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The Rheological Models of Becker, Scott Blair, Kolsky, Lomnitz and Jeffreys Revisited, and Implications for Wave Attenuation and Velocity Dispersion

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Abstract
The rheological models of Lomnitz and Jeffreys have been widely used in earthquake seismology (to simulate a nearly constant Q medium) and to describe the creep and relaxation behavior of rocks as a function of time. Other similar models, such as those of Becker, Scott Blair and Kolsky, show similar properties, particularly the Scott Blair model describes a perfectly constant Q as a function of frequency. We first give a historical overview of the main scientists and the development and versions of the various models and priorities of discovery. Then, we clarify the relationship between the different versions of these models in terms of mathematical expressions of the complex modulus and calculate the phase velocity and quality factor Q as a function of frequency, illustrating the various special cases. In addition, we give useful hints for the numerical calculation of these moduli, which include special cases of the hypergeometric function.

Keywords Stress–strain relations · Becker · Scott Blair · Kolsky · Lomnitz and Jeffreys laws · Phase velocity · Attenuation factor · Q

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Article Highlights

- Five rheological (stress/strain) models for the description of media with constant Q used in seismology are reviewed and revised
- We give a historical overview of the development of the models and clarify the relationship between the different versions
- The Scott Blair stress–strain relationship is the only one that describes the perfect constant quality factor, Q, across all frequencies
- In view of the similarities and differences, it is concluded that it is more correct to speak of an “almost linear attenuation factor" than of an “almost constant quality factor"

1 Introduction

The relationship between stress and strain can be expressed by creep and relaxation functions in the time domain. The stress–strain laws can be applied at various temporal scales, from wave propagation to geological processes, implying relatively high and low temporal frequencies in the Fourier domain, respectively. For example, Hanyga (2014) considered several laws to study wave propagation, and Lomnitz (1956) used a logarithmic law to model creep measurements in igneous rocks, while Jeffreys (1972) used a generalization of Lomnitz law to study the elastic behavior of the Earth and planets. In addition, the study of the rheology of the Earth's crust and mantle and the observation of continental drift requires models that show a transition from the initial elastic behavior (“terrestrial monopole") to the viscous behavior required to support convection (e.g., Peltier 1984).

In general, the analysis of the various anelastic models represented by the stress–strain relationship is performed in terms of creep or relaxation functions (e.g., Strick and Mainardi 1982; Mainardi 2022). Here, we focus on velocity dispersion and attenuation as a function of frequency, i.e., phase velocity and attenuation and quality factors for seismic applications, e.g., hydrocarbon exploration [(5,300) Hz], in terms of complex and frequency-dependent moduli and velocities, and compare the different versions of the models that are seemingly dissimilar (based on different notation in different papers) and use the Fourier transform instead of the Laplace transform (e.g., Hanyga 2014).

This study mainly analyzes the models of Lomnitz and Jeffreys (and related similar versions: Becker, Kolsky and Scott Blair laws) in the context of wave propagation, focusing on the complex mathematical tools to formulate the equations in the frequency domain. We also provide a historical overview of the development of the various models and priorities of discovery. The paper is organized as follows. Section 2 contains historical notes on the main scholars and their theories. Section 3 explains the basic wave theory in order to understand the mathematics of the various models presented in Sect. 4. Finally, Sect. 5 presents the results illustrating the comparison between the different models.

2 Historical Notes

In many cases, scientists rediscover things already known or overlook earlier theories or misuse them. Some examples are given below. A historical review of science can help scientists contextualize their research. The following individuals were involved in the major discoveries of the rheological models discussed in this study.

2.1 Richard Becker

The first pioneer to work on Lomnitz-type attenuation models (before Lomnitz) was Richard Becker (1887–1955), a German theoretical physicist who made important contributions to thermodynamics, statistical mechanics, electromagnetism, superconductivity, and quantum electrodynamics. He was a professor first in Berlin and then in Göttingen. For more details, see his biography at https://en.wikipedia.org/wiki/Richard_Becker_(physicist).

