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Wave propagation simulation in a linear viscoelastic medium

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SUMMARY

A new formulation for wave propagation in an anelastic medium is developed. The phenomenological theory of linear viscoelasticity provides the basis for describing the attenuation and dispersion of seismic waves. The concept of a spectrum of relaxation mechanisms represents a convenient description of the constitutive relation of linear viscoelastic solids; however, Boltzmann's superposition principle does not have a straightforward implementation in time-domain wave propagation methods. This problem is avoided by the introduction of memory variables which circumvent the convolutional relation between stress and strain.

The formulae governing wave propagation are recast as a first-order differential equation in time, in the vector represented by the displacements and memory variables. The problem is solved numerically and tested against the solution of wave propagation in a homogeneous viscoelastic medium, obtained by using the correspondence principle.

Key words: viscoelasticity, attenuation, dispersion, wave propagation simulation

1 INTRODUCTION

Until recently, the interpretation of seismic data obtained from geophysical surveys has been based on comparatively simple representations of the earth structure. The existing algorithms that simulate the process of wave propagation are mainly based on the acoustic wave equation, which considers the medium to behave as an ideal fluid. This approximation does not account for all arrivals and does not predict wave amplitudes correctly. The next step in improving upon the acoustic assumption is the use of an isotropic elastic material to approximate the earth structure (e.g. Blake, Bond & Downie 1982; Kosloff, Reshef & Loewenthal 1984). This type of description yields better accuracy than the acoustic one for the determination of wave amplitudes and distinguishes between P- and S-waves.

However, wave propagation in the Earth has always been known to be anelastic. Consequently, simulations which attempt at accurate amplitude reconstruction must be able to account for the effects of attenuation and dispersion. Moreover, the physical characteristics of the wave field that propagates in anelastic media differ from those of elastic media. A simple model like a plane interface separating two different materials is enough to show that in the anelastic case several physical phenomena exist that are not encountered for elastic waves.

Just to see the importance of considering the anelastic effects in seismic wave propagation, we show some of the differences between viscoelastic and elastic wave propagation. Several authors (Lockett 1962; Cooper & Reiss 1966; Cooper 1967; Buchen 1971), and more recently Borcherdt (Borcherdt 1973, 1977, 1982; Borcherdt & Wenneberg 1985), dedicated effort to the study of the physical characteristics of plane waves in anelastic media, and mainly to their behaviour on plane boundaries separating two linear viscoelastic materials. They found that a special type of wave, termed a generalized inhomogeneous wave, can be generated there. There is a distinct difference between the inhomogeneous wave of elastic media (interface waves) and that of viscoelastic media. In the former case the direction of attenuation is normal to the direction of propagation, whereas for inhomogeneous viscoelastic waves the angle between these two directions must be less than $\pi/2$. Furthermore, for viscoelastic inhomogeneous waves the

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energy does not propagate normal to the wavefront and the particle motions are elliptical. The phase velocity is less than that of a corresponding homogeneous wave; critical angles do not exist in general, only under particular circumstances (special value of the angle between the directions of propagation and attenuation); and the phase velocity and attenuation depend on the angle of incidence. The latter physical property implies that the velocity and attributes of the wave field are raypath dependent. It was proved by Borcherdt (1982) that, in general, a wave travelling through a layered media has angular dependence of attenuation and dispersion, where the more oblique direction has more energy dissipation and lower velocity.

This paper considers wave propagation simulation in a general heterogeneous anelastic medium within the framework of the theory of linear viscoelasticity, and represent a further step in improving upon the viscoacoustic description of wave propagation (Carcione, Kosloff & Kosloff 1988a,b). Growing evidence suggests a linear attenuation mechanism (with or without constant Q) for seismic strains and upper crustal conditions (Jones 1986). The phenomenological theory of linear viscoelasticity provides a general framework for such behaviour.

The concept of a spectrum of relaxation mechanisms is used to define the constitutive relation (Liu, Anderson & Kanamori 1976). A wave propagating in a real material induces a non-instantaneous deformation, but not all of the energy can be recovered, as in the case of a purely elastic solid. Also, the energy that is not dissipated is delivered in a finite time. This relaxation time may be a consequence of many processes such as grain boundary relaxation, thermoelasticity, diffusional motion of dislocations and point defects, etc. The standard linear solid element explains these processes very well (Zener 1948). Some of them can be modelled with one mechanism and others using a spectrum of relaxation mechanisms.

While the viscoacoustic constitutive relation can be expressed in a simple equation through the relation between the pressure and dilatation fields and one relaxation function, in the viscoelastic case two relaxation functions are needed which describe the dilatational and shear behaviour of the medium. The constitutive equations for the viscoelastic medium relates the traces and the deviatoric components of the stress and strain tensors corresponding to dilatational and shear deformations (respectively).

As in the case of viscoacoustic wave propagation (Carcione *et al.* 1988a), Boltzmann's superposition principle is implemented by the introduction of memory variables which circumvent the convolutional relation between the stress and strain tensors. The solution of the two-dimensional wave propagation problem implies the introduction of three memory variables, one for each dilatational relaxation mechanism and two for each shear relaxation mechanism, unlike the viscoacoustic problem where only one is needed for each mechanism.

The new theory explains, within the framework of the most general linear relation between stress and strain, the correct changes in the phase and spectrum of the wave field. Any type of frequency-dependent complex modulus function can be incorporated. The theory includes, as special cases, linear models which describe elastic wave propagation through porous media (Murphy, Winkler & Kleimberg 1986; Biot 1956a,b; Burridge & Keller 1981; de la Cruz & Spanos 1986).

The first section presents the constitutive relation of the linear viscoelastic medium. In the following two sections, the dilatational and shear relaxation functions are introduced, and the quality factor and phase and group velocities are calculated. Then the equation of motion is derived and solved by using a new pseudo-spectral time integration scheme based on the work of Tal-Ezer (1986), which was successfully applied to solve the viscoacoustic equations of motion (Carcione *et al.* 1988a). Finally, wave propagation simulation in a homogeneous 2-D medium is performed, and the numerical algorithm is tested against the analytical solution. This is based on a two-dimensional viscoelastic Green's function which is derived from the correspondence principle.

2 CONSTITUTIVE RELATIONS OF THE LINEAR VISCOELASTIC MEDIUM

A realistic representation of the Earth may be achieved by combining the mechanical properties of elastic solids and of viscous fluids. In the resulting material the stress depends both on the strain and the rate of strain together, as well as higher time derivatives of the strain. Such a medium which combines solid-like and liquid-like behaviour is called viscoelastic. For an anisotropic linear viscoelastic material, the most general relation between the components of the stress tensor σ_{ij} and the components of the strain tensor ϵ_{kl} is (Christensen 1982)

$$\sigma_{ij}(\mathbf{x}, t) = \psi^c_{ijkl}(\mathbf{x}, t) * \dot{\epsilon}_{kl}(\mathbf{x}, t), \qquad k, l, \ldots, n, \tag{1}$$

where t is time, $\mathbf{x}$ is the position vector, and ψ^c_{ijkl} is a fourth-order tensorial relaxation function. The dot above a variable represents a time derivative. The usual Cartesian tensor notation is employed, with the Latin indices i and j lying in the range $1, \ldots, n$, where n is the dimension of space. Repeated indices imply summation. The time convolution of two functions $f(t)$ and $g(t)$ is expressed by

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\, d\tau,$$

and the definition

$$f^c(t) = f(t)H(t)$$

is used, where $H(t)$ denotes the Heaviside function. Equation (1) is the formulation of Boltzmann's superposition principle, in which the current stress is determined by the superposition of the responses at previous times. The material is considered to have a memory because the current stress depends on the full strain history.

The most general isotropic fourth-order tensor is given by (e.g. Christensen 1982)

$$\psi^c_{ijkl}(t) = \frac{1}{n}[\psi^c_1(t) - \psi^c_2(t)]\delta_{ij}\delta_{kl} + \tfrac{1}{2}[\psi^c_2(t)](\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{2}$$

where $\psi^c_1(t)$ and $\psi^c_2(t)$ are relaxation functions and δ_{ij} is the Kronecker delta. Substituting (2) into (1) gives

$$\sigma_{ij}(t) = \frac{1}{n}[\psi^c_1(t) - \psi^c_2(t)]\delta_{ij} * \dot{\epsilon}_{kk} + \psi^c_2(t) * \dot{\epsilon}_{ij}. \tag{3}$$

We introduce

$$T_{ij}^{\nu} = \sigma_{kk}\delta_{\nu 1} + \sigma_{ij}^{d}\delta_{\nu 2} \tag{4a}$$

and

$$E_{ij}^{\nu} = \epsilon_{kk}\delta_{\nu 1} + \epsilon_{ij}^{d}\delta_{\nu 2}, \tag{4b}$$

where

$$\sigma_{ij}^{d} = \sigma_{ij} - \frac{1}{n}\,\delta_{ij}\sigma_{kk}, \tag{5a}$$

and

$$\epsilon_{ij}^{d} = \epsilon_{ij} - \frac{1}{n}\,\delta_{ij}\epsilon_{kk} \tag{5b}$$

are the deviatoric components of the stress and strain tensors, respectively. Hereafter, $\nu = 1$ defines variables related to states of dilatation, and $\nu = 2$ defines variables related to states of deviatoric deformation. From (3) we obtain the relation

$$T_{ij}^{\nu} = \psi_{\nu}^{c} * \dot{E}_{ij}^{\nu} = E_{ij}^{\nu} * \dot{\psi}_{\nu}^{c}, \qquad \nu = 1, 2. \tag{6}$$

We conclude, from (4) and (6), that $\psi_1^c(t)$ and $\psi_2^c(t)$ correspond to the relaxation function characteristics of the states of dilatation and shear, respectively. Equation (6) represents the convolutional form of the constitutive relation for the linear isotropic viscoelastic medium.

3 RELAXATION FUNCTIONS

Another way to establish the constitutive relation involves differential operators (Christensen 1982):

$$\sum_{k=0}^{m_\nu} c_k^{\nu} \frac{d^k}{dt^k} T_{ij}^{\nu} = \sum_{k=0}^{m_\nu} d_k^{\nu} \frac{d^k}{dt^k} E_{ij}^{\nu}, \qquad m_\nu \in N, \tag{7}$$

where c_k^{ν}, and d_k^{ν} are coefficients related to the material properties of the medium, subjected to the following constraints on the initial conditions:

$$\sum_{r=k}^{m_\nu} c_r^{\nu} T_{ij}^{\nu(r-k)}(0) = \sum_{r=k}^{m_\nu} d_r^{\nu} E_{ij}^{\nu(r-k)}(0), \tag{8}$$

with $T_{ij}^{\nu(r-k)}(0)$ and $E_{ij}^{\nu(r-k)}(0)$ indicating the $(r-k)$-order derivative of the stress and strain tensor components evaluated at $t = 0$. In the Laplace-transform domain the stress and strain components are related by a rational function as can easily be seen from equation (7). Decomposing this rational function in partial fractions and going back to the time domain, we obtain an explicit form for the relaxation functions $\psi_\nu(t)$ (Fung 1965). These can be expressed as (Liu *et al* 1976),

$$\psi_\nu(t) = \mathrm{M}_\nu\left[1 - \sum_{l=1}^{L_\nu}\left(1 - \frac{\tau_{\epsilon l}^{\nu}}{\tau_{\sigma l}^{\nu}}\right)e^{-(t/\tau_{\sigma l}^{\nu})}\right], \qquad \nu = 1, 2, \tag{9}$$

where $\tau_{\sigma l}^{\nu}(\mathbf{x})$ and $\tau_{\epsilon l}^{\nu}(\mathbf{x})$, denote material relaxation times for the lth mechanism, L_ν is the number of relaxation mechanisms, $\mathrm{M}_\nu(\mathbf{x})$ is the elastic or relaxed modulus of the medium, corresponding to dilatational ($\nu = 1$) or shear ($\nu = 2$) behaviour of the medium.

