Elastic medium equivalent to Fresnel's double-refraction crystal

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In 1821, Fresnel obtained the wave surface of an optically biaxial crystal, assuming that light waves are vibrations of the ether in which longitudinal vibrations (P waves) do not propagate. An anisotropic elastic medium mathematically analogous to Fresnel's crystal exists. The medium has four elastic constants: a P-wave modulus, associated with a spherical P wave surface, and three elastic constants, c_{44}, c_{55}, and c_{66}, associated with the shear waves, which are mathematically equivalent to the three dielectric permittivity constants ϵ_{11}, ϵ_{22}, and ϵ_{33} as follows: $\mu_0\epsilon_{11} \Leftrightarrow \rho/c_{44}$, $\mu_0\epsilon_{22} \Leftrightarrow \rho/c_{55}$, $\mu_0\epsilon_{33} \Leftrightarrow \rho/c_{66}$, where μ_0 is the magnetic permeability of vacuum and ρ is the mass density. These relations also represent the equivalence between the elastic and electromagnetic wave velocities along the principal axes of the medium. A complete mathematical equivalence can be obtained by setting the P-wave modulus equal to zero, but this yields an unstable elastic medium (the hypothetical ether). To obtain stability the P-wave velocity has to be assumed infinite (incompressibility). Another equivalent Fresnel's wave surface corresponds to a medium with anomalous polarization. This medium is physically unstable even for a nonzero P-wave modulus.
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I. INTRODUCTION

Scientists of the 19th century have made extensive use of the analogy between light and elastic waves to study the behavior of light in matter. In particular, they have exploited the mathematical equivalence between the electromagnetic constitutive equations and the elastic rheological equations. In many cases, this practice led to important discoveries. For instance, Fresnel's formulas and Maxwell's equations were obtained from mathematical analogies—and physical analogies to a lesser degree—with shear wave propagation and Hooke's law, respectively.

As early as the 17th century it was known that light waves and elastic waves are of a similar nature. Hooke believed light to be a vibratory displacement of a medium (the ether), through which it propagates at finite speed. Later, in the 19th century, Maxwell and Lord Kelvin made extensive use of physical and mathematical analogies to study wave phenomena in elastodynamics and electromagnetism. In many cases, this formal analogy becomes a complete mathematical equivalence such that the problems in both fields can be solved by using the same analytical (or numerical) methodology.

Green[1] made the analogy between elastic waves in an incompressible solid (the ether) and light waves. One of the most remarkable analogies in science is the equivalence between electric and elastic displacements used by Maxwell to obtain his famous electromagnetic equations. Fresnel showed that if light were a transverse wave, then it would be possible to develop a theory accommodating the polarization of light. Therefore, the study of acoustic wave propagation and light propagation are intimately related, and this fact is reflected in the course of scientific research. With the advent of the theory of relativity, the concept of the ether was abandoned. However, the fact that electromagnetic waves are transverse waves is very useful.

Carcione and Cavallini[2] showed that the two-dimensional Maxwell equations describing propagation of the transverse-magnetic mode in anisotropic media are mathematically equivalent to the SH wave equation in an anisotropic-viscoelastic solid where attenuation is described with the Maxwell model. The problem of energy definition in the time domain, particularly for lossy media, has been discussed by Carcione[3] using mathematical analogies. Later, Carcione and Robinson[4] established the analogy for the reflection-transmission problem, showing that contrasts in compressibility yield the reflection coefficient for light polarized perpendicular to the plane of incidence (Fresnel's sine law—the electric vector perpendicular to the plane of incidence), and density contrasts yield the reflection coefficient for light polarized in the plane of incidence (Fresnel's tangent law).

Fresnel[5] read a summary of a memoir to the Academy of Sciences in Paris on 26 November 1821. He presented the wave surface of an optically biaxial purely dielectric medium—a crystal such as calcite or iceland spar. Referring to his equation he writes: "If in the construction that Huygens made to determine the direction of the refracted rays by Iceland spar, and which can be applied to any waveform, one substitutes the sphere and the ellipsoid of revolution by the surface composed of the two terms represented by this last

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equation, and operating elsewhere in the same way, one will have two tangent planes at the points of contact, joined at the center of the wave, which will give the direction of the ordinary ray and that of the extraordinary ray." It constitutes, therefore, a generalization of the isotropic equations to the anisotropic case, where double refraction occurs. In this work, we obtain Fresnel's double-refraction equivalent anisotropic elastic medium, starting from the Newton–Euler equations and the stress–strain relations of an orthorhombic elastic medium. The examples illustrate the different cases where the Fresnel wave surface can be obtained.

II. ELECTROMAGNETISM

A. Maxwell's equations

Maxwell's equations, for a purely dielectric (lossless) medium, in the absence of electric and magnetic sources, are[6]

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B},$$

$$\nabla \times \mathbf{H} = \partial_t \mathbf{D}, \tag{1}$$

where $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$ are the electric vector, the magnetic vector, the electric displacement, and the magnetic induction, respectively, the multiplication sign denotes vector product and ∂_t represents the time derivative. Constitutive equations are needed to relate $\mathbf{D}$ and $\mathbf{B}$ to the field vectors. For a dielectrically anisotropic medium we have

$$\mathbf{D} = \boldsymbol{\epsilon} \cdot \mathbf{E},$$

$$\mathbf{B} = \mu_0 \mathbf{H}, \tag{2}$$

where $\boldsymbol{\epsilon}$ is the dielectric-permittivity matrix, μ_0 is the magnetic permeability of vacuum, and the dot indicates matrix product.