In 1925, Becker introduced a creep law to treat the deformation of certain viscoelastic and plastic bodies (Becker 1925). The Lomnitz model turns out to be very similar, if not equivalent, to the Becker model from a practical point of view. The creep law proposed by Becker on the basis of empirical arguments has found a number of applications, first in ferromagnetism, as documented by Becker and Doring (1939) at that time, and in the mathematical theory of linear viscoelasticity (Gross 1953), where we find references to applications in dielectrics in the 1950s. In 1956, Jellinek and Brill (1956) proposed a model for the primary creep of ice based on the Becker model, while Orowan (1967) revisited the Becker model to obtain a quality factor that is almost independent of frequency, as observed in most rheological materials. Considering this weak dependence of the Q factor in seismology, Strick and Mainardi (1982) were the first to compare the Becker and Lomnitz models and note their similarity. More recently, Mainardi and Spada (2012a) compared the creep functions of the Becker and Lomnitz models. Unfortunately, we have to mention that the Becker model, despite its advantages, is almost neglected in the rheological literature. Nevertheless, the Becker law in linear viscoelasticity was rediscovered (independently) by Lubliner and Panoskaltsis (1992) as a modification of the Kuhn law of 1947, but the priority of Becker with respect to Kuhn is not up for discussion. More recently, Mainardi et al. (2019) revised and generalized this model for the creep and relaxation function, including the expression of the quality factor versus frequency.

2.2 George William Scott Blair

George William Scott Blair (1902–1987) was born in Surrey, England. After his education at Winchester College, southern England, he studied chemistry at Oxford University from 1920 to 1923, receiving a BA degree. He earned his Ph.D. from the University of London. Scott Blair played a prominent and active role in the development of rheology. He was a founding member and later President of the British Society of Rheology. Scott Blair's law follows the work of Nutting (1921), who reported that the mechanical strains of viscoelastic materials decrease as power functions with time. This conclusion was in strong contradiction to the standard exponential law and Maxwell's linear law.

It seems that Scott Blair was not aware of the constant Q properties of his model when he introduced it (Scott Blair and Caffyn 1942) and even before the 1960s. This is because he was concerned with the creep function in the time domain to describe the quasi-static deformation of rocks and not to study wave propagation. Probably Bland (1960, p. 54, Eq. 111) was the first to determine the complex modulus in the frequency domain, referring to Scott Blair but not citing any of his works. From Bland's equation it can be readily deduced that Q is constant over all frequencies. Notably, Futterman (1962) and Knopoff (1964) missed the Scott Blair model, the latter in his section "Loss Models for Constant Q," and if the models of Lomnitz and Futterman were to describe constant Q from the 1960 s onward, then, the perfect candidate was known beforehand, i.e., the Scott Blair model.

Kjartansson (1979) appears to be the first to use the Scott Blair model for constant Q, citing (Bland 1960), and to use it in seismology. In fact, three years before (Liu et al. 1976) do not report the model and propose a near-constant Q approach based on a spectrum of Zener relaxation peaks (Carcione 2022, Section 2.4.5). This approach has been used extensively to simulate wave propagation, based on memory variables (Carcione 2022, Section 3.9). On the other hand, Carcione et al. (2002) were the first to simulate seismic wave propagation with the Scott Blair model in the time domain, using fractional derivatives (Carcione 2022, Section 2.5.2).

2.3 Herbert Kolsky

Herbert Kolsky (1916–1992) was an eminent researcher and educator in the fields of applied physics, mathematics, and engineering. He was born in London and graduated from Imperial College in 1937. He earned a doctorate in philosophy in 1940 and a doctorate in science in 1957, both from the University of London. Kolsky developed a device called the Split-Hopkinson Bar that can measure permanent deformations of objects subjected to sudden, intense pressure. The device is used in laboratories for projects ranging from aircraft development to weapons and armor. He wrote the book "Stress Waves in Solids" (1953), a major work on the subject, and published 90 papers on such topics as the origin of cracks in brittle material, the effects of dynamic conditions on metals and polymers, and the nature of fiber-reinforced material. Early in his career in England, he worked in military research at Imperial College, headed the physics department of the Imperial Chemical Industries Laboratories for nine years, and was chief scientific officer of the Royal Armament Research and Development Establishment for two years. He joined the faculty of Brown College in 1960 and retired there in 1983. His obituary appeared in the New York Times on May 16, 1992, Section 1, Page 10 of the National edition with the headline: "Herbert Kolsky, 75; Noted for Research On Applied Physics". Kolsky developed one of the first so-called nearly constant Q models (Kolsky 1956, 1964), and we show in this paper that there are two versions of his model.