The constitutive relation (6) is analogous to the purely elastic relation when viewed in the frequency domain. Then, the stress transform is merely a multiplication of the time Fourier transforms of the strain field and the time derivative of the relaxation function, respectively. The latter quantity is identified as the complex modulus, which is calculated in Appendix A together with the complex Lamé constants, velocities and wavenumbers.

Basically, the relaxation functions given by (9) represent L_ν numbers of standard linear elements connected in parallel for each deformation state (dilatational and shear), and also include as special cases the Maxwell and Kelvin–Voigt behaviour of the material.

4 QUALITY FACTORS AND VELOCITY DISPERSION FOR THE VISCOELASTIC SOLID

To calculate the quality factors and phase velocity of the linear viscoelastic medium, we first review some of the main concepts of general viscoelasticity (see Buchen 1971, and Borcherdt 1973). Using the same notation as Buchen, we define the complex quantities

$$\Omega_\nu(\omega) = \Omega_\nu^{R} + i\Omega_\nu^{I} = k_\nu^2, \qquad \nu = 1, 2, \tag{10}$$

where k_ν is defined by (A28) and Ω_ν^R and Ω_ν^I are real numbers. Let

$$\mathbf{k}_\nu = \boldsymbol{\kappa}_\nu - i\boldsymbol{\alpha}_\nu, \tag{11}$$

with $\boldsymbol{\kappa}_\nu$ and $\boldsymbol{\alpha}_\nu$ real vectors, indicating the direction and magnitude of propagation and attenuation, respectively. From (11) and (A28) we obtain

$$\boldsymbol{\kappa}_\nu^2 - \boldsymbol{\alpha}_\nu^2 - 2i\boldsymbol{\kappa}_\nu \cdot \boldsymbol{\alpha}_\nu = k_\nu^2 = \frac{\omega_0^2}{V_\nu^2}. \tag{12}$$

Because V_ν is a complex quantity, $\boldsymbol{\kappa}_\nu \cdot \boldsymbol{\alpha}_\nu$ must be different from zero for a viscoelastic medium. Only for the elastic case, in which the velocity is real, can the angle between the direction of propagation and attenuation be equal to $\pi/2$; this is the case of interface inhomogeneous waves. If we call γ_ν the angle between $\boldsymbol{\kappa}_\nu$ and $\boldsymbol{\alpha}_\nu$, it must hold that $0 \le \gamma_\nu < \pi/2$ for a general viscoelastic wave [if $\gamma_\nu = \pi/2$, from (10) and (12), $-2\boldsymbol{\kappa}_\nu \cdot \boldsymbol{\alpha}_\nu = \Omega_\nu^I = 0$, which is the elastic case, see also Buchen (1971)]. This means that the amplitude of the wave does not increase in the direction of propagation and, from (10) and (12), that $\Omega_\nu^I < 0$. This condition can be deduced from (A6), (A28) and (10) if we consider that the dissipation of energy by a travelling wave implies that the imaginary part of M_ν^C must be greater than zero (Flugge 1960, p. 58).

From equations (10) and (11) we have

$$\kappa_\nu^2 = \frac{1}{2}\left[\Omega_\nu^R + \left(\Omega_\nu^{R^2} + \frac{\Omega_\nu^{I^2}}{\cos^2\gamma_\nu}\right)^{1/2}\right] \tag{13}$$

and

$$\alpha_\nu^2 = \frac{1}{2}\left[-\Omega_\nu^R + \left(\Omega_\nu^{R^2} + \frac{\Omega_\nu^{I^2}}{\cos^2\gamma_\nu}\right)^{1/2}\right]. \tag{14}$$

Without loss of generality, we consider the x_2-axis perpendicular to the plane formed by the vectors $\boldsymbol{\kappa}_\nu$ and $\boldsymbol{\alpha}_\nu$. Then the phase of the plane wave is independent of the variable x_2. We consider now a viscoelastic plane wave for the potential field:

$$\phi^{\nu} = \phi_0^{\nu} \exp\{i(\omega t - \mathbf{k}_\nu \cdot \mathbf{x})\}, \tag{15}$$

where ϕ_0^ν is a constant complex quantity, and the condition

$$\nabla \cdot \mathbf{e}_2 \phi^{(2)} = 0 \tag{16}$$

must hold because $\phi^{(2)}$ is a purely rotational field. $\mathbf{e}_2$ is the unit vector in the x_2 direction. Defining the vector

$$\mathbf{v}_\nu = \mathbf{k}_1 \delta_{\nu 1} + (\mathbf{e}_2 \times \mathbf{k}_2)\delta_{\nu 2}, \tag{17}$$

where '×' denotes vectorial product, the displacement field is then obtained by using the Helmholtz decomposition theorem,

$$\begin{aligned}\mathbf{u}^\nu &= \nabla \phi^{(1)} + \nabla \times \phi^{(2)} \\ &= -i\phi_0^\nu \mathbf{v}_\nu \exp\{i(\omega t - \mathbf{k}_\nu \cdot \mathbf{x})\}.\end{aligned} \tag{18}$$

Substitution of (11) into (15) yields

$$\phi^\nu = \phi_0^\nu \exp\{-\boldsymbol{\alpha}_\nu \cdot \mathbf{x}\} \exp\{i(\omega t - \boldsymbol{\kappa}_\nu \cdot \mathbf{x})\} \tag{19}$$

Alternatively, we can express (19) as

$$\begin{aligned}\phi^\nu = \phi_0^\nu &\exp\{-\alpha_\nu(x_1 \sin\beta_\nu + x_3 \cos\beta_\nu)\} \\ &\times \exp\{i\omega t - i\kappa_\nu(x_1 \sin\theta_\nu + x_3 \cos\theta_\nu)\},\end{aligned} \tag{20}$$

where θ_ν and β_ν are the angles between the direction of propagation and the direction of attenuation with the x_3-axis, respectively. Let us introduce the angle

$$\gamma_\nu = \theta_\nu - \beta_\nu. \tag{21}$$

The wave defined by (20) is called an inhomogeneous or general plane wave since, as γ_ν is different from zero, the amplitude along the wavefront is variable. These kinds of waves are necessary to describe the reflection and refraction of viscoelastic waves in stratified media. In the special case $\gamma_\nu = 0$, the waves are called homogeneous or simple plane waves; in this case, the amplitude is constant along the wavefront.

From (19) we identify the real phase velocity $c_\nu(\omega)$ of the viscoelastic plane wave

$$\mathbf{c}_\nu(\omega) = \omega \frac{\boldsymbol{\kappa}_\nu}{\kappa_\nu^2}, \tag{22}$$

where $\kappa_\nu(\omega)$ is defined by (13). To describe the energy loss of a wave travelling through the medium, we use the definition of the quality factor given by Borcherdt (1973); that is, the ratio of the peak energy density stored per cycle of forced oscillation to the loss in energy density during the cycle. Borcherdt derived expressions for the dissipated and stored energy densities from an explicit energy conservation relation, valid for an arbitrary steady-state viscoelastic radiation field. The energy loss per cycle is given by

$$\begin{aligned}\Delta E_\nu = 2\pi &|\phi_0^\nu|^2 \exp\{-2\boldsymbol{\alpha}_\nu \cdot \mathbf{x}\} \\ &\times [\rho\omega^2 \boldsymbol{\kappa}_\nu \cdot \boldsymbol{\alpha}_\nu + 2\mu^I |\boldsymbol{\kappa}_\nu \times \boldsymbol{\alpha}_\nu|^2].\end{aligned} \tag{23}$$

The peak energy density stored per cycle is the maximum value of the potential energy density, and is expressed by

$$\begin{aligned}P_\nu^{\max} = \tfrac{1}{2} &|\phi_0^\nu|^2 \exp\{-2\boldsymbol{\alpha}_\nu \cdot \mathbf{x}\} \\ &\times [\rho\omega^2(\kappa_\nu^2 - \alpha_\nu^2) + 2\mu^R |\boldsymbol{\kappa}_\nu \times \boldsymbol{\alpha}_\nu|^2].\end{aligned} \tag{24}$$

Then the quality factor is

$$Q_\nu = 2\pi \frac{P_\nu^{\max}}{\Delta E_\nu} = \frac{\rho\omega^2(\kappa_\nu^2 - \alpha_\nu^2) + 2\mu^R |\boldsymbol{\kappa}_\nu \times \boldsymbol{\alpha}_\nu|^2}{2\rho\omega^2 \boldsymbol{\kappa}_\nu \cdot \boldsymbol{\alpha}_\nu + 4\mu^I |\boldsymbol{\kappa}_\nu \times \boldsymbol{\alpha}_\nu|^2} \tag{25}$$

where μ^R and μ^I are the real and imaginary parts of μ. As can be seen from (25), the quality factor for inhomogeneous viscoelastic waves is not an intrinsic property of the medium because of its dependence on the angle between the direction of propagation and the direction of attenuation. For homogeneous waves $\boldsymbol{\kappa}_\nu \times \boldsymbol{\alpha}_\nu = 0$ and, by (10) and (12), the expression (25) takes the simple form

$$Q_\nu = -\frac{\Omega_\nu^R}{\Omega_\nu^I}, \tag{26}$$

which can be expressed in terms of the dilatational and shear complex moduli. From (10) and (26) the quality factor can be written as

$$Q_\nu = -\frac{\operatorname{Re}(k_\nu^2)}{\operatorname{Im}(k_\nu^2)}. \tag{27}$$

Replacing equation (A20a) in (27) results in

$$Q_\nu = -\frac{\operatorname{Re}(V_\nu^{-2})}{\operatorname{Im}(V_\nu^{-2})} = -\frac{\operatorname{Re}(V_\nu^{2*})}{\operatorname{Im}(V_\nu^{2*})} = \frac{\operatorname{Re}(V_\nu^2)}{\operatorname{Im}(V_\nu^2)}. \tag{28}$$

Finally, substituting equations (A20a) and (A20b) in (28) and using (A7) and (A8), we obtain

$$Q_1 = \frac{\operatorname{Re}[\mathrm{M}_1^C + (n-1)\mathrm{M}_2^C]}{\operatorname{Im}(\mathrm{M}_1^C + (n-1)\mathrm{M}_2^C]} \tag{29}$$

for P-waves and

$$Q_2 = \frac{\operatorname{Re}(\mathrm{M}_2^C)}{\operatorname{Im}(\mathrm{M}_2^C)} \tag{30}$$

for S-waves.