For optically biaxial media, the dielectric-permittivity matrix is given by[2]

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{pmatrix}. \tag{3}$$

B. Kelvin–Christoffel matrix and slowness surface

Assume harmonic plane waves with a phase factor

$$\exp[i\omega(t - \mathbf{s} \cdot \mathbf{x})], \tag{4}$$

where $i = \sqrt{-1}$, ω is the angular frequency, $\mathbf{s}$ is the slowness vector, and $\mathbf{x}$ is the position vector. We use the following correspondence between space–time and slowness–frequency domains:

$$\nabla \times \ \to -i\omega \mathbf{s} \times, \quad \partial_t \to i\omega. \tag{5}$$

Substituting the plane wave (4) into Maxwell's equations (1) and using Eqs. (2) and (5) gives

$$\mathbf{s} \times \mathbf{E} = \mathbf{B} = \mu_0 \mathbf{H}, \tag{6a}$$

$$\mathbf{s} \times \mathbf{H} = -\mathbf{D} = -\boldsymbol{\epsilon} \cdot \mathbf{E}. \tag{6b}$$

The vector product of Eq. (6b) with $\mathbf{s}$ and use of Eq. (6a) yields

$$\mathbf{s} \times [(\boldsymbol{\epsilon})^{-1} \cdot \mathbf{s} \times \mathbf{H}] + \mu_0 \mathbf{H} = 0. \tag{7}$$

In terms of components we have

$$(\epsilon_{ijk} s_j (\epsilon_{kl})^{-1} \epsilon_{lpq} s_p + \mu_0 \delta_{iq}) H_q = 0, \quad i = 1, \ldots, 3, \tag{8}$$

where ϵ_{ijk} are the components of the Levi-Civita tensor. Consider the case given by Eq. (3). Then, the equivalent of the elastic Kelvin–Christoffel equation (e.g., Refs. 7 and 8) for the magnetic vector is

$$\boldsymbol{\Gamma} \cdot \mathbf{H} = \mathbf{0}, \tag{9}$$

where the Kelvin–Christoffel matrix is

$$\boldsymbol{\Gamma} = \begin{pmatrix} 1 - \left(\dfrac{s_2^2}{\eta_3} + \dfrac{s_3^2}{\eta_2} \right) & \dfrac{s_1 s_2}{\eta_3} & \dfrac{s_1 s_3}{\eta_2} \\ \dfrac{s_1 s_2}{\eta_3} & 1 - \left(\dfrac{s_1^2}{\eta_3} + \dfrac{s_3^2}{\eta_1} \right) & \dfrac{s_2 s_3}{\eta_1} \\ \dfrac{s_1 s_3}{\eta_2} & \dfrac{s_2 s_3}{\eta_1} & 1 - \left(\dfrac{s_1^2}{\eta_2} + \dfrac{s_2^2}{\eta_1} \right) \end{pmatrix}, \tag{10}$$

where

$$\eta_i = \mu_0 \epsilon_{(i)i}. \tag{11}$$

Equation (9) has two solutions, which represent light waves transversally polarized.[6,7,9]

The dispersion relation (i.e., the vanishing of the determinant of the Kelvin–Christoffel matrix) is the slowness surface:

$$(\eta_1 s_1^2 + \eta_2 s_2^2 + \eta_3 s_3^2)(s_1^2 + s_2^2 + s_3^2) - (\eta_1 \zeta_1 s_1^2 + \eta_2 \zeta_2 s_2^2 + \eta_3 \zeta_3 s_3^2) + \eta_1 \eta_2 \eta_3 = 0, \tag{12}$$

where

$$\zeta_i = \eta_j + \eta_k, \quad j \neq k \neq i, \tag{13}$$

There are only quartic and quadratic terms of the slowness components in the dispersion relation of an anisotropic medium.

C. Fresnel's wave surface

Now, we use the following property:

$$\mathbf{s} \cdot \mathbf{v} = 1, \tag{14}$$

where $\mathbf{v}$ is the group or energy velocity.[7,8] Equation (14) states that the slowness and wave-velocity surfaces are polar reciprocal, i.e.,

$$x s_1 + y s_2 + z s_3 = 1, \tag{15}$$

where we have assumed $\mathbf{x} = (x, y, z) = \mathbf{v}t$, with $t = 1$; x, y, and z define a point in the wave surface. The wave surface can be obtained by standard methods, from the group and energy velocities,[10,7] or using the duality principle.[6] Using these two different methods, it is shown in the Appendix that the wave surface is given by

$$\left(\frac{x^2}{\eta_1}+\frac{y^2}{\eta_2}+\frac{z^2}{\eta_3}\right)(x^2+y^2+z^2) - \left[\frac{x^2}{\eta_1}\left(\frac{1}{\eta_2}+\frac{1}{\eta_3}\right)+\frac{y^2}{\eta_2}\left(\frac{1}{\eta_1}+\frac{1}{\eta_3}\right) + \frac{z^2}{\eta_3}\left(\frac{1}{\eta_1}+\frac{1}{\eta_2}\right)\right]+\frac{1}{\eta_1\eta_2\eta_3}=0, \quad (16)$$

This is Fresnel's wave surface,[5]

$$(a^2x^2+b^2y^2+c^2z^2)(x^2+y^2+z^2)-[a^2x^2(b^2+c^2) + b^2y^2(a^2+c^2)+c^2z^2(a^2+b^2)]+a^2b^2c^2=0, \quad (17)$$

where

$$a=\frac{1}{\sqrt{\eta_1}}, \quad b=\frac{1}{\sqrt{\eta_2}}, \quad c=\frac{1}{\sqrt{\eta_3}} \quad (18)$$

are wave (light) velocities along the principal axes of the crystal ("axes optiques," according to Fresnel). Quoting Fresnel: "The three semi-axes a,b,c, represent here the propagation velocities of the parallel vibrations."