2.4 Cinna Lomnitz

Cinna Lomnitz (1925–2016) was a German-Chilean-Mexican geophysicist. He completed his doctoral thesis under the supervision of Hugo Benioff and Beno Gutenberg. At Berkeley he attended the courses of Carl Anderson, Nobel Prize winner and discoverer of positrons, and of Richard Feynman. In 1955, Lomnitz was the first Latin American to obtain a PhD in Geophysics. For more information, see https://en.wikipedia.org/wiki/Cinna_Lomnitz. He was a professor at the University of California-Berkeley and in 1968 moved to the Instituto de Geofísica at the Universidad Nacional Autónoma de México, where he worked until the end of his life. Lomnitz (1957) introduced a creep law to describe flow in igneous rocks. Moreover, this law was also used by Lomnitz to explain the damping of the Earth's free core nutation (Chandler wobble) and the behavior of seismic S-waves (Lomnitz 1957, 1962).

As mentioned above, Lomnitz was not aware of Becker's model, but reported on the logarithmic law of Griggs (1939), from whom he probably took the form of his model (Lomnitz 1956). The logarithmic law proposed by Griggs for creep of limestones under pressure has also been shown to be valid for long-term creep at high temperatures and pressures.

Moreover, Lomnitz was unaware of Scott Blair's model. The myth of "constant Q" arose after the works of Kolsky (1956), dealing with experiments on attenuation in polythene filaments, and McDonal et al. (1958), in which these authors made attenuation measurements in homogeneous media and found that the attenuation factor behaved linearly with frequency. The latter authors performed measurements of the P-wave Q in a relatively narrow frequency band [100, 500] Hz (see their Fig. 9), and Q can be nearly constant due to various damping mechanisms. If the intention of Lomnitz (1957) was to simulate a constant Q, his model is an approximation, and he overlooked the Scott Blair model (Scott Blair and Caffyn 1942), which gives a perfectly constant Q. In fact, we show below that the Lomnitz model is a special case (approximation) of the Scott Blair model.

On the other hand, the Futterman (1962) model is similar to Lomnitz's in the sense that experimental discrimination is unlikely, as noted by Savage and O'Neill (1975). However, the Lomnitz paper is not cited by Futterman.

2.5 Harold Jeffreys

In Earth rheology, transient creep is often described by what is known as the Jeffreys-Lomnitz power law (Jeffreys 1958), a creep law proposed by Sir Harold Jeffreys that includes an additional parameter (the exponent r, see Eq. (13) below) that generalizes Lomnitz's logarithmic law. Harold Jeffreys (1891–1989) was a British applied mathematician who was also interested in problems of geophysics and astronomy. Jeffreys became a fellow of St. John's College at Cambridge University in 1914 and retained that fellowship until his death 75 years later (https://en.wikipedia.org/wiki/Harold_Jeffreys). At Cambridge University, he taught mathematics, then geophysics, and was eventually appointed Plumian Professor of Astronomy. His interests as a mathematician spanned several areas, including probability theory, asymptotic expansions, and tensor calculus, as his books attest. In the field of geophysics, his Treatise on the Earth has been reprinted several times since the first edition to keep it up to date (Jeffreys 1976). One of us (FM) had the opportunity to meet him personally when he was awarded a PhD studentship at the Institute of Applied Mathematics and Theoretical Physics, Cambridge University (1973–74). In particular, he was invited by him to lunch together at St. John's College. At that time, Jeffreys was a retired professor but visited the department from time to time.

For $r = 0$ (in the limiting case) Jeffreys' law reduces to Lomnitz' logarithmic law, while for $r = 1$, it goes back to the simple linear law of the viscoelastic Maxwell body. According to Jeffreys, his equation fits better the data on creep and dissipation in the Earth for seismological purposes. Indeed, the Jeffreys law is well recognized in rock rheology as it interpolates creep data between a logarithmic and a linear law (Jeffreys 1976).