The quality factor in bulk Q_k is obtained from (A9) in the same way as the previous ones. This yields

$$Q_k = \frac{\operatorname{Re}(\mathrm{M}_1^C)}{\operatorname{Im}(\mathrm{M}_1^C)}. \tag{31}$$

The homogeneous phase velocities are given by (22) considering that for $\gamma_\nu = 0$ in (12) it holds that $\kappa_\nu = \operatorname{Re}[k_\nu]$. Replacing (A20a) and (A20b) in (A28), the homogeneous phase velocities are expressed in terms of the complex moduli as

$$c_1 = \operatorname{Re}^{-1}\left[\left(\frac{\rho}{E}\right)^{1/2}\right] \tag{32a}$$

and

$$c_2 = \operatorname{Re}^{-1}\left[\left(\frac{2\rho}{\mathrm{M}_2^C}\right)^{1/2}\right], \tag{32b}$$

with E given by (A22). At zero frequency (32a) and (32b) become the elastic velocities given by (A25a) and (A25b). Having the phase velocities, the group velocities of the wave field are obtained as the derivative of the frequency with respect to the real wavenumber, $\kappa_\nu = \omega/c_\nu(\omega)$,

$$c_{\nu_g}(\omega) = \frac{d\omega}{d\kappa_\nu} = \left\{\frac{d}{d\omega}\left[\frac{\omega}{c_\nu(\omega)}\right]\right\}^{-1}. \tag{33}$$

Replacing the phase velocities in equation (33) we obtain

$$c_{1_g} = \operatorname{Re}^{-1}\left\{\left(\frac{\rho}{E}\right)^{1/2}\left[1 - \frac{1}{2}\omega\frac{1}{E}\frac{dE}{d\omega}\right]\right\} \tag{34a}$$

for P-waves and

$$c_{2_g} = \mathrm{Re}^{-1}\left\{\left(\frac{2\rho}{\mathrm{M}_2^C}\right)^{1/2}\left[1 - \frac{1}{2}\omega\frac{1}{\mathrm{M}_2^C}\frac{d(\mathrm{M}_2^C)}{d\omega}\right]\right\} \tag{34b}$$

for S-waves. The derivatives in (34a) and (34b) are given by

$$\frac{d\mathrm{M}_2^c(\omega)}{d\omega} = \mathrm{M}_2\sum_{l=1}^{L_2}\frac{i(\tau_{\epsilon l}^{(2)} - \tau_{\sigma l}^{(2)})}{(1 + i\omega\tau_{\sigma l}^{(2)})^2}$$

and

$$\frac{dE(\omega)}{d\omega} = \mathrm{M}_1\sum_{l=1}^{L_1}\frac{i(\tau_{\epsilon l}^{(1)} - \tau_{\sigma l}^{(1)})}{(1 + i\omega\tau_{\sigma l}^{(1)})^2} + (n-1)\frac{d\mathrm{M}_2^C}{d\omega}.$$

When $\tau_{\sigma_l} = \tau_{\epsilon_l}$ or at the zero-frequency limit, the phase and group velocities are constant and equal to the relaxed or elastic velocities.

5 EQUATIONS OF MOTION AND MEMORY VARIABLES

The description of wave propagation in a general medium is based on the equation of momentum conservation combined with the constitutive relations, which contain complete information about the rheology of the material. However, implementation of Boltzmann's superposition principle in the time domain is not straightforward because of the presence of convolutional kernels in the stress–strain relation (3). Consequently, in this section the rheological relations are reformulated to yield a more convenient description.

In an n-dimensional continuous medium, the linearized equation of momentum conservation is given by

$$\rho\ddot{\mathbf{u}}(\mathbf{x}, t) = \nabla\cdot\Sigma(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t), \tag{35}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the displacement field, $\Sigma(\mathbf{x}, t)$ is the stress tensor, $\mathbf{f}(\mathbf{x}, t)$ represents the body forces, $\rho(\mathbf{x})$ is the density and '$\nabla\cdot$' is the divergence operator.

As in the viscoacoustic case (Carcione *et al.* 1988a) we will now see that by the introduction of memory variables, the convolutional integral in the constitutive relation (3) can be avoided. This equation can also be expressed as

$$\sigma_{ij} = \frac{1}{n}\delta_{ij}\dot{\psi}_1^c * \epsilon_{kk} + \dot{\psi}_2^c * \epsilon_{ij}^d. \tag{36}$$

Performing the time derivatives in (36), and using (9), yields

$$\sigma_{ij} = \frac{1}{n}\delta_{ij}(\mathrm{M}_{u_1} + \Phi_1^c *)\epsilon_{kk} + (\mathrm{M}_{u_2} + \Phi_2^c *)\epsilon_{ij}^d, \tag{37}$$

where

$$\mathrm{M}_{u_\nu} \equiv \psi_\nu^c(0) = \mathrm{M}_\nu\left[1 - \sum_{l=1}^{L_\nu}\left(1 - \frac{\tau_{\epsilon_l}^\nu}{\tau_{\sigma_l}^\nu}\right)\right] \tag{38}$$

is the unrelaxed modulus and

$$\Phi_\nu(t) \equiv \dot{\psi}_\nu(t) = \sum_{l=1}^{L_\nu}\phi_{\nu_l}(t) \tag{39}$$

is called the response function of the medium, with

$$\phi_{\nu_l}(t) = \frac{\mathrm{M}_\nu}{\tau_{\sigma_l}^\nu}\left(1 - \frac{\tau_{\epsilon_l}^\nu}{\tau_{\sigma_l}^\nu}\right)\exp\left\{-\frac{t}{\tau_{\sigma_l}^\nu}\right\}. \tag{40}$$

We now define the memory variables

$$e_{1_l} = \phi_{1_l}^c * \epsilon_{kk}, \qquad l = 1, \ldots, L_1 \tag{41}$$

and

$$e_{ijl} = \phi_{2_l}^c * \epsilon_{ij}^d, \qquad l = 1, \ldots, L_2 \tag{42}$$

Because the strain tensor is symmetric and considering that $e_{iil} = 0$, the number of independent memory variables for the n-dimensional viscoelastic solid is one for each relaxation mechanism corresponding to states of dilatational deformation and $l_s = [n(n+1)/2] - 1$ for each relaxation mechanism corresponding to states of shear deformation. The total number of memory variables for L_1 dilatational mechanisms and L_2 shear mechanisms is then $m = L_1 + l_s L_2$. Replacing the memory variables in (37) we obtain

$$\sigma_{ij} = \frac{1}{n}\delta_{ij}\left(\mathrm{M}_{u_1}\epsilon_{kk} + \sum_{l=1}^{L_1} e_{1_l}\right) + \left(\mathrm{M}_{u_2}\epsilon_{ij}^d + \sum_{l=1}^{L_2} e_{ijl}\right). \tag{43}$$

Taking derivatives with respect to time in (41) and (42) gives

$$\dot{e}_{1_l} = \epsilon_{kk}(t)\phi_{1_l}^c(0) - \frac{e_{1_l}(t)}{\tau_{\sigma_l}^{(1)}}, \qquad l = 1, \ldots, L_1 \tag{44a}$$

and

$$\dot{e}_{ijl} = \epsilon^d_{ij}(t)\phi^c_{2l}(0) - \frac{e_{ijl}(t)}{\tau^{(2)}_{\sigma_l}}, \qquad l = 1, \ldots, L_2 \tag{44b}$$

Equations (35), (43), (44a) and (44b) fully describe the response of the viscoelastic solid and will be the basis for the numerical solution algorithm. After substitution of (43) in (35), considering the definition of the strain tensor as a function of the displacement field (Aki & Richards 1981, p. 13) and using (44a) and (44b), we obtain a first-order differential equation in time,

$$\dot{\mathbf{U}} = \mathbf{M}\mathbf{U} + \mathbf{F}, \tag{45}$$

where $\mathbf{M}$ is a spatial operator matrix of dimension $M = 2n + m$ given by

$$\mathbf{M} = \begin{array}{c} \\ n \\ n \\ L_1 \\ L_2 \\ L_2 \\ \vdots \\ L_2 \\ \vdots \\ L_2 \end{array} \begin{array}{c} \begin{array}{cccccccccc} n & n & L_1 & L_2 & L_2 & & L_2 & & L_2 \end{array} \\ \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{D}^u & \mathbf{0} & \mathbf{D} & \mathbf{D}^{11} & \mathbf{D}^{12} & \cdots & \mathbf{D}^{ij} & \cdots & \mathbf{D}^{nn} \\ \mathbf{R} & \mathbf{0} & \mathbf{T} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{R}^{11} & \mathbf{0} & \mathbf{0} & \mathbf{T}^{11} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{R}^{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{T}^{12} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \mathbf{R}^{ij} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{T}^{ij} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \mathbf{R}^{nn} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{T}^{nn} \end{bmatrix} \end{array} \tag{46}$$

with

$$\{\mathbf{D}^u\}_{kl} = \frac{\delta_{kj}}{\rho}\frac{\partial}{\partial x_j}\lambda_u\frac{\partial}{\partial x_l} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\mu_u\left[\delta_{kl}\frac{\partial}{\partial x_j} + \delta_{jl}\frac{\partial}{\partial x_k}\right] \tag{47a}$$

$$\{\mathbf{D}\}_{kt} = \frac{1}{n\rho}\frac{\partial}{\partial x_k} \tag{47b}$$

$$\{\mathbf{D}^{ij}\}_{kr} = \frac{1}{\rho}\frac{\partial}{\partial x_j}\delta_{ik} \tag{47c}$$

$$\{\mathbf{R}\}_{sl} = \phi^c_{1s}(0)\frac{\partial}{\partial x_l} \tag{47d}$$

$$\{\mathbf{T}\}_{st} = -\frac{\delta_{st}}{\tau^{(1)}_{\sigma_s}} \tag{47e}$$

$$\{\mathbf{R}^{ij}\}_{rl} = \frac{1}{n}\phi^c_{2r}(0)\left[\frac{n}{2}\left(\delta_{li}\frac{\partial}{\partial x_j} + \delta_{jl}\frac{\partial}{\partial x_i}\right) - \delta_{ij}\frac{\partial}{\partial x_l}\right] \tag{47f}$$

$$\{\mathbf{T}^{ij}\}_{rv} = -\frac{\delta_{rv}}{\tau^{(2)}_{\sigma_r}} \tag{47g}$$

where $k, l = 1, \ldots, n$, $s, t = 1, \ldots, L_1$, $r, v = 1, \ldots, L_2$, and

$$\lambda_u = \frac{1}{n}(\mathrm{M}_{u_1} - \mathrm{M}_{u_2}) \tag{48}$$

and

$$\mu_u = \frac{\mathrm{M}_{u_2}}{2} \tag{49}$$

are the unrelaxed Lamé constants of the n-dimensional solid, with M_{u_1} and M_{u_2} defined by equation (38).