It can easily be shown that at the coordinate planes the wave surface is factorable into two factors, and each factor is an ellipse. Also, the intersection of the slowness surface with the Cartesian planes are ellipses, since the polar reciprocal of an ellipse is another ellipse.

The extension of the theory to the lossy case is given in Born and Wolf,[6] and for instance, in Carcione and Schoenberg,[9] and Carcione.[7]

III. ELASTODYNAMICS

In the following, we obtain the elastic medium equivalent to Fresnel's double-refraction crystal. Fresnel chose his medium to be orthorhombic, thus it is reasonable to look for the mechanical equivalent in a similar medium. Moreover, the electromagnetic field is governed by a tensor of rank two, thus the slowness and wave surfaces are closed surfaces of order 2, i.e., they are ellipsoids (and thus have three mutually perpendicular symmetry planes). Elastic wave fields are governed by tensors of rank 4, thus slowness surfaces are of order 6 and wave surfaces of order n with n an even number with $6\leq n\leq 150$. Thus, geometrically similar slowness and wave surfaces in an elastic medium can exist only if the medium has at least orthorhombic symmetry (to guarantee three symmetry planes), and if given conditions are satisfied to force the surfaces to be (three-axial) ellipsoids [see Eq. (32)].

A. Newton–Euler equation

In the absence of body forces, the Newton–Euler equation governing the dynamic of continuum media is

$$\nabla\cdot\boldsymbol{\sigma}=\rho\partial_{tt}^2\mathbf{u}, \quad (19)$$

where ρ is the density,

$$\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6)^T \quad (20)$$

is the stress vector,

$$\mathbf{u}=(u_1,u_2,u_3)^T \quad (21)$$

is the displacement vector, and

$$\bar{\boldsymbol{\nabla}}=\begin{pmatrix}\partial_1 & 0 & 0 & 0 & \partial_3 & \partial_2\\ 0 & \partial_2 & 0 & \partial_3 & 0 & \partial_1\\ 0 & 0 & \partial_3 & \partial_2 & \partial_1 & 0\end{pmatrix} \quad (22)$$

is the gradient operator.[10,7]

The strain–displacement relation can be written as

$$\mathbf{e}=\bar{\boldsymbol{\nabla}}^T\cdot\mathbf{u}, \quad (23)$$

where

$$\mathbf{e}=(e_1,e_2,e_3,e_4,e_5,e_6)^T \quad (24)$$

is the strain vector

The stress–strain relation reads

$$\boldsymbol{\sigma}=\rho\mathbf{C}\cdot\mathbf{e}, \quad (25)$$

with the elasticity matrix for orthorhombic media—normalized by the density—given by

$$\mathbf{C}=\begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{pmatrix}, \quad (26)$$

where $C_{IJ}=c_{IJ}/\rho$ and c_{IJ} are the elastic constants.

B. Kelvin–Christoffel matrix and slowness surface

Combining Eqs. (19), (23), and (25)yields

$$\bar{\boldsymbol{\nabla}}\cdot\mathbf{C}\cdot\bar{\boldsymbol{\nabla}}^T\cdot\mathbf{u}=\partial_{tt}^2\mathbf{u}. \quad (27)$$

Substituting the plane wave (4) into Eq. (27) and using $\partial_t\rightarrow i\omega$ and

$$\bar{\boldsymbol{\nabla}}\rightarrow -i\omega\begin{pmatrix}s_1 & 0 & 0 & 0 & s_3 & s_2\\ 0 & s_2 & 0 & s_3 & 0 & s_1\\ 0 & 0 & s_3 & s_2 & s_1 & 0\end{pmatrix}\equiv -i\omega\mathbf{S}, \quad (28)$$

we obtain

$$\boldsymbol{\Gamma}\cdot\mathbf{u}=\mathbf{0} \quad (29)$$

where

$$\boldsymbol{\Gamma}=\mathbf{I}-\mathbf{S}\cdot\mathbf{C}\cdot\mathbf{S}^T, \quad (30)$$

is a symmetric Kelvin–Christoffel matrix, and $\mathbf{I}$ is the 3×3 identity matrix.

Equation (29) is analogous to Eq. (9), with

$$\mathbf{\Gamma} = \begin{pmatrix} 1-(C_{11}s_1^2+C_{66}s_2^2+C_{55}s_3^2) & -(C_{12}+C_{66})s_1s_2 & -(C_{13}+C_{55})s_1s_3 \\ -(C_{12}+C_{66})s_1s_2 & 1-(C_{66}s_1^2+C_{22}s_2^2+C_{44}s_3^2) & -(C_{23}+C_{44})s_2s_3 \\ -(C_{13}+C_{55})s_1s_3 & -(C_{23}+C_{44})s_2s_3 & 1-(C_{55}s_1^2+C_{44}s_2^2+C_{33}s_3^2) \end{pmatrix}. \quad (31)$$

The vanishing of the determinant of the Kelvin–Christoffel matrix $\mathbf{\Gamma}$ yields the slowness surface. The three solutions of Eq. (29) in normally polarized media[7] correspond to the qP wave and two qS waves.