The laws of Lomnitz and Jeffreys have been used to refute the theory of plate tectonics. Jeffreys rejected continental drift in the 1920s and plate tectonics in the 1970s. He believed that the solid Earth was too rigid to allow mantle convection and crustal motion. His view had a mathematical basis (Jeffreys 1970, 1972). He estimated the exponential parameter of his creep function from seismic attenuation and the Chandler wobble and concluded that it was about 0.2. The law worked surprisingly well, even though the data set covers a frequency range of 10^7. Then, Jeffreys extrapolated the time scale to geologic time by skipping another factor of 10^7. Even more surprising was the fact that Jeffreys' creep law explained the existence of low harmonics in the gravity field and of mountain ranges. He applied his law to the Moon and was able to explain the figure of the Moon and its lack of rotation (Jeffreys 1972). Because the theory is incompatible with plate tectonics, Jeffreys rejected the ideas of mantle convection and continental drift.

2.6 Ellis Strick

Although in Jeffreys' original work and in most of the work on the Jeffreys law the exponent r was not explicitly restricted to positive values, the applications with $r < 0$ were only subsequently introduced in Earth rheology. In this context, it is important to remember Ellis Strick (1922–2005), a professor emeritus of geophysics in the Department of Geology and Planetary Sciences at the University of Pittsburgh (US). Strick taught at the university from 1968 to 1990 and published relevant papers on seismology, geophysics, physics, and nuclear physics. For more details, see his obituary at https://www.utimes.pitt.edu/archives/?p=46724. We note that the late Professor Strick, in his 1984, paper (Strick 1984) had already extended the Jeffreys creep law into the range $-1 \leq r \leq +1$ and introduced hypergeometric functions to describe the frequency dependence of the complex compliance. Moreover, he was interested in the representation of the extended Jeffreys creep law in terms of a suitable ladder network of springs and dashpots. In his work, Strick was motivated by some experimental observations pointing to negative values of the exponent r.

Strick (1967, Section 4) introduced a three-parameter model (later revisited by Müller (1983)) of which the Scott Blair model of constant Q is a special case. However, this model is not mentioned, and Strick rejected it on the grounds that infinite velocity at the high-frequency limit violates causality. During the activities of the late Prof. Strick, we are not aware of any reaction of the geophysical community to his results. However, we have subsequently noted some published works in which the Jeffreys creep law with $r < 0$ was applied to fit experimental geophysical data (Crough and Burford 1977; Spencer 1981; Wesson 1988; Darby and Smith 1990). More recently, Mainardi and Spada (2012b) have studied the extended Jeffrey law in detail. One of us (FM) fondly recalls his personal contacts with Prof. Strick (1980–1984) when he invited him to the Euromech 127 in “Wave Propagation in Linear Viscoelastic Media" (Taormina, Sicily, Italy, April 1980), and then in Bologna, a visiting professor on subsequent visits until 1984.

3 Wave Theory

The basic principles of linear viscoelasticity are illustrated in well known treatises (e.g., Gross 1953; Christensen, 1982). Here, we mostly follow the notation of the recent books of Carcione (2022), Mainardi (2022) and Gurevich et al. (2022).

The complex and frequency-dependent creep compliance is defined as

$$J(\omega) = \mathcal{F}[\dot{\chi}(t)], \tag{1}$$

where χ is the creep function, ω is the angular frequency (we consider positive values in this study), t is the time variable, $\mathcal{F}$ is the Fourier-transform operator, and the overdot indicates time derivative. The complex modulus or stiffness is

$$M(\omega) = [J(\omega)]^{-1} \tag{2}$$

(e.g., Carcione 2022, Eqs. 2.42 and 2.43).

We consider the viscoelastic plane-wave kernel $\exp[\mathrm{i}(\omega t - kx)]$, where

$$k = \frac{\omega}{v_p} - \mathrm{i}\alpha = \frac{\omega}{v_c} \tag{3}$$

is the complex wavenumber, with v_p being the phase velocity, α the attenuation factor,

$$v_c(\omega) = \frac{M(\omega)}{\rho} \tag{4}$$

is the complex velocity, ρ is the mass density, x the spatial variable, and $\mathrm{i} = \sqrt{-1}$ (e.g., Carcione 2022, Section 3.2).