The vector $\mathbf{U}$ is given by

$$\mathbf{U}^T = [\mathbf{u}, \dot{\mathbf{u}}, (e_{1_l}, l = 1, \ldots, L_1), (e_{ijl}, l = 1, \ldots, L_2)], \tag{50}$$

and the body force vector is expressed by

$$\mathbf{F}^T = [\mathbf{0}, \mathbf{f}/\rho, \mathbf{0}, \mathbf{0}]. \tag{51}$$

Equation (45) represents the equation of motion governing the viscoelastic response of the medium. It correctly describes the anelastic effects observed in wave propagation, namely attenuation and dispersion, within the framework of the linear response theory. The model can describe wave propagation through any kind of linear viscoelastic material, for example porous rocks (waves of the first kind can be approximated by standard linear solid rheology (Geertsma & Smit, 1961)), provided the complex moduli of the porous media are given as a function of frequency. By fitting the observed to the viscoelastic complex moduli given by equation (A6), the corresponding relaxation times and relaxed moduli can be obtained for any frequency range.

Storage requirements increase with the number of memory variables, i.e. with the number of relaxation mechanisms. Depending on the accuracy required, constant-Q materials in the seismic exploration band (say between 5 and 100 Hz) can be obtained by using two or more sets of relaxation times for each mode. Besides the curve-fitting procedure, which is used in this work, optimal relaxation times can be obtained using the Padé approximant method derived by Day & Minster (1984).

For the spatial derivative operators in equation (45) we use the Fourier pseudo-spectral method (Kosloff & Baysal, 1982), which consists of a discretization of space and calculation of spatial derivatives using the Fast Fourier Transform (FFT).

The propagation in time is done by a new pseudo-spectral time integration technique (Tal-Ezer 1986). A detailed description can also be found in Carcione *et al.* (1988a), where a modification was introduced to apply the method to the viscoacoustic equation of motion. The same considerations on convergence, resolution and stability given in that paper apply here, because the method is completely general for equations of the type given in (45). In virtue of the complete accuracy both in time and space for band-limited functions, it can be concluded that numerical dispersion is not present in this numerical algorithm. This is very important in anelastic wave propagation, where numerical dispersion can be confused as physical dispersion.

6 2-D WAVE PROPAGATION

For the 2-D case it can be seen from (41) and (42) that three sets of independent memory variables are obtained. These are given by

$$e_{1_l} = \phi^c_{1_l} * (\epsilon_{11} + \epsilon_{22}), \qquad l = 1, \ldots, L_1 \tag{52a}$$

$$e_{11l} = -e_{22l} = \tfrac{1}{2}\phi^c_{2_l} * (\epsilon_{11} - \epsilon_{22}), \qquad l = 1, \ldots, L_2 \tag{52b}$$

and

$$e_{12l} = \phi^c_{2_l} * \epsilon_{12}, \qquad l = 1, \ldots, L_2. \tag{52c}$$

The equations of motion can be obtained from (45) with $n = 2$, to yield

$$\rho\frac{\partial^2 u_1}{\partial t^2} = \frac{\partial}{\partial x_1}\left[E_u\frac{\partial u_1}{\partial x_1} + \lambda_u\frac{\partial u_2}{\partial x_2} + \frac{1}{2}\sum_{l=1}^{L_1} e_{1_l} + \sum_{l=1}^{L_2} e_{11_l}\right] + \frac{\partial}{\partial x_2}\left[\mu_u\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) + \sum_{l=1}^{L_2} e_{12l}\right] \tag{53a}$$

and

$$\rho\frac{\partial^2 u_2}{\partial t^2} = \frac{\partial}{\partial x_2}\left[\lambda_u\frac{\partial u_1}{\partial x_1} + E_u\frac{\partial u_2}{\partial x_2} + \frac{1}{2}\sum_{l=1}^{L_1} e_{1_l} - \sum_{l=1}^{L_2} e_{11_l}\right] + \frac{\partial}{\partial x_1}\left[\mu_u\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) + \sum_{l=1}^{L_2} e_{12l}\right], \tag{53b}$$

where E_u is given by

$$E_u = \lambda_u + 2\mu_u. \tag{54}$$

To complete the scheme we need the equivalent expressions to equations (44a) and (44b), which are

$$\dot{e}_{1_l} = -\frac{e_{1_l}}{\tau^{(1)}_{\sigma_l}} + \phi^c_{1_l}(0)\left[\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right], \tag{55a}$$

$$\dot{e}_{11_l} = -\frac{e_{11_l}}{\tau^{(2)}_{\sigma_l}} + \frac{1}{2}\phi^c_{2l}(0)\left[\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right], \tag{55b}$$

and

$$\dot{e}_{12_l} = -\frac{e_{12_l}}{\tau^{(2)}_{\sigma_l}} + \frac{1}{2}\phi^c_{2l}(0)\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right]. \tag{55c}$$

The equations of motion for an isotropic elastic solid are a special case of (53a) and (53b). In the limit $\tau^{\nu}_{\epsilon_l} = \tau^{\nu}_{\sigma_l}$, the unrelaxed constants λ_u and E_u approach the relaxed or elastic ones (see (A24) and (38)), and the memory variables are identically zero (this can be deduced from (40), (41) and (42)).

6.1 Wave propagation in a homogeneous medium

The example that we choose involves wave propagation in a two-dimensional viscoelastic medium, which may represent a near-surface unconsolidated material or a strongly anelastic porous rock. The calculation uses a 99×99 grid with a grid spacing $DX = DZ = 20$ m. The motion is initiated by a vertical force located in the centre of the homogeneous region. For the directional force we use the source time history defined by (B6) with $\eta = 0.5$, $\epsilon = 1$, $t_0 = 0.06$ s and a cutoff frequency $f_0 = 50$ Hz. The material is defined by the relaxed moduli, denisty and relaxation times given in Table 1, which yield the relaxed or elastic velocities $v_1 = 3000$ and $v_2 = 2000$ m s^{-1}, and the unrelaxed velocities $V_{u_1} = 3190$ and $V_{u_2} = 2175$ m s^{-1}; therefore, we expect considerable velocity dispersion.

Fig. 1 shows the P-wave, S-wave and bulk quality factors, for the material defined in Table 1. The dashed line in Fig. 1 indicates the amplitude spectrum of the source. It is clear that the relaxation times give almost constant-Q values in the source frequency band, with the shear mode attenuating more than the dilatational mode. Fig. 2(a) and (b) shows the phase and group velocity dispersion for the P- and S-waves, respectively. As mentioned previously, the velocity dispersion is strong, ranging from the relaxed velocities (low-frequency limit) to the unrelaxed velocities (high-frequency limit). The shear mode presents more dispersion, as a consequence of the lower Q value, than for the dilatational mode. This fact can also be deduced from the Kramers–Kronig dispersion relations which are completely valid for linear viscoelastic solids. Ben-Menahem & Singh (1981, p. 892) used these relations to show that for a constant-Q solid the quality factor is inversely proportional to the phase velocity.

Figures 3 and 5 (elastic), and 4 and 6 (viscoelastic) display the u_1 component (a) and u_2 component (b) for two different times. At $t = 0.22$ s the S-wave (inner wavefront) in the viscoelastic snapshot still appears stronger than the

Table 1. Material properties.

M_1 (GPa)	M_2 (GPa)	density (kg m^{-3})	l	$\tau_{\epsilon_l}^{(1)}$ (s)	$\tau_{\sigma_l}^{(1)}$ (s)	$\tau_{\epsilon_l}^{(2)}$ (s)	$\tau_{\sigma_l}^{(2)}$ (s)
20	16	2000	1	0.0325305	0.0311465	0.0332577	0.0304655
			2	0.0032530	0.0031146	0.0033257	0.0030465

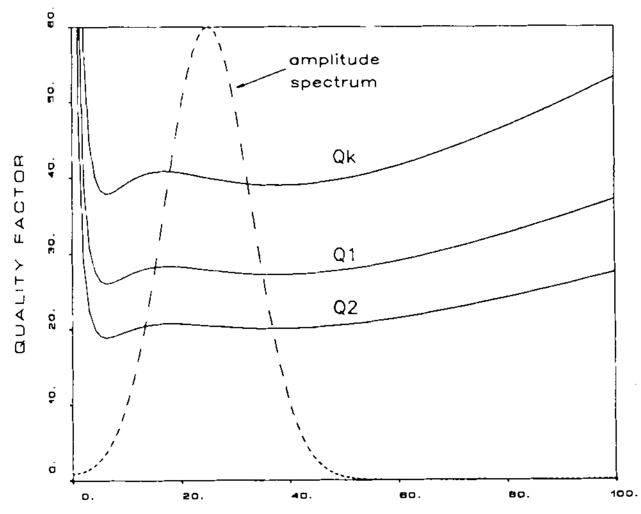

Figure 1. P-wave, S-wave and bulk quality factors versus frequency for the medium defined in Table 1. The dashed line represents the amplitude spectrum of the source.

P-wave, but at $t = 0.32$ s the amplitudes are almost comparable, due to the higher attenuation acting on the shear mode. Comparison between the elastic and viscoelastic wavefronts reveals the strong wave attenuation at relatively small propagation time.

Fig 7(a) and 7(b) shows the comparison between the elastic and viscoelastic time histories at station 1 of Table 2. The coordinates indicate horizontal and vertical distances from the source position. Several physical effects can be observed. First, the viscoelastic wave field arrives earlier than the elastic one. This is a consequence of the velocity dispersion curves in which the elastic or non-dispersive behaviour is defined at zero frequency. Second, the shear mode is relatively faster than the dilatational mode; another consequence of the velocity dispersion function. Finally, as was mentioned above, the amplitudes of the shear and dilatational modes are comparable, in contrast to the elastic wave field where the S-wave has more amplitude (see also Figs 5 and 6).

The analytical solution of the problem is obtained in Appendix B. The approximations carried out to obtain the viscoelastic wave field, i.e. considering $\bar{\phi}_1(r, 0) \equiv \bar{\phi}_2(r, 0) \equiv 0$, and the numerical inverse Fourier transform, can be tested in order to ensure that the solution is exact up to a given number of digits. The evaluation is performed in the elastic case, where the exact solution is obtained as a convolution of the time-domain Green's function with the source time history. Comparing the solution obtained through the frequency domain versus the time convolution for the P-wave peak amplitude gives an accuracy of at least four digits provided that a long operator is used for the inverse Fourier transform (at least 4000 points). Greater accuracy can be obtained by using double precision arithmetic.