C. Elastic medium equivalent to an optically biaxial crystal

We know from the acoustic-electromagnetic analogy[2] that light waves are mathematically analogous to shear waves. Then, in order to find the elastic medium mathematically equivalent to an optically biaxial medium corresponding to Fresnel's equation (17), we have to decouple the P wave from the S waves. These relations are necessary and sufficient:[11]

$$C_{11} = C_{22} = C_{33} = C_{12} + 2C_{66} = C_{13} + 2C_{55} = C_{23} + 2C_{44} \equiv E, \quad (32)$$

where E is a P wave modulus (normalized by the density). Substituting these relations into Eq. (31) gives

$$\mathbf{\Gamma} = \begin{pmatrix} 1-(C_{66}s_2^2+C_{55}s_3^2) & C_{66}s_1s_2 & C_{55}s_1s_3 \\ C_{66}s_1s_2 & 1-(C_{66}s_1^2+C_{44}s_3^2) & C_{44}s_2s_3 \\ C_{55}s_1s_3 & C_{44}s_2s_3 & 1-(C_{55}s_1^2+C_{44}s_2^2) \end{pmatrix} - E\begin{pmatrix} s_1^2 & s_1s_2 & s_1s_3 \\ s_1s_2 & s_2^2 & s_2s_3 \\ s_1s_3 & s_2s_3 & s_3^2 \end{pmatrix}. \quad (33)$$

It can easily be shown that the dispersion relation obtained from Eq. (33) gives two coupled S waves and a decoupled isotropic compressional wave, with a spherical wave surface and velocity $\sqrt{E}$. Anisotropic elastic materials having spherical wave surfaces are discussed by Ting.[12] Other anisotropic media propagating pure longitudinal waves are studied by Rychlewski[13] and Ostrosablin.[14] Note that for $E=0$, S waves propagate only. In isotropic media $\rho E=\lambda+2\mu$, where λ and μ are the Lamé constants. If $E=0$, the compression modulus $k=\lambda+(2/3)\mu=-(4/3)\mu<0$. Such a medium would be unstable (see the following).

The mathematical analogies between elastic and light waves were known to scientists of the 19th century. In fact, Fresnel, Green, MacCullagh, and Cauchy, among others, obtained expressions of wave surfaces and reflection coefficients in crystals by using the elastodynamic equations, assuming that light waves are the vibrations of a medium called ether. Fresnel[5] states "I consider a medium endowed with the double refraction having different elasticities along the various directions."

MacCullagh,[15] (see also Ref. 16, p. 141), in a conference held at the Royal Irish Academy in 1839, presented an isotropic medium, whose potential energy is only based on rotation of the volume elements, thus ignoring pure dilatations from the beginning. The result is a rotationally elastic ether and the wave equation for shear waves. The corresponding reflection and refraction coefficients coincide with Fresnel's formulae.[6] Green[1] assumed the P-wave velocity to be infinite and dismissed a zero P-wave velocity on the basis that the medium would be unstable since the compression modulus is negative (from the previous discussion, please note that the potential energy must be positive). Cauchy,[17] (see also Ref. 16) neglecting this fact considered that P waves have zero velocity ($E=0$), and obtained the sine law and tangent law of Fresnel. He assumed the shear modulus to be the same for both media. Cauchy's ether is known as the *contractile* or *labile aether*. It corresponds to an elastic medium of negative compressibility, as shown earlier.

Fresnel[5] states "The light vibrations perform only by following the directions parallel to the wave surface. In the note already cited I have presented this hypothesis with some development, I showed that it is enough to admit that the ether has a resistance so high to compression to conceive the absence of longitudinal vibrations."

Rudzki[11] investigated the condition for the separate propagation of "dilatational" (rot u=0) and "torsional" (div u=0) waves. The second condition leads immediately to

$$s_1u_1 + s_2u_2 + s_3u_3 = 0. \quad (34)$$

In this way, he obtains the dispersion relation for the shear waves. In fact, Eq. (29) results in

$$[1-(Es_1^2+C_{66}s_2^2+C_{55}s_3^2)]u_1 + (C_{66}-E)s_1s_2u_2 + (C_{55}-E)s_1s_3u_3 = 0,$$

$$(C_{66}-E)s_1s_2u_1 + [1-(C_{66}s_1^2+Es_2^2+C_{44}s_3^2)]u_2 + (C_{44}-E)s_2s_3u_3 = 0,$$

$$(C_{55}-E)s_1s_3u_1+(C_{44}-E)s_2s_3u_2+[1-(C_{55}s_1^2+C_{44}s_2^2 + Es_3^2)]u_3=0, \quad (35)$$

or, in view of Eq. (34),

$$[1-(C_{66}s_2^2+C_{55}s_3^2)]u_1+C_{66}s_1s_2u_2+C_{55}s_1s_3u_3=0,$$

$$C_{66}s_1s_2u_1+[1-(C_{66}s_1^2+C_{44}s_3^2)]u_2+C_{44}s_2s_3u_3=0,$$

$$C_{55}s_1s_3u_1+C_{44}s_2s_3u_2+[1-(C_{55}s_1^2+C_{44}s_2^2)]u_3=0, \quad (36)$$

which correspond to the first matrix of the right-hand side in Eq. (33), describing the behavior of the coupled S waves only.