In n-D space ($n = 1, 2, 3$) and only dilatational waves, the wave equation is a combination of the constitutive equation

$$\sigma = M\epsilon, \tag{5}$$

where $\sigma = \sigma_1 = \sigma_2 = \sigma_3$ is the component of stress, and ϵ is the trace of the strain tensor, and the equation of momentum conservation

$$\partial_i \sigma = \rho \ddot{u}_i, \quad i =, 1 \ldots, 3, \tag{6}$$

where u_i are the displacement components, such that $\epsilon = \partial_i u_i$ (Einstein summation), ∂_i is a spatial derivative with respect to the variable x_i, and an overdot denotes a temporal derivative (Carcione 2022, Section 3.2). We have assumed no external forces. Substituting (5) into (6), we obtain the wave equation,

$$v_c^2 \Delta \sigma = \ddot{\sigma}, \tag{7}$$

where we have assumed that the density is spatially constant.

In wave theory, the phase velocity v_p and quality factor Q are key properties to describe the physics of wave propagation:

$$v_p = \left[\mathrm{Re}\left(\frac{1}{v_c}\right)\right]^{-1} \quad \text{and} \quad Q = \frac{\mathrm{Re}(M)}{\mathrm{Im}(M)} = \frac{\mathrm{Re}(v_c^2)}{\mathrm{Im}(v_c^2)} = -\frac{\mathrm{Re}(k^2)}{\mathrm{Im}(k^2)} \tag{8}$$

(Carcione 2022, Eqs. 2.123 and 2.124), where Q is obtained from an energy balance, defined as twice the strain energy divided by the dissipated energy (Carcione 2022; Eq. 2.122). Defining Q as the total energy divided by the dissipated energy (see Carcione 2022, Eq. 2.127), we have the dissipation factor

$$Q = \frac{\omega}{2\alpha v_p}, \tag{9}$$

where α is the attenuation factor (see next equation). This Q is approximately equal to that in Eq. (8) for $Q \gg 1$. The relation between the two quality factors is given in Eq. (2.128) of Carcione (2022), although here we use the same symbol. The dependence of the velocity with frequency is termed velocity dispersion and implies spreading of the wave pulse.

Moreover, the attenuation factor quantifies the spatial decay as $\exp(-\alpha x)$ (e.g., Carcione 2022, Eqs. 2.85 and 2.88). We have

$$\alpha = -\omega \mathrm{Im}\left(\frac{1}{v_c}\right). \tag{10}$$

As stated above, for $Q \gg 1$ the relation $\alpha = \pi f/(v_p Q)$ holds (Carcione 2022, Eq. 2.126).

The real and imaginary parts of M and the phase velocity and attenuation factor are related by the Kramers–Kronig relations (e.g., Carcione 2022, Sections 2.2.4 and 2.3.2, respectively; Carcione et al. 2019), and for low-loss media, the complex modulus can be obtained as

$$M(\omega) = \rho v_p^2(\omega)\left(1 + \frac{\mathrm{i}}{Q(\omega)}\right). \tag{11}$$

(Carcione 2022 Eq. 2.142).

The working frequency in seismology, particularly in seismic applications, is f, whose unit is Hz, such that $\omega = 2\pi f$. Then, we express the equations in terms of f, more precisely in terms of a dimensionless variable

$$\bar{f} = \frac{f}{f_0} = \frac{\omega}{2\pi f_0}, \tag{12}$$

where f_0 is a reference frequency.

4 Lomnitz and Jeffreys Creep Functions

The Jeffreys creep function $\chi(t)$ as a function of time is

$$\frac{\chi(t)}{\chi_0} = 1 + \frac{2}{\pi Q_0}\begin{cases} \dfrac{(1+\omega_0 t)^r - 1}{r}, & -\infty < r < 1, \quad \text{Jeffreys}, \\ \ln(1+\omega_0 t), & r = 0, \quad \text{Lomnitz}, \\ \omega_0 t, & r = 1, \quad \text{Maxwell}, \\ 0, & r = -\infty, \quad \text{Hooke} \end{cases} \tag{13}$$

(e.g., Strick 1984; Hanyga 2014) where

$$\chi_0 = \frac{1}{\rho v_0^2}, \tag{14}$$

v_0 and Q_0 are the phase velocity and quality factor at approximately ω_0, when using the approximation of the Lomnitz model by Savage and O'Neill (1975) (see Eq. (38) below and Figure 2). The logarithmic law ($r = 0$) is due to Lomnitz (1957). Jeffreys (1958) considered positive values of r, while (Strick and Mainardi 1982) also negative values.