Figures 8–11 compare numerical and analytical time histories at the stations indicated in Table 2; (a) u_1 components, and (b) u_2 components. The time histories are normalized with respect to the maximum peak amplitude recorded at station 3. The u_1 components of the wave field are zero at stations 2 and 3 (as evidenced by insignificant numbers in the numerical results), in agreement with equation (B1a). As the figures show, the comparison is excellent, with the shear and dilatational modes having the correct polarities at the

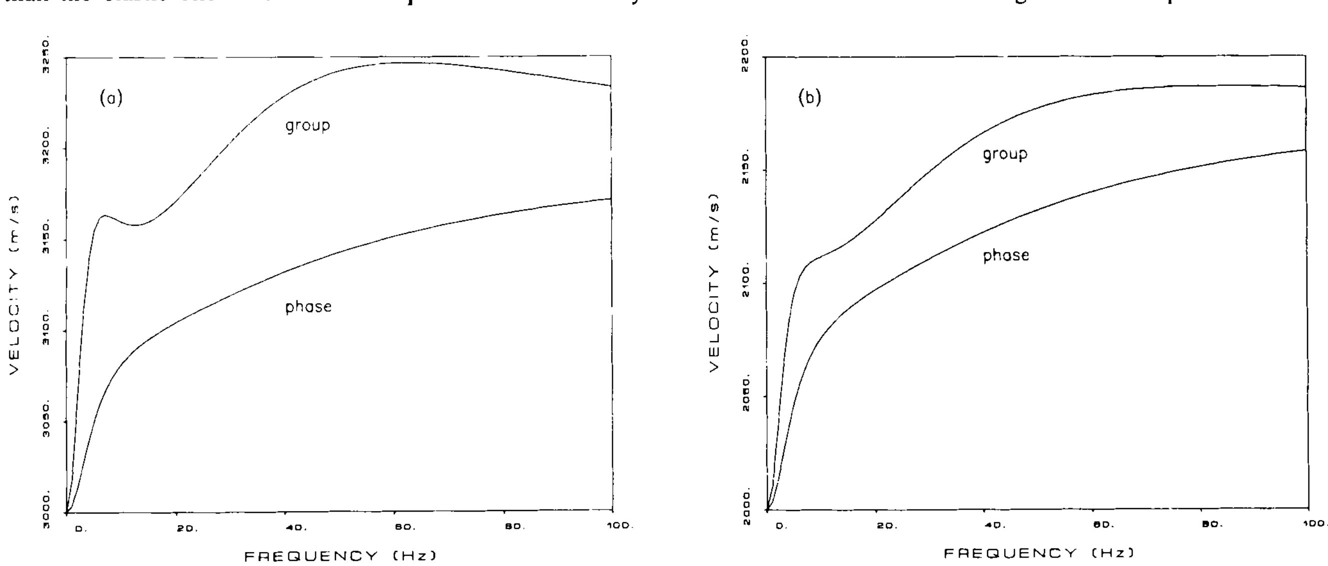

Figure 2. Phase and group velocities for the medium defined in Table 1. (a) P-wave. (b) S-wave.

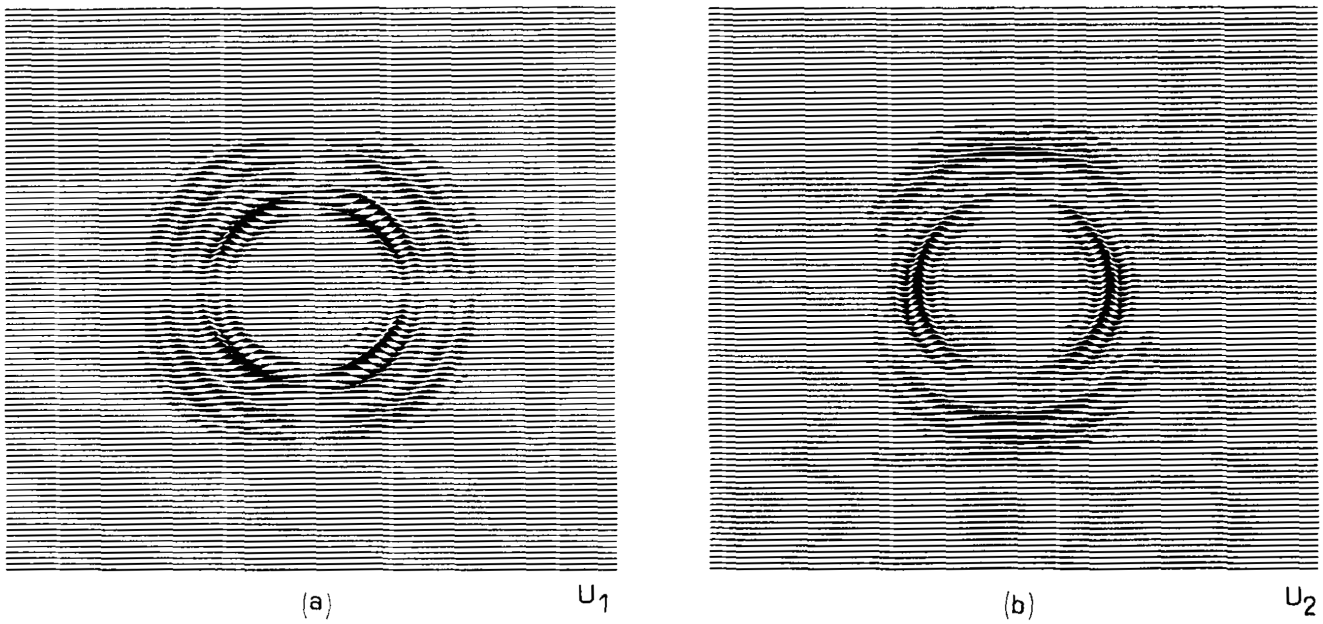

Figure 3. Elastic u_1 component (a) and u_2 component (b) at $t = 0.22$ s due to a vertical force in the homogeneous medium defined in Table 1. The S-wave amplitude (inner wavefront) is stronger than the P-wave amplitude (outer wavefront).

four stations. The deviations at long observation times are due to the approximations used to display the analytical results. The wave field presents similar characteristics to the elastic case. Due to symmetry considerations, the u_1 component should show antisymmetric behaviour across $z = 0$, and the u_2 component should show symmetric behaviour across the same line. These effects can be appreciated at stations 1 and 4, where u_1 undergoes a polarity change, and u_2 keeps the same sign at both sides of $z = 0$. Moreover, because station 2 lies in the direction of the force, the recorded wave field is mainly compressional. Conversely, station 3 records a shear behaviour.

CONCLUSIONS

We have presented a model for wave propagation simulation in a general heterogeneous anelastic medium within the framework of the theory of linear viscoelasticity. The formulation of the model is based on the introduction of memory variables and is a more suitable approach for the

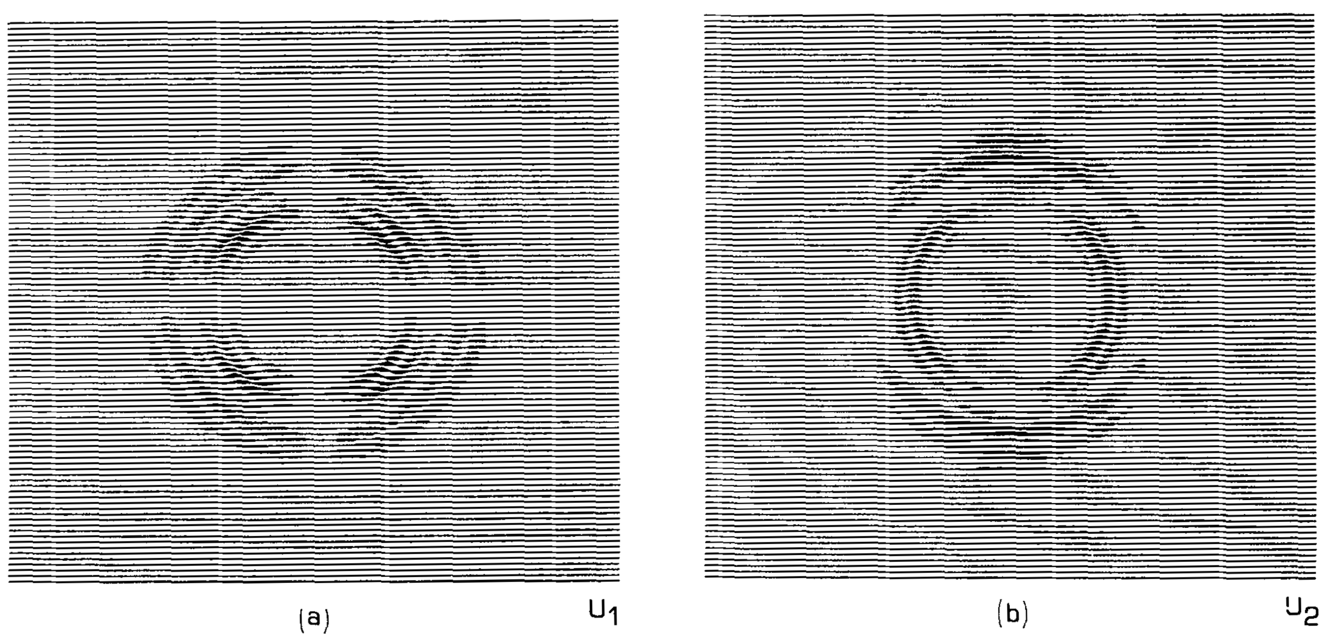

Figure 4. Viscoelastic u_1 component (a) and u_2 component (b) at $t = 0.22$ s due to a vertical force in the homogeneous medium defined in Table 1. The S-wave (inner wavefront) still appears stronger than the P-wave (outer wavefront).