D. Equivalence and Fresnel's slowness and wave surfaces

Like Cauchy, we set $E=0$ in Eq. (33) and use the following mathematical analogies:

$$C_{44}=\frac{c_{44}}{\rho} \Leftrightarrow \frac{1}{\eta_1}=\frac{1}{\mu_0\epsilon_{11}},$$

$$C_{55}=\frac{c_{55}}{\rho} \Leftrightarrow \frac{1}{\eta_2}=\frac{1}{\mu_0\epsilon_{22}},$$

$$C_{66}=\frac{c_{66}}{\rho} \Leftrightarrow \frac{1}{\eta_3}=\frac{1}{\mu_0\epsilon_{33}}. \quad (37)$$

Substituting these relations into Eq. (33), we obtain Fresnel's slowness surface (12), associated with Fresnel's wave surface (17), where $\sqrt{C_{44}}$, $\sqrt{C_{55}}$, and $\sqrt{C_{66}}$ are the wave velocities of the shear waves along the principal axes of the medium. Strictly, it is not necessary to set $E=0$, leading to an unstable medium, to obtain Fresnel's wave surface, because the P wave is decoupled from the S waves. However, the mathematical equivalence is not complete. Physically, the hypothetical medium (the ether), supporting only transverse vibrations, is an elastically unstable medium.

Another way to get rid of the P wave is to assume incompressibility. This problem was of the utmost importance to the proponents of the elastic ether. Green,[1] for instance, chose infinite velocity for the outer wave front. With infinite stiffness, there is an immediate signal (infinite velocity) with vanishing displacement amplitude (infinite impedance), while with vanishing stiffness the signal arrives after infinite time with vanishing pressure amplitude (zero impedance).

E. Anomalously polarized medium

There are media with the same slowness and wave surfaces but drastically different polarization behavior. Such media are kinematically identical but dynamically different. Examples of anomalous polarization have been discussed for transverse isotropy by Helbig and Schoenberg,[18] and for orthorhombic symmetry by Carcione and Helbig.[19]

We do not repeat the details of the theory here,[7] but give a specific example. Let us consider the following Kelvin–Christoffel matrix:

$$\mathbf{\Gamma}=\begin{pmatrix} 1-(C_{11}s_1^2+C_{66}s_2^2+C_{55}s_3^2) & (C_{12}^*+C_{66})s_1s_2 & -(C_{13}+C_{55})s_1s_3 \\ (C_{12}^*+C_{66})s_1s_2 & 1-(C_{66}s_1^2+C_{22}s_2^2+C_{44}s_3^2) & (C_{23}^*+C_{44})s_2s_3 \\ -(C_{13}+C_{55})s_1s_3 & (C_{23}^*+C_{44})s_2s_3 & 1-(C_{55}s_1^2+C_{44}s_2^2+C_{33}s_3^2) \end{pmatrix}. \quad (38)$$

It is clear that the dispersion equation associated with this matrix is the same as that corresponding to Eq. (31), where C_{12}^* and C_{23}^* are the elastic constants of the anomalously polarized medium, which differ from those of the normally polarized medium. Comparing the two matrices we obtain the following relations, instead of Eq. (32):

$$C_{11}=C_{22}=C_{33}=-C_{12}^*=C_{13}+2C_{55}=-C_{23}^*\equiv E. \quad (39)$$

Then, we have

$$\mathbf{\Gamma}=\begin{pmatrix} 1-(C_{66}s_2^2+C_{55}s_3^2) & -C_{66}s_1s_2 & C_{55}s_1s_3 \\ -C_{66}s_1s_2 & 1-(C_{66}s_1^2+C_{44}s_3^2) & -C_{44}s_2s_3 \\ C_{55}s_1s_3 & -C_{44}s_2s_3 & 1-(C_{55}s_1^2+C_{44}s_2^2) \end{pmatrix} - E\begin{pmatrix} s_1^2 & -s_1s_2 & s_1s_3 \\ -s_1s_2 & s_2^2 & -s_2s_3 \\ s_1s_3 & -s_2s_3 & s_3^2 \end{pmatrix}, \quad (40)$$

instead of Eq. (33). Such a medium is anomalously polarized, because the polarization vectors are not perpendicular and tangent to the P- and S-wave slowness surfaces, respectively, as it is for a normally polarized medium corresponding to the elastic constants given in Eq. (32). An example is shown in Sec. IV.

Although the S waves of this medium are described by the Fresnel wave surface, the medium is physically unstable since the stability conditions[8] are not satisfied; for instance, $C_{11}C_{22}-C_{12}^2>0$ implies $0>0$, etc.

IV. EXAMPLES

Here, the theoretical results are verified by means of full-wave three-dimensional numerical simulations. The modeling code is based on the Fourier pseudospectral method for computing the spatial derivatives and a Chebyshev expansion of the evolution operator as time-integration technique. This algorithm possesses spectral accuracy for band-limited signals and is not affected by temporal or spatial numerical dispersion. The details can be found in Carcione *et al.*[20] The source is an additional term on the right-hand side of Eq. (19), i.e., $\rho\partial_{tt}^2\mathbf{u}+\mathbf{f}$, where $\mathbf{f}=(f_1,f_2,f_3)$, for directional forces, $\mathbf{f}=\nabla\phi$, where $\phi=\phi(x,y,z)$ for a pressure source, and $\mathbf{f}=\nabla\times\mathbf{A}$, where $\mathbf{A}=(A_1,A_2,A_3)$, $A_i=a_i\phi(x,y,z)$, for a shear source; ϕ is a Gaussian function.