5 Velocity Dispersion and Attenuation of All Models

5.1 Becker

Becker (1925) introduced the creep function

$$\chi(t) = \chi_0\left\{1 + \frac{2}{\pi Q_0}\left[\ln(\gamma\omega_0 t) + \mathrm{E}_1(\omega_0 t)\right]\right\} = \chi_0\left[1 + \frac{2}{\pi Q_0}\mathrm{E}_{\mathrm{in}}(\omega_0 t)\right], \tag{15}$$

where E_1 is the generalized exponential integral of order one, and E_{in} is the modified exponential integral (Strick and Mainardi 1982; Mainardi and Masina 2018; Hanyga 2014). Gross (1953) has shown that

$$M(\bar{f}) = \rho v_0^2\left[1 + \frac{2}{\pi Q_0}\ln\left(1 + \frac{1}{\mathrm{i}\bar{f}}\right)\right]^{-1}, \tag{16}$$

such that

$$Q(\bar{f}) = \frac{1}{2}\,\frac{\pi Q_0 + \ln(1 + \bar{f}^{-2})}{\arctan\,(\bar{f}^{-1})} \tag{17}$$

(Strick and Mainardi 1982; Eqs. 6b and 8).

5.2 Scott Blair

As will be seen in the plots below, all the models have not exactly a constant Q. The creep function corresponding to a perfectly constant-Q model was introduced by Scott Blair and Caffyn (1942) based on Nutting (1921) (see Rogosin and Mainardi 2014). This creep function is

$$\chi(t) = \frac{\chi_0}{\Gamma(1+2\nu)}\left(\omega_0 t\right)^{2\nu}, \quad \nu = \frac{1}{\pi}\arctan\left(\frac{1}{Q_0}\right), \tag{18}$$

and has the complex modulus

$$M(\bar{f}) = \rho v_0^2\left(\mathrm{i}\bar{f}\right)^{2\nu} \tag{19}$$

(Carcione et al. 2002; Carcione 2022; Mainardi 2022), with $Q = Q_0$ at all frequencies.

A simple extension of the Scott Blair creep function (18) has been proposed by Jaishankar and McKinley (2012),

$$\chi(t) = \frac{1}{2}\chi_0\left[\frac{1}{\Gamma(1+2\nu_1)}(\omega_0 t)^{2\nu_1} + \frac{1}{\Gamma(1+2\nu_2)}(\omega_0 t)^{2\nu_2}\right]. \tag{20}$$

The complex modulus is

$$M(\bar{f}) = 2\rho v_0^2 \cdot \frac{(\mathrm{i}\bar{f})^{2\nu_1}\cdot(\mathrm{i}\bar{f})^{2\nu_2}}{(\mathrm{i}\bar{f})^{2\nu_1} + (\mathrm{i}\bar{f})^{2\nu_2}}. \tag{21}$$

If $\nu_1 = \nu_2 = \nu$, we obtain the Scott Blair model. Note a typo in Jaishankar and McKinley (2012), where V_s and G_s should be interchanged in the denominator of their Eq. 4.3. These authors have used this four-parameter model to fit the viscoelastic properties of Acacia gum (see their Fig. 4a).

5.3 Kolsky

Kolsky introduced a nearly constant Q model that preceded the other authors, with the exception of Becker. According to Ursin and Toverud (2002) and using our notation, its wave properties are

$$v_p(\bar{f}) = v_0\left[1 + \frac{1}{\pi Q_0}\ln\left(\frac{1}{\bar{f}}\right)\right]^{-1} = v_0\left[1 - \frac{1}{\pi Q_0}\ln\left(\bar{f}\right)\right]^{-1}, \tag{22}$$

$$\alpha(f) = \frac{\pi f}{v_0 Q_0} \tag{23}$$

and

$$Q(\bar{f}) = \frac{\omega}{2\alpha v_p} = Q_0 + \frac{1}{\pi}\ln\left(\frac{1}{\bar{f}}\right). \tag{24}$$