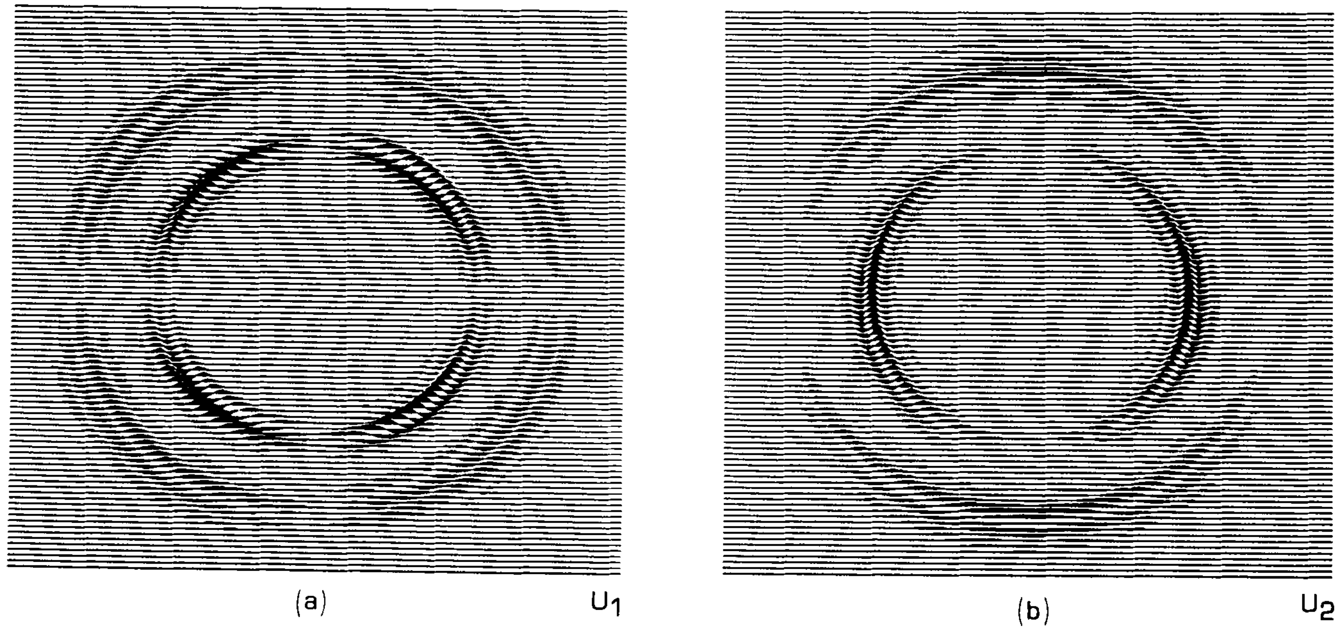

Figure 5. Elastic u_1 component (a) and u_2 component (b) at $t = 0.32$ s due to a vertical force in the homogeneous medium defined in Table 1. The S-wave amplitude (inner wavefront) remains, as expected, stronger than the P-wave amplitude (outer wavefront).

treatment of the convolutional form of Boltzmann's superposition principle in the time domain. The constitutive relation is based on a spectrum of multiple relaxation mechanisms; a model which can explain the anelastic effects caused by any linear relaxation phenomena, particularly those which affect wave propagation in porous media.

The equations of motion are solved using the same approach applied to the viscoacoustic equations of motion (Carcione *et al.* 1987a). The method is very accurate, therefore the problem of numerical dispersion is avoided. This is very important in anelastic wave propagation where numerical dispersion could be confused with real physical dispersion.

The validity of the theory is not restricted to any particular choice of the parameters (relaxed moduli, density and relaxation times). The examples shown in this work belong to the seismic exploration band, but problems in other frequency ranges, i.e. in the fields of global seismology, acoustic logging and material science for instance, can be solved with the same effectiveness.

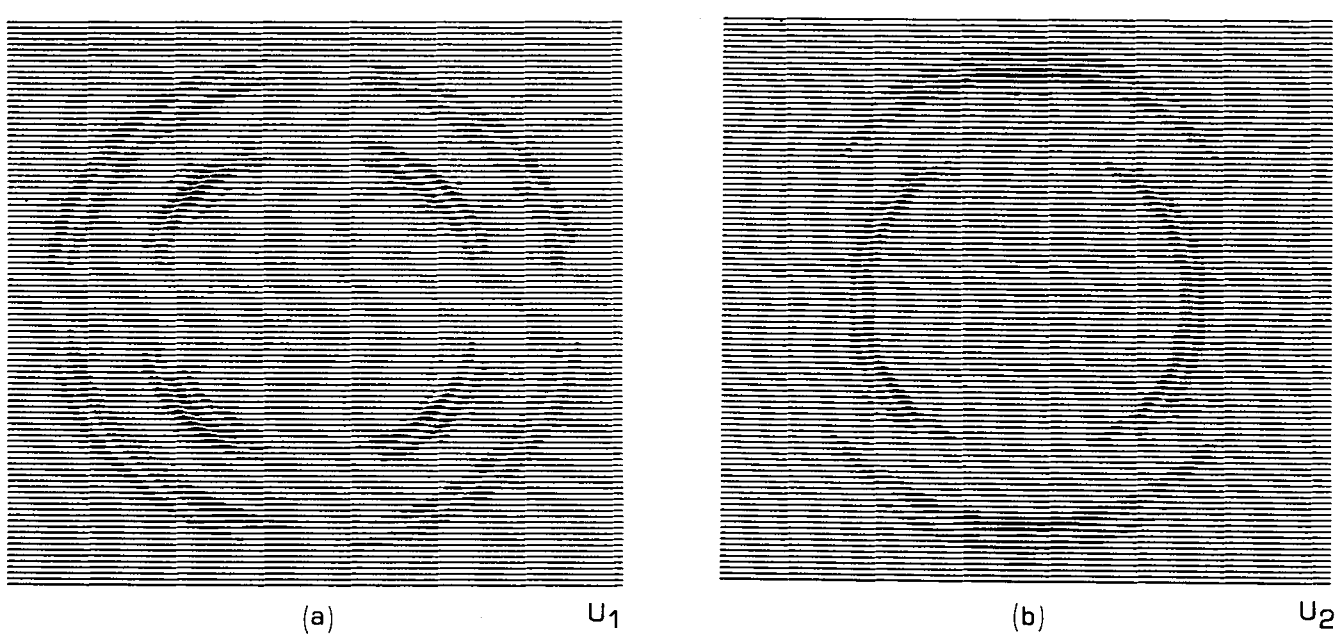

Figure 6. Viscoelastic u_1 component (a) and u_2 component (b) at $t = 0.32$ s due to a vertical force in the homogeneous medium defined in Table 1. S- and P-wave amplitudes (inner and outer wavefronts, respectively) are almost comparable, due to the higher attenuation acting on the shear mode.

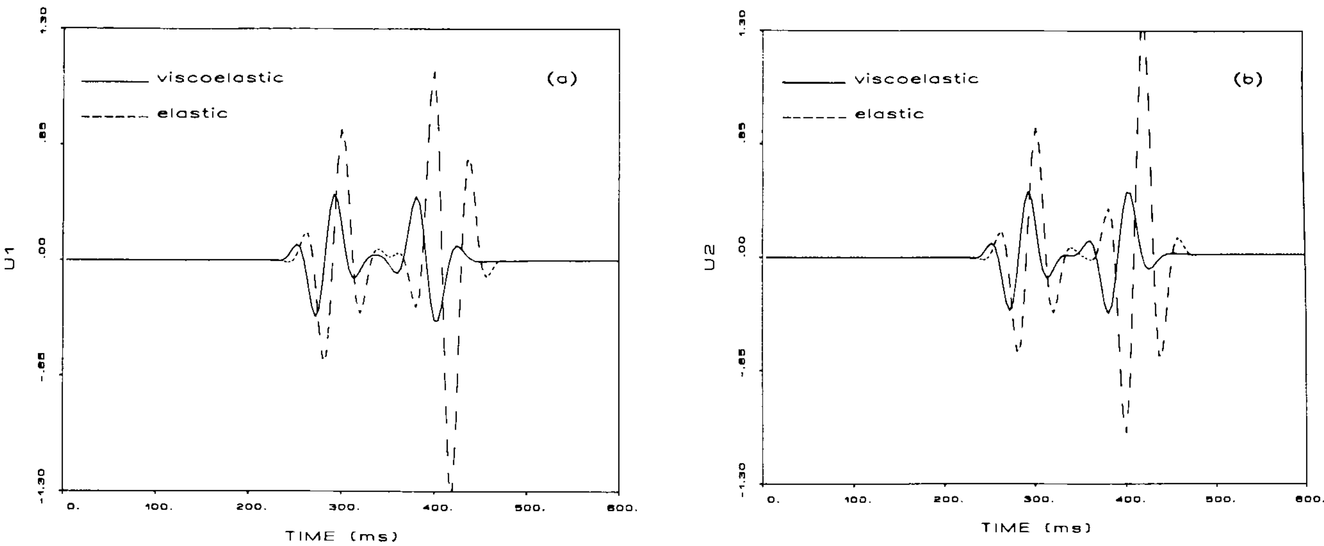

Figure 7. Time history comparison between the u_1 components (a) and u_2 components (b) of the viscoelastic and elastic simulations at station 1 of Table 2. The medium is defined in Table 1. Important differences in amplitudes and arrival times can be appreciated.

Possible limitations are concerned with the numerical method used to solve equation (45). The pseudo-spectral method implemented here was originally designed to solve the elastic equation of motion and, although accurate, it is not very effective in terms of computer time. To overcome this problem, a new algorithm is being developed which will reduce the computer time to the same levels of the elastic case (Tal-Ezer *et al.*, 1988).

Wave propagation simulation in a 2-D homogeneous strongly anelastic medium has been performed. Comparisons between elastic and viscoelastic simulations show important differences in the amplitudes of the wave field and in the arrival times. The accuracy of the numerical algorithm is verified in comparisons with the theoretical solution based on a 2-D viscoelastic Green's function.

Table 2. Horizontal and vertical distances from the source position at 4 stations.

Station	x_1 (m)	x_2 (m)
1	500	500
2	0	500
3	500	0
4	−500	500

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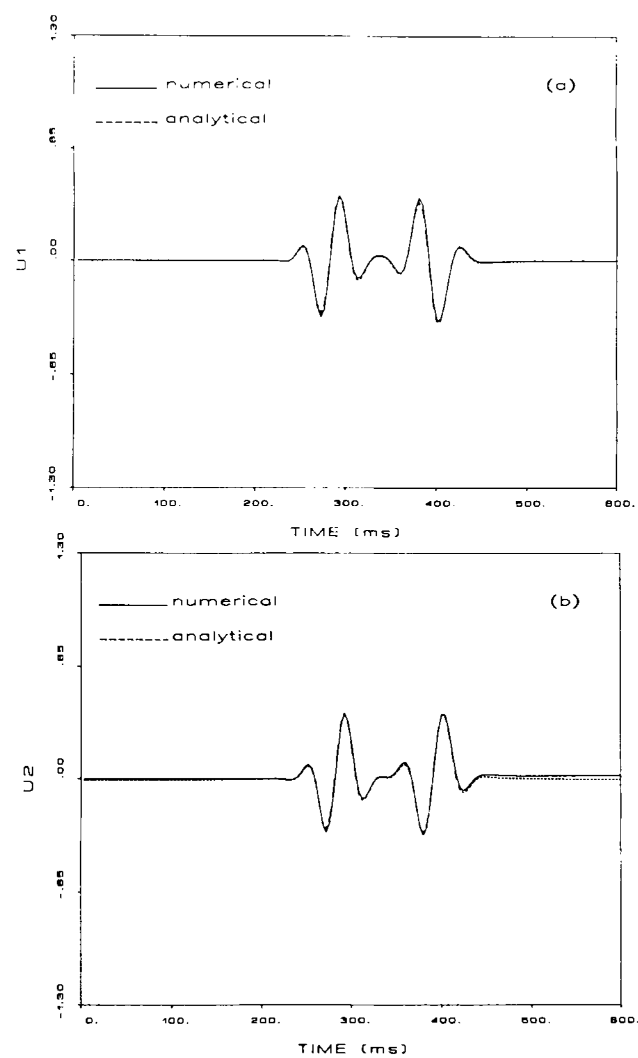

Figure 8. Theoretical and numerical u_1 component (a) and u_2 component (b) time histories at station 1 of Table 2, for the homogeneous medium defined in Table 1.