We consider a medium with $E=10$, $C_{44}=3$, $C_{55}=2$, and $C_{66}=1$ ($C_{12}=8$, $C_{13}=6$ and $C_{23}=4$), where these elastic constants are normalized by $\rho\times$MPa, with ρ given in kg/m^3. Figure 1 shows sections of the slowness and wave (energy or group-velocity) surfaces on the symmetry planes. The polarization is indicated in the velocity sections. The outer curve (dashed line) represents the P-wave spherical front, and the inner curves (solid lines) correspond to the shear waves, equivalent to Fresnel's electromagnetic waves. The conical point is located in the plane defined by the maximum and minimum principal axes, i.e., the (x,z) plane.

The size of the numerical mesh to perform the simulations is 165×165 grid points, and the motion is initiated by a directional force making an angle $\pi/4$ with the principal axes. A Ricker pulse is used.[7] Figure 2 shows the snapshots of the components of the displacement vector on the three symmetry planes of the medium.

The anomalously polarized medium has the same elastic constants, except $C_{12}^*=C_{23}^*=-10$. The wave surface and snapshots are shown in Figs. 3 and 4, respectively. The shape of the slowness and wave surfaces are the same as those of Fig. 1. Figure 5 represents the wave field in terms of the polarization vector $\mathbf{u}$. As can be seen, the polarization presents anomalous behavior, changing from perpendicular to tangent to the wave front in the same mode. The medium has anomalous polarization in the (x,y) and (y,z) planes, while the polarizations in the (x,z) plane are unaltered.

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Thank you to F. Cavallini for the valuable collaboration.

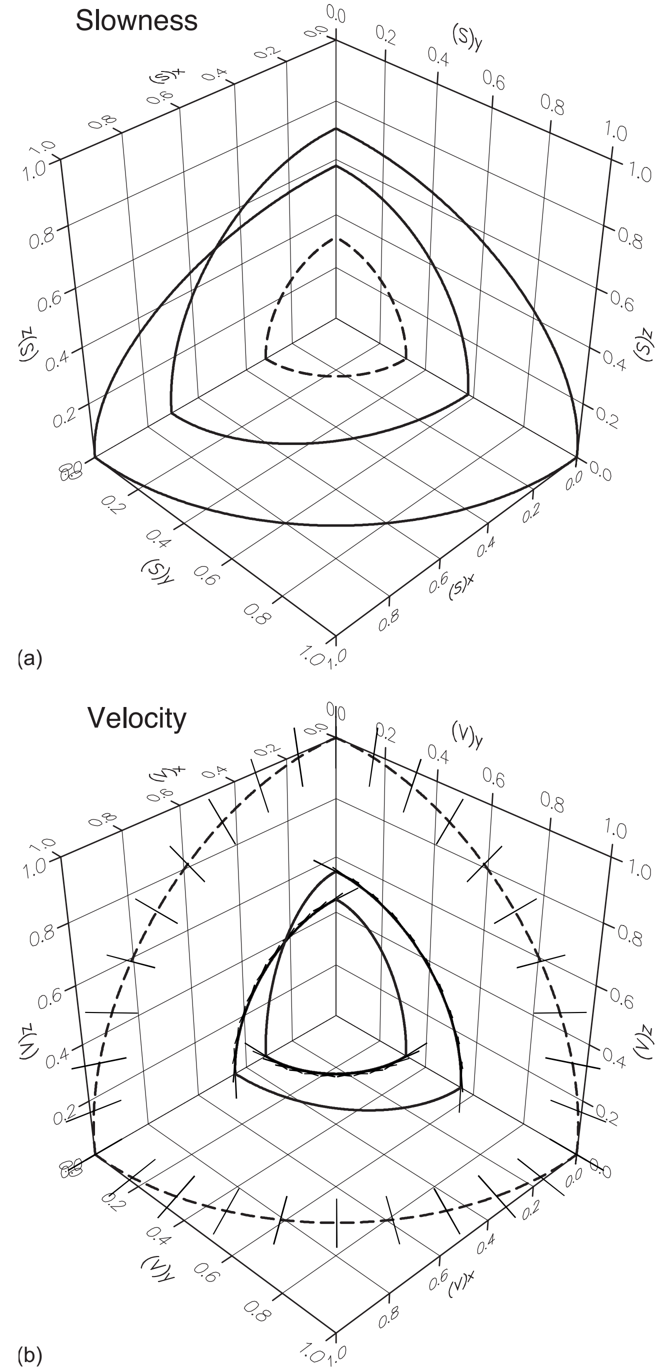

FIG. 1. Normalized sections of the slowness (a) and wave (velocity) (b) surfaces of an elastic anisotropic medium equivalent to an optically biaxial dielectric medium. One octant is shown because of symmetries. The spherical sections (dashed line) correspond to the P wave. The polarization is indicated for each wave mode.

APPENDIX: CALCULATION OF THE WAVE SURFACE

We obtain Fresnel's wave surface by using two different methods.