In Kolsky (1956; Eq. 12; 1964, Eqs. 10 and 11), Futterman (1962) and Kjartansson (1979, Eq. 58), the original equations are slightly different:

$$v_p(\bar{f}) = v_0\left[1 + \frac{1}{\pi Q_0}\ln(\bar{f})\right], \tag{25}$$

$$\alpha(\bar{f}) = \frac{\pi f}{v_0 Q_0}\left[1 - \frac{1}{\pi Q_0}\ln(\bar{f})\right] \tag{26}$$

and

$$Q(\bar{f}) = \frac{Q_0}{1 - \left[\frac{1}{\pi Q_0}\ln(\bar{f})\right]^2}, \tag{27}$$

or equivalently computed from Eq. (9). [There is a probable typo in Kolsky (1964) in the equation above his Eq. 11: $\tan(\delta/2)$ must be replaced by $(\tan\delta)/2$]. These two versions of the same model make a difference in the plots, despite the approximation based on $Q_0 \gg 1$ and that involving the reference frequency, i.e., $(\pi Q_0)^{-1}\ln(\bar{f} = f/f_0) \ll 1$ (see Kjartansson 1979, Eq, 44).

Hao and Greenhalgh (2021, Eq. 21) report the complex modulus

$$M(\bar{f}) = \rho v_0^2\left[1 + \frac{1}{Q_0}\left(\frac{2}{\pi}\ln(\bar{f}) - \mathrm{i}\,\mathrm{sgn}(\omega)\right)\right], \quad -\infty \le \omega \le \infty, \tag{28}$$

using the opposite convention for the Fourier transform [$\exp(-\mathrm{i}\omega t)$]. This modulus [as well as the Scott Blair (Kjartansson) modulus, their Eq. 19] gives a negative Q value for $\omega > 0$ when using their Eq. 3. The correct definition of the quality factor in their case is $Q = -\mathrm{sgn}(\omega)M_R/M_I$ if positive and negative frequencies are considered. Nevertheless, their model differs from the 1964 Kolsky model.

Then, the model (or approximations) considered by Ursin and Toverud (2002), and Hao and Greenhalgh (2021) do not correspond exactly to the Kolsky model. Even though these models could be an approximation of exact formulas, they differ and lead to different results. It can be shown that the phase velocity and Q factor of the two versions of this model differ from those of the other authors.

Some of the models do not satisfy causality, i.e., Kramers–Kronig relationships (e.g., Carcione et al. 2019; Carcione 2022). This property is not discussed in the present study. For the Kolsky and Futterman models, see (Ursin and Toverud 2003).

5.4 Lomnitz

Lomnitz (1957) has introduced an attenuation model whose phase velocity and attenuation factors are

$$v_p(\bar{f}) = v_0 D(\bar{f}), \tag{29}$$

and

$$\alpha(\bar{f}) = \frac{2fD(f)}{v_0 Q_0}\left[\left(\frac{\pi}{2} - \mathrm{Si}(\bar{f})\right)\cos\bar{f} + \mathrm{Ci}(\bar{f})\sin\bar{f}\right], \tag{30}$$

respectively, where

$$D(\bar{f}) = \left\{1 + \frac{2}{\pi Q_0}\left[\left(\frac{\pi}{2} - \mathrm{Si}(\bar{f})\right)\sin\bar{f} - \mathrm{Ci}(\bar{f})\cos\bar{f}\right]\right\}^{-1/2}, \quad \bar{f} = \frac{f}{f_0} = \frac{\omega}{\omega_0}, \tag{31}$$

where Ci and Si are the sine and cosine integral functions (see Appendix A). The above equation holds for $Q_0 \gg 1$.

Defining quality factor Q as the total energy divided by the dissipated energy (see Carcione 2022, Eq. 2.127), we have the dissipation factor

$$\frac{1}{Q(\bar{f})} = \frac{2\alpha v_p}{\omega} = \frac{2}{\pi Q_0}\left[\left(\frac{\pi}{2} - \mathrm{Si}(\bar{f})\right)\cos\bar{f} + \mathrm{Ci}(\bar{f})\sin\bar{f}\right]D^2 \tag{32}$$

Note a typo in Eq. (33) in Lomnitz (1957), where the square root in the denominator has to be removed. Eq. (32) is also reported in Strick and Mainardi (1982, Eq. 5).