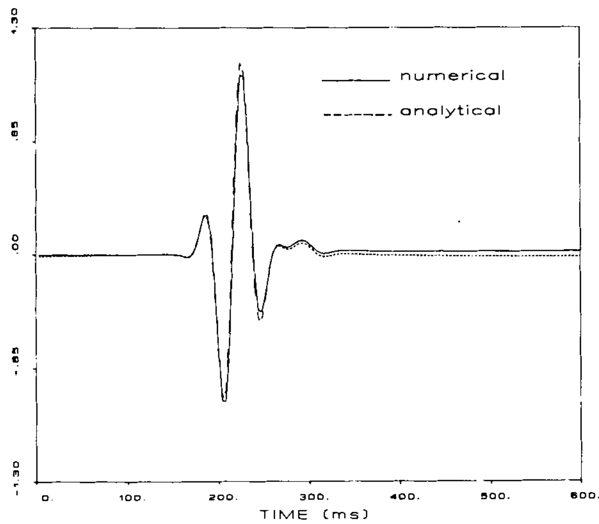

Figure 9. Theoretical and numerical u_2 component time histories at station 2 of Table 2, for the homogeneous medium defined in Table 1.

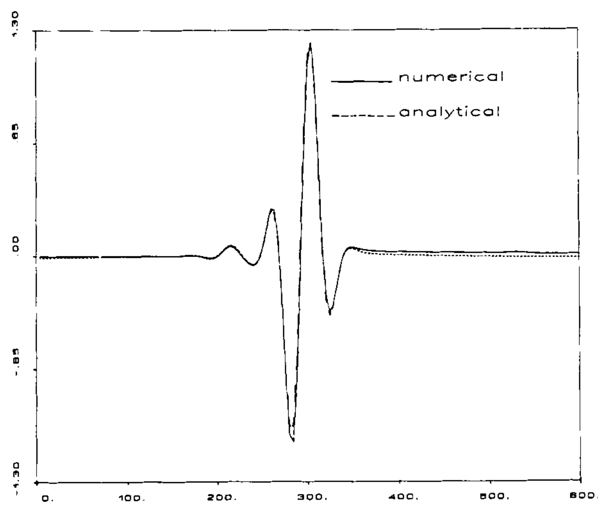

Figure 10. Theoretical and numerical u_2 component time histories at station 3 of Table 2, for the homogeneous medium defined in Table 1.

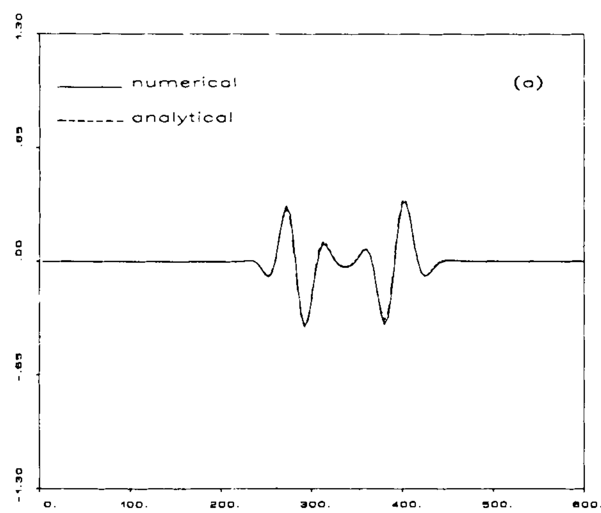

Figure 11. Theoretical and numerical u_1 component (a) and u_2 component (b) time histories at station 4 of Table 2, for the homogeneous medium defined in Table 1.

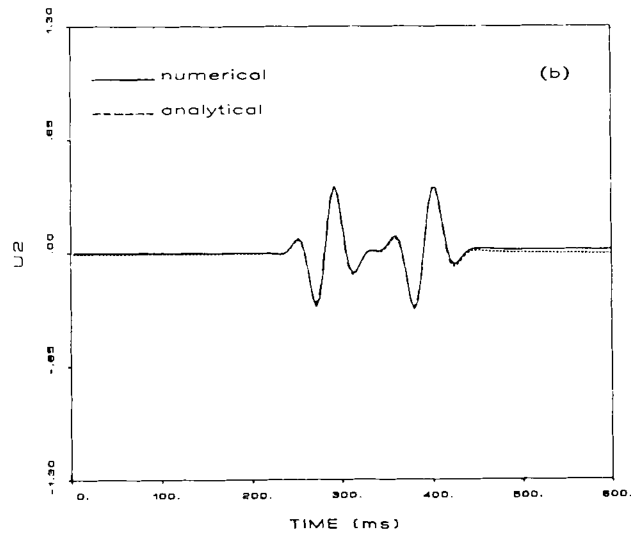

Figure 11. (continued)

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APPENDIX A: COMPLEX MODULI, LAMÉ CONSTANTS, VELOCITIES AND WAVENUMBER FOR THE VISCOELASTIC MEDIUM

Applying the convolutional theorem to equation (6), the rheological relation in the frequency domain takes the form

$$\tilde{T}^{\nu}_{ij}(\omega) = \tilde{E}^{\nu}_{ij}(\omega)\mathbf{F}[\dot{\psi}^{c}_{\nu}(t)], \tag{A1}$$

where ω is the angular frequency, the tilde means time Fourier transform and the operator $\mathbf{F}$ performs the time Fourier transform. From (A1) we identify the dilatational and shear complex moduli of the medium as

$$\mathrm{M}^{C}_{\nu} = \mathbf{F}[\dot{\psi}^{c}_{\nu}(t)]. \tag{A2}$$

Using the derivative of $\psi^{c}_{\nu}(t)$ defined by (9), equation (A2) can be expressed as

$$\mathrm{M}^{C}_{\nu}(\omega) = \mathbf{F}\left[\delta(t)\psi_{\nu}(t) + H(t) \times \sum_{l=1}^{L_{\nu}} A_{\nu_l} \exp\left\{\frac{-t}{\tau^{\nu}_{\sigma_l}}\right\}\right], \tag{A3}$$

where $\delta(t)$ is Dirac's function, and A_{ν_l} is given by

$$A_{\nu_l} = \frac{M_{\nu}}{\tau^{\nu}_{\sigma_l}}\left(1 - \frac{\tau^{\nu}_{\epsilon_l}}{\tau^{\nu}_{\sigma_l}}\right). \tag{A4}$$

Taking the Fourier transform we find that

$$\mathrm{M}^{C}_{\nu}(\omega) = \psi_{\nu}(0) + \sum_{l=1}^{L_{\nu}} A_{\nu_l}\int_0^{\infty} \exp\left\{-\left(i\omega + \frac{1}{\tau^{\nu}_{\sigma_l}}\right)t\right\} dt. \tag{A5}$$

Performing the integration we obtain, after some calculation, the complex moduli

$$\mathrm{M}^{C}_{\nu}(\omega) = \mathrm{M}_{\nu}\left[1 - L_{\nu} + \sum_{l=1}^{L_{\nu}} \frac{1 + i\omega\tau^{\nu}_{\epsilon_l}}{1 + i\omega\tau^{\nu}_{\sigma_l}}\right], \qquad \nu = 1, 2. \tag{A6}$$

We define the complex Lamé constants as

$$\lambda(\omega) = \frac{1}{n}[\mathrm{M}^{C}_{1}(\omega) - \mathrm{M}^{C}_{2}(\omega)] \tag{A7}$$

and

$$\mu(\omega) = \tfrac{1}{2}\mathrm{M}^{C}_{2}(\omega). \tag{A8}$$

The complex bulk modulus of the medium is then

$$k(\omega) = \lambda(\omega) + \frac{2}{n}\mu(\omega) = \frac{1}{n}\mathrm{M}^{C}_{1}(\omega). \tag{A9}$$

We will now see that these complex Lamé constants play an analogous role here to that in the elastic case concerning the dynamic behaviour of the medium.

The equation of motion of the viscoelastic medium without body forces in the ω-domain is (Borcherdt 1977),

$$\tilde{\sigma}_{ij,j}(\omega) + \omega^2\rho\tilde{u}_i(\omega) = 0, \tag{A10}$$

where $\tilde{u}_i(\mathbf{x}, \omega)$ are the components of the displacement vector, $\rho(\mathbf{x})$ is the density and the notation $\tilde{\sigma}_{ij,j} = \partial\tilde{\sigma}_{ij}/\partial x_j$ is used.

Applying the convolutional theorem to equation (3) implies that

$$\tilde{\sigma}_{ij} = \tilde{\epsilon}_{kk}\delta_{ij}\frac{1}{n}\mathbf{F}[(\dot{\psi}^{c}_{1} - \dot{\psi}^{c}_{2})] + \tilde{\epsilon}_{ij}\mathbf{F}[\dot{\psi}^{c}_{2}]. \tag{A11}$$

Substituting (A2) with $\nu = 1, 2$ into (A11) gives

$$\tilde{\sigma}_{ij} = \frac{1}{n}[\mathrm{M}^{C}_{1} - \mathrm{M}^{C}_{2}]\delta_{ij}\tilde{\epsilon}_{kk} + \mathrm{M}^{C}_{2}\tilde{\epsilon}_{ij}, \tag{A12}$$

or, in terms of the complex Lamé constants defined by (A7) and (A8),

$$\tilde{\sigma}_{ij} = \lambda\delta_{ij}\tilde{\epsilon}_{kk} + 2\mu\tilde{\epsilon}_{ij}. \tag{A13}$$

We can see the complete analogy with the elastic case, in agreement with the correspondence principle which establishes that the solution for a dynamic problem in a viscoelastic medium can be obtained by replacing the elastic constants by the corresponding viscoelastic complex moduli in the frequency domain (Bland 1960, p. 96).