(1) Let us consider the s_1 and x components, and Eq. (12) for $s_2=s_3=0$. Then,

$$s_1^4-(\eta_2+\eta_3)s_1^2+\eta_2\eta_3=0. \tag{A1}$$

Using Eq. (15), we obtain

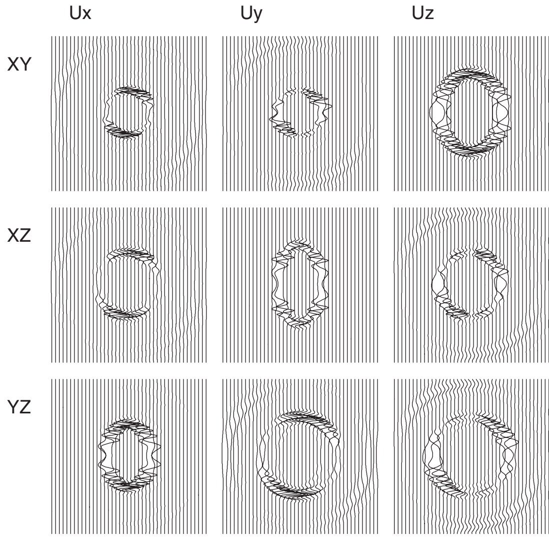

FIG. 2. Snapshots of the displacement vector corresponding to the case shown in Fig. 1. The displacement fields u_x, u_y, and u_z are shown for the three symmetry planes of the medium.

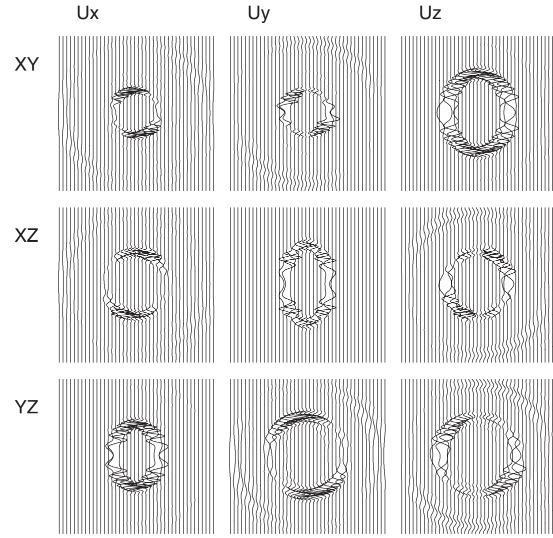

FIG. 4. Anomalous polarization. Snapshots of the displacement vector corresponding to the case shown in Fig. 3. The displacement fields u_x, u_y, and u_z are shown for the three symmetry planes of the medium.

$$x^4 - \left(\frac{1}{\eta_2} + \frac{1}{\eta_3}\right)x^2 + \frac{1}{\eta_2 \eta_3} = 0. \tag{A2}$$

Dividing this equation by η_1 and performing a similar procedure for the other components, we have

$$\frac{x^4}{\eta_1} - \left[\frac{1}{\eta_1}\left(\frac{1}{\eta_2} + \frac{1}{\eta_3}\right)\right]x^2 + \frac{1}{\eta_1 \eta_2 \eta_3} = 0,$$

$$\frac{y^4}{\eta_2} - \left[\frac{1}{\eta_2}\left(\frac{1}{\eta_1} + \frac{1}{\eta_3}\right)\right]y^2 + \frac{1}{\eta_1 \eta_2 \eta_3} = 0,$$

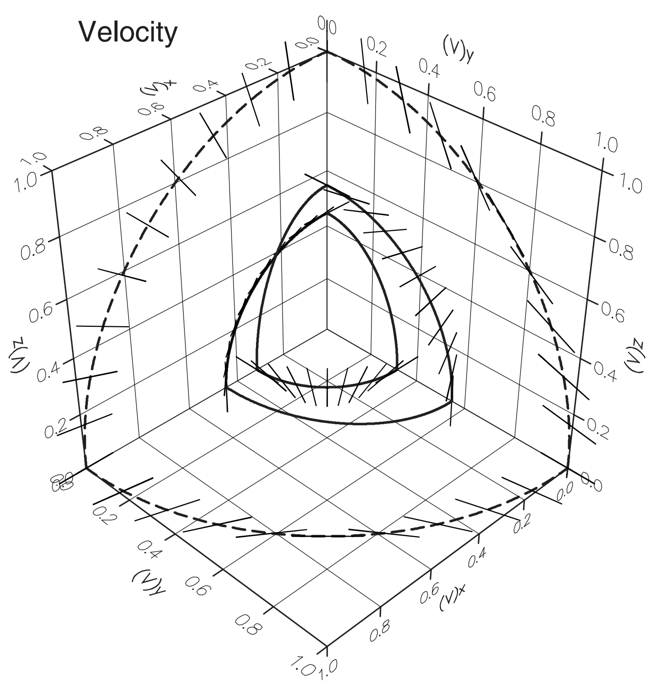

FIG. 3. Wave (velocity) surface corresponding to the anomalously polarized medium.

$$\frac{z^4}{\eta_3} - \left[\frac{1}{\eta_3}\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)\right]z^2 + \frac{1}{\eta_1 \eta_2 \eta_3} = 0. \tag{A3}$$

Since the slowness surface is a quadratic surface[21] and the slowness and velocity vectors are reciprocal, this implies that the wave surface has the same functional form as the slowness surface (12). Then

$$(a_1 x^2 + a_2 y^2 + a_3 z^2)(x^2 + y^2 + z^2) - (b_1 x^2 + b_2 y^2 + b_3 z^2) + d = 0. \tag{A4}$$

Comparing Eqs. (A3) and (A4), we have $a_1 = \eta_1^{-1}$, $a_2 = \eta_2^{-1}$, $a_3 = \eta_3^{-1}$, $b_1 = \eta_1^{-1}(\eta_2^{-1} + \eta_3^{-1})$, $b_2 = \eta_2^{-1}(\eta_1^{-1} + \eta_3^{-1})$, $b_3 = \eta_3^{-1}(\eta_1^{-1} + \eta_2^{-1})$ and $d = (\eta_1 \eta_2 \eta_3)^{-1}$. Hence, Eq. (16) is obtained.

