sufficient for this determination only for members of the subset Θ_s+, with $A_{IJ} = c_{KL} + c_{QQ} > 0$ (no summation), where $K, L \in \{1, 2, 3\}$; $K \neq L$; $Q = 9 - K - L$. Up to eight members of Θ_s can share the same slowness distribution in the symmetry planes (the precise number is determined by stability considerations), but will have different polarization distributions. The eight "companion media" fall into two groups, each with its own (distinct) slowness distribution over all directions. The first group — which includes the member of Θ_s+ — has anomalous polarizations in an even number (zero or two) of symmetry planes, the second has anomalous polarization in an odd number (one or three) of symmetry planes.

Anomalous companion media are characterized by strongly negative off-diagonal stiffnesses. The existence of anomalous companion media is controlled by the diagonal stiffnesses, viz., by the magnitude of the three terms $c_{QQ}^2/(c_{KK}c_{LL}) < 1$ (no summation). The smaller these terms are, the larger the corresponding subset of media that have companion media. The subset of the companion media in the second group is always a subset of the companion media of the first group.

In a symmetry plane, normal polarization means that the polarization angle α increases on average with increasing propagation angle β. If a stable medium c_{IJ}^* with the same slowness distribution in this symmetry plane exists, its polarization angle α^* decreases on average with increasing propagation angle. Along the symmetry axes, the two polarizations are identical; off the symmetry axes, $\alpha^* = -\alpha$, with the symmetry axis bisecting the two polarization directions.

Off the symmetry planes, for media of the first group, the bisecting axis is defined by the two planes with anomalous polarization. If anomalous polarization exists in the (x_i, x_k)- and in the (x_k, x_l)-planes $(i \neq k \neq l)$, the x_k-axis lies in the plane of the two polarization vectors, and bisects the angle between these vectors. Correspondingly, the regions of anomalous polarization $(|\alpha_P - \beta| > \pi/4)$ in this group are restricted to "zones" centered on the axis, where α_P is the polarization angle of the P wave. For media of the second group with anomalous polarization in only one plane, the regions of anomalous polarization are not zones, but "islands" centered about the region of anomalous polarization in the single plane of symmetry. If there is anomalous polarization in all three planes, the three islands are connected in the region about the spatial diagonal.

Anomalous polarization is a phenomenon that does not exist only in the high-frequency limit. It can be simulated by modeling codes and thus should be observable if a medium with a corresponding stiffness tensor can be found (or "constructed"). That such media can be found is unlikely. However, a synthetic study of slowness and polarization distributions in orthorhombic media cannot be complete without taking this possibility into account.

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REFERENCES

Carcione, J. M., Kosloff, D., and Kosloff, R., 1988, Wave propagation simulation in an anisotropic (transversely isotropic) medium: Q. J. Mech. Appl. Math., **41**, 320–345.

Carcione, J. M., Kosloff, D., Behle, A., and Seriani, G., 1992, A spectral scheme for wave propagation simulation in 3-D elastic-anisotropic media: Geophysics, **57**, 1593–1607.

Crampin, S., 1981, A review of wave motion in anisotropic and cracked elastic media: Wave Motion, **3**, 343–391.

Helbig, K. and Schoenberg, M., 1987, Anomalous polarization of elastic waves in transversely isotropic media: J. Acoust. Soc. Am., **81**, 1235-1245.

Helbig, K., 1993, Longitudinal directions in media of arbitrary anisotropy: Geophysics, **58**, 680–691.

———1994, Foundations of anisotropy for exploration seismics: Pergamon Press, Inc.

Musgrave, M. J. P., 1981, On the elastodynamic classification of orthorhombic media: Proc., Roy. Soc. London, Ser. A, **374**, 401–429.

———1990, On the constraints of positive-definite strain energy in anisotropic elastic media: Q. J. Mech. Appl. Math., **43**, 605–621.

Nye, J. F., 1987, Physical properties of crystals—Their representation by tensors and matrices: Clarendon Press.

Rudzki, M. P., 1911, Parametrische Darstellung der elastischen Welle in anisotropen Medien: Anzeiger der Akademie der Wissenschaften, Krakau Univ., 503–536.

Schoenberg, M. and Helbig, K., 1997, Orthorhombic media: Modeling elastic wave behavior in a vertically fractured earth: Geophysics, **62**, 1954–1974.