Let us consider a viscoelastic plane wave of the form

$$u_i(t) = U_i \exp\{i(\omega_0 t - \mathbf{k}\cdot\mathbf{x})\}, \tag{A14}$$

where U_i, $i = 1, \ldots, n$ are complex constants, $\mathbf{k}$ is the complex wavenumber vector and ω_0 is the angular frequency. Fourier transforming (A14) gives

$$\tilde{u}_i(\omega) = 2\pi U_i\delta(\omega - \omega_0)\exp\{-i\mathbf{k}\cdot\mathbf{x}\}. \tag{A15}$$

Considering the definition of the strain tensor as a function of the displacement field (Aki & Richards 1981, p. 13) and replacing (A15) into (A13) results in

$$\tilde{\sigma}_{ij} = -2i\pi\delta(\omega - \omega_0)[\lambda k_l U_l\delta_{ij} + \mu(U_i k_j + U_j k_i)] \times \exp\{-k\mathbf{k}\cdot\mathbf{x}\}. \tag{A16}$$

Taking the divergence of (A16) and replacing it in (A10), implies that

$$[(\lambda+\mu)k_i k_l + \mu\delta_{il}k_j k_j - \rho\omega_0^2\delta_{il}]U_l = 0. \quad \text{(A17)}$$

If we take $U_l = U_0 k_l$, then

$$\rho\frac{\omega_0^2}{k_1^2} = \lambda + 2\mu. \quad \text{(A18)}$$

This defines the dilatational viscoelastic wave, where k_1 is the P-wavenumber. Letting $U_l K_l = 0$ results in the shear viscoelastic wave for which

$$\rho\frac{\omega_0^2}{k_2^2} = \mu, \quad \text{(A19)}$$

with k_2 the S-wavenumber. From (A18) and (A19) we can define the complex P- and S-velocities:

$$V_1(\omega_0) = pv\left(\frac{\lambda(\omega_0)+2\mu(\omega_0)}{\rho}\right)^{1/2} \quad \text{(A20a)}$$

and

$$V_2(\omega_0) = pv\left(\frac{\mu(\omega_0)}{\rho}\right)^{1/2}, \quad \text{(A20b)}$$

where pv means that we have to take the principal value of the square root, i.e., the solution with a positive real part. Replacing (A7) and (A8) in (A20a) and (A20b), we obtain the complex velocities in terms of the complex moduli:

$$V_1 = \left(\frac{E}{\rho}\right)^{1/2} \quad \text{(A21a)}$$

and

$$V_2 = \left(\frac{\mathrm{M}_2^C}{2\rho}\right)^{1/2}, \quad \text{(A21b)}$$

where

$$E = \frac{\mathrm{M}_1^C + (n-1)\mathrm{M}_2^C}{n}. \quad \text{(A22)}$$

As $\omega \to \infty$, we have

$$V_1 \to V_{u_1} = \left(\frac{E_u}{\rho}\right)^{1/2} \quad \text{(A23a)}$$

and

$$V_2 \to V_{u_2} = \left(\frac{\mathrm{M}_{u_2}}{2\rho}\right)^{1/2} \quad \text{(A23b)}$$

the unrelaxed wave velocities, E_u being the unrelaxed value of E,

$$E_u = \frac{\mathrm{M}_{u_1} + (n-1)\mathrm{M}_{u_2}}{n}, \quad \text{(A24)}$$

with M_{u1} and M_{u_2} given by (38).

Real materials behave elastically at both very low and very high frequencies. The relaxation functions (9), which are based on general standard linear solid rheology, describe correctly this behaviour (Liu *et al.* 1976). In this work we choose the elastic behaviour in the low-frequency limit. For a standard linear solid mechanical model the elastic limit is reached when the dashpot is eliminated; that implies $\tau_{\epsilon_l} \to 0$ and $\tau_{\sigma_l} \to 0$ (Ben-Menahem & Singh 1981, p. 856). This is equivalent to $\omega \to 0$, as can be seen from equation (A6); hence, the relaxed and elastic moduli coincide. In practice, however, we do not need to restrict the representation of real materials to mechanical models, therefore we can choose the acoustic or 'non-dispersive' behaviour in the high-frequency limit. This is the case discussed by Ben-Mehahem & Singh (1981, p. 873).

When $\omega \to 0$, it is equivalent to take the limit $\tau_{\epsilon_l} \to \tau_{\sigma_l}$. We obtain

$$V_1 \to v_1 = \left(\frac{E_r}{\rho}\right)^{1/2} \quad \text{(A25a)}$$

and

$$V_2 \to v_2 = \left(\frac{\mathrm{M}_2}{2\rho}\right)^{1/2}, \quad \text{(A25b)}$$

the relaxed or elastic wave velocities of the medium, where

$$E_r = \frac{\mathrm{M}_1 + (n-1)\mathrm{M}_2}{n} \quad \text{(A26)}$$

is the relaxed value of E. The relaxed moduli M_1 and M_2 are expressed as functions of the elastic wave velocities as

$$\mathrm{M}_1 = \rho(nv_1^2 - 2(n-1)v_2^2) \quad \text{(A27a)}$$

$$\mathrm{M}_2 = 2\rho v_2^2. \quad \text{(A27b)}$$

The complex wavenumber is

$$k_\nu(\omega) = \frac{\omega}{V_\nu(\omega)}, \qquad \nu = 1, 2 \quad \text{(A28)}$$

APPENDIX B: CALCULATION OF THE 2-D GREEN'S FUNCTION USING THE CORRESPONDENCE PRINCIPLE

The solution of the wave field propagation generated by an impulsive point force in an elastic medium is given by Eason, Fulton & Sneddon (1956) and Pilant (1979, p. 59). For an impulsive force acting in the positive x_2 direction, this solution can be expressed as

$$u_1(r,t) = \frac{F}{2\pi\rho}\cdot\frac{x_1x_2}{r^2}[G_1(r,t) + G_2(r,t)] \quad \text{(B1a)}$$

and

$$u_2(r,t) = \frac{F}{2\pi\rho}\cdot\frac{1}{r^2}[x_2^2G_1(r,t) - x_1^2G_2(r,t)], \quad \text{(B1b)}$$

where F is a constant which gives the magnitude of the force, $r = (x_1^2 + x_2^2)^{1/2}$ and

$$G_1(r,t) = \frac{1}{v_1^2}(t^2-\tau_1^2)^{-1/2}H(t-\tau_1) + \frac{1}{r^2}(t^2-\tau_1^2)^{1/2}H(t-\tau_1) - \frac{1}{r^2}(t^2-\tau_2^2)^{1/2}H(t-\tau_2) \quad \text{(B2a)}$$

and

$$G_2(r,t) = -\frac{1}{v_2^2}(t^2-\tau_2^2)^{-1/2}H(t-\tau_2) + \frac{1}{r^2}(t^2-\tau_1^2)^{1/2}H(t-\tau_1) - \frac{1}{r^2}(t^2-\tau_2^2)^{1/2}H(t-\tau_2), \tag{B2b}$$

where $\tau_\nu = r/v_\nu$, $\nu = 1,2$, with v_1 and v_2 the compressional and shear elastic wave velocities, and H represents the Heaviside function. In order to apply the correspondence principle we must take a time Fourier transform of (B2a), (B2b), and replace the elastic Lamé constants by the corresponding viscoelastic Lamé constants given by (A7) and (A8). This is equivalent to replacing the elastic wave velocities v_1 and v_2 by the complex velocities given by (A20a) and (A20b).

Using the transform pairs of the zero- and first-order Hankel functions of the second kind,

$$\frac{1}{\tau^2}(t^2-\tau^2)^{1/2}H(t-\tau) \leftrightarrow \frac{i\pi}{2\omega\tau}H_1^{(2)}(\omega\tau)$$

and

$$(t^2-\tau^2)^{-1/2}H(t-\tau) \leftrightarrow -\frac{i\pi}{2}H_0^{(2)}(\omega\tau),$$

we obtain the time Fourier transform of the wave field:

$$\bar{u}_1(r,\omega,v_1,v_2) = \frac{F}{2\pi\rho}\cdot\frac{x_1x_2}{r^2}[\bar{G}_1(r,\omega,v_1,v_2) + \bar{G}_2(r,\omega,v_1,v_2)] \tag{B3a}$$

and

$$\bar{u}_2(r,\omega,v_1,v_2) = \frac{F}{2\pi\rho}\cdot\frac{1}{r^2}[x_2^2\bar{G}_1(r,\omega,v_1,v_2) - x_1^2\bar{G}_2(r,\omega,v_1,v_2)], \tag{B3b}$$

where

$$\bar{G}_1(r,\omega,v_1,v_2) = -i\frac{\pi}{2}\left[\frac{1}{v_1^2}H_0^{(2)}\left(\frac{\omega}{rv_1}\right) + \frac{1}{\omega rv_2}H_1^{(2)}\left(\frac{\omega}{rv_2}\right) - \frac{1}{\omega rv_1}H_1^{(2)}\left(\frac{\omega}{rv_1}\right)\right] \tag{B4a}$$

and

$$\bar{G}_2(r,\omega,v_1,v_2) = i\frac{\pi}{2}\left[\frac{1}{v_2^2}H_0^{(2)}\left(\frac{\omega}{rv_2}\right) - \frac{1}{\omega rv_2}H_1^{(2)}\left(\frac{\omega}{rv_2}\right) + \frac{1}{\omega rv_1}H_1^{(2)}\left(\frac{\omega}{rv_1}\right)\right]. \tag{B4b}$$

Using the correspondence principle we replace the elastic wave velocities in (B3a) and (B3b) by the viscoelastic velocities given by (A20a) and (A20b). The two-dimensional viscoelastic Green's function in ω-space can be expressed as

$$\bar{u}_1^v(r,\omega) \equiv \begin{cases} \bar{u}_1(r,\omega,V_1,V_2) & \omega \geq 0 \\ \bar{u}_1^*(r,-\omega,V_1,V_2) & \omega \leq 0 \end{cases} \tag{B5a}$$

and

$$\bar{u}_2^v(r,\omega) \equiv \begin{cases} \bar{u}_2(r,\omega,V_1,V_2) & \omega \geq 0 \\ \bar{u}_2^*(r,-\omega,V_1,V_2) & \omega \leq 0 \end{cases}. \tag{B5b}$$

The definitions of (B5a) and (B5b) ensure that the time Fourier transform of the viscoelastic Green's function is real. Because the Hankel functions in (B5a) and (B5b) have singularities at $\omega = 0$, we multiply the viscoelastic Green's function by the time Fourier transform of a shifted zero-phase Ricker wavelet defined by

$$F(t) = \exp\{-nf_0^2(t-t_0^2)\}\cos\epsilon\pi f_0(t-t_0), \tag{B6}$$

with $f_0 = 2\pi\Omega_0$ the cutoff frequency, t_0 is the time shift and η and ϵ are constants. The Fourier transform of $F(t)$ is

$$\bar{F}(\omega) = \pi\left(\frac{\pi}{\eta}\right)^{1/2}\frac{1}{\Omega_0}\exp\{i\omega t_0\} \times\left[\exp\left\{-\frac{\pi^2}{\eta}\left(\frac{\epsilon}{2}-\frac{\omega}{\Omega_0}\right)^2\right\} + \exp\left\{-\frac{\pi^2}{\eta}\left(\frac{\epsilon}{2}+\frac{\omega}{\Omega_0}\right)^2\right\}\right]. \tag{B7}$$

Multiplying the transformed Green's function (B5a), (B5b) by $\bar{F}(\omega)$ we obtain

$$\bar{\Phi}_1(r,\omega) \equiv \begin{cases} \bar{u}_1^v(r,\omega)\bar{F}(\omega) & \omega \neq 0 \\ 0 & \omega = 0 \end{cases} \tag{B8a}$$

$$\bar{\Phi}_2(r,\omega) \equiv \begin{cases} \bar{u}_2^v(r,\omega)\bar{F}(\omega) & \omega \neq 0 \\ 0 & \omega = 0 \end{cases} \tag{B8b}$$

avoiding, with this definition, the singularities (actually, this is an approximation because strictly $\bar{F}(0) \neq 0$). Because the inverse Fourier transforms of $\bar{\Phi}_1(r,\omega)$ and $\bar{\Phi}_2(r,\omega)$ do not have exact analytical expressions, we invert them numerically by using the discrete Fast Fourier Transform.