(2) A more rigorous method is the so-called principle of duality.[6] It is known that in lossless—pure dielectrics—anisotropic media, **E** and **H** are tangent to the slowness surface and that the group-velocity vector **v** is perpendicular to that surface. Therefore

$$\mathbf{v} \cdot \mathbf{E} = 0, \quad \mathbf{v} \cdot \mathbf{H} = 0.$$

Let us consider the property $\mathbf{v} \times (\mathbf{s} \times \mathbf{A}) = -(\mathbf{v} \cdot \mathbf{s})\mathbf{A} + (\mathbf{v} \cdot \mathbf{A})\mathbf{s} = -\mathbf{A}$, where **A** is **E** or **H** and Eq. (14) has been used. Taking the vector product of Eq. (6) with **v** and using that property gives

$$\mathbf{v} \times \mathbf{D} = \mu_0^{-1} \mathbf{B},$$

$$\mathbf{v} \times \mathbf{B} = -\boldsymbol{\epsilon}^{-1} \cdot \mathbf{D}. \tag{A5}$$

Equations (6) and (A5) are dual equations. This means that the dispersion relation for **v** (the wave surface) can be obtained from the slowness surface by the following substitutions:

$$\mathbf{s} \to \mathbf{v}, \quad \mu_0 \to \mu_0^{-1}, \quad \boldsymbol{\epsilon} \to \boldsymbol{\epsilon}^{-1}.$$

The duality property also implies

$$\mathbf{s} \cdot \mathbf{E} = 0, \quad \mathbf{s} \cdot \mathbf{H} = 0,$$

and that **D** and **B** are tangent to the wave surface.

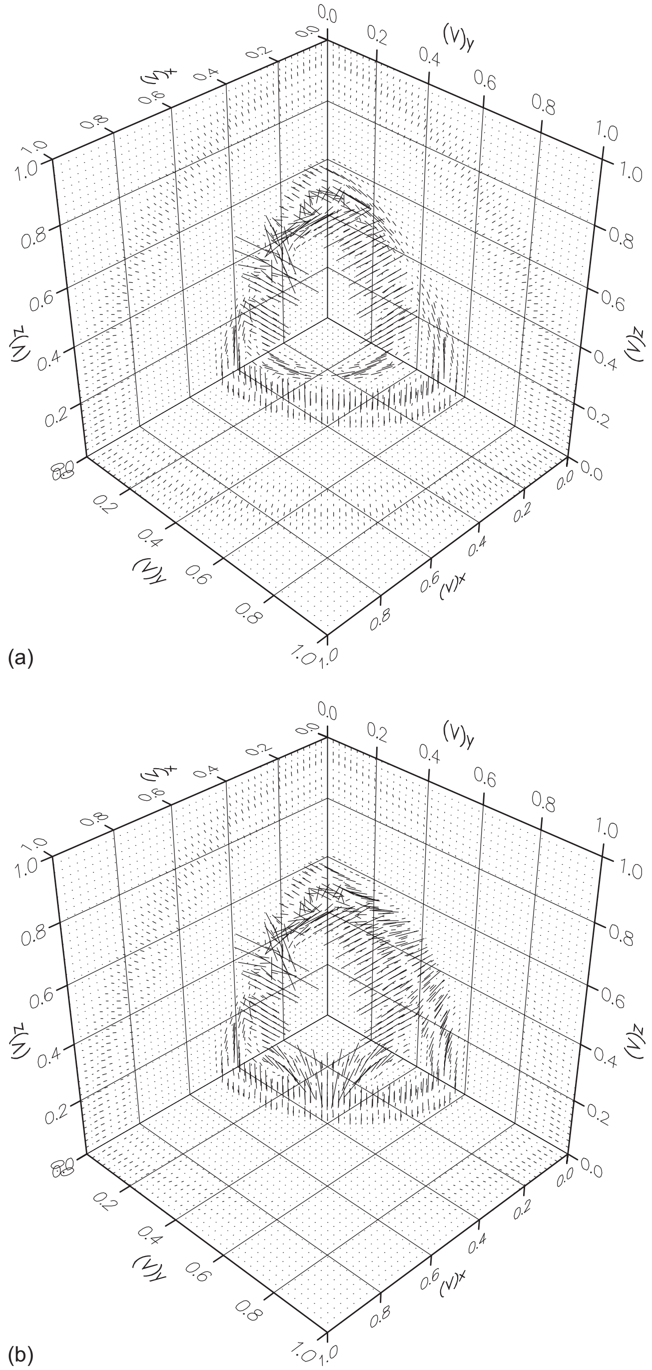

FIG. 5. Snapshots of the polarization vector for the normally (a) and the anomalously (b) polarized media.

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