5.5 Savage and O'Neill

The Lomnitz Eqs. (29), (30) and (32) can be simplified if we take $\bar{f} \ll 1$, i.e., f_0 very large. Expanding the integral function in power series, Savage and O'Neill (1975, Table 1) obtained

$$v_p(\bar{f}) = v_0\left[1 - \frac{2}{\pi Q_0}\ln(\gamma\bar{f})\right]^{-1/2}, \tag{33}$$

$$\alpha(\bar{f}) = \frac{\omega}{2v_0 Q_0}\left[1 - \frac{2}{\pi Q_0}\ln(\gamma\bar{f})\right]^{-1/2}, \tag{34}$$

$$Q(\bar{f}) = \frac{\omega}{2\alpha v_p} = Q_0\left[1 - \frac{2}{\pi Q_0}\ln(\gamma\bar{f})\right], \tag{35}$$

where $\ln\gamma = 0.577216$ is Euler's constant [$\gamma = \exp(0.577216) \approx 1.78$]. It can easily be shown that these properties correspond to the complex wavenumber

$$k(\bar{f}) = \frac{\omega}{v_0}\sqrt{1 - \frac{2}{\pi Q_0}\ln(\mathrm{i}\gamma\bar{f})} \tag{36}$$

(for $\bar{f} \ll 1$ and $Q_0 \gg 1$) (Shibuya 1977, Eq. 14; Shibuya's complex wavenumber $\bar{k}$ is $-\mathrm{i}k$), with the complex velocity

$$v_c(\bar{f}) = \frac{\omega}{k} = v_0\left[1 - \frac{2}{\pi Q_0}\ln(\mathrm{i}\gamma\bar{f})\right]^{-1/2}, \tag{37}$$

such that the complex modulus is

$$M(\bar{f}) = \rho v_c^2 = \rho v_0^2\left[1 - \frac{2}{\pi Q_0}\ln(\mathrm{i}\gamma\bar{f})\right]^{-1}. \tag{38}$$

5.6 Jeffreys

Jeffreys (1958) extended the Lomnitz model by adding the parameter $r \le 1$. Hanyga (2014, Eq. 56) reports the complex compliance (divided by p) in terms of the Laplace variable $p = \mathrm{i}\omega$. The reciprocal of the compliance is the complex modulus:

$$M(\bar{f}) = \rho v_0^2\left\{1 + \frac{2}{\pi Q_0 r}\left[\mathrm{i}\bar{f}\exp(\mathrm{i}\bar{f})\mathrm{E}_{-r}(\mathrm{i}\bar{f}) - 1\right]\right\}^{-1}, \tag{39}$$

where E_{-r} is the generalized exponential integral of order $-r$. Alternatively,

$$M(\bar{f}) = \rho v_0^2\left\{1 + \frac{2}{\pi Q_0 r}\left[\frac{1}{(\mathrm{i}\bar{f})^r}\exp(\mathrm{i}\bar{f})\Gamma(1 + r, \mathrm{i}\bar{f}) - 1\right]\right\}^{-1}, \tag{40}$$

where Γ is the incomplete Gamma function, since

$$\mathrm{E}_{-r}(\mathrm{i}z) = \frac{1}{(\mathrm{i}z)^{1+r}}\Gamma(1 + r, \mathrm{i}z), \quad z \text{ real}, \tag{41}$$

(Olver 1994; Temme 1994).

In terms of the Kummer (or Tricomi) confluent hypergeometric function of the second kind U, we have

$$M(\bar{f}) = \rho v_0^2\left\{1 + \frac{2}{\pi Q_0 r}\left[\mathrm{i}\bar{f}\mathrm{U}(1, 2 + r, \mathrm{i}\bar{f}) - 1\right]\right\}^{-1}, \tag{42}$$

since

$$\mathrm{E}_{-r}(\mathrm{i}x) = \exp(-\mathrm{i}z)\mathrm{U}(1, 2 + r, \mathrm{i}z) \tag{43}$$

(Navas-Palencia 2018). Details for the numerical calculation of U in terms of the hypergeometric function are given in Appendix A. Moreover, this function can be used to compute also E_1, E_{-r}, Ci and Si if codes related to these functions are required to be tested.

5.7 Lomnitz from Jeffreys Equation

If $r = 0$, the Lomnitz modulus is obtained from Eq. (42),

$$M(\bar{f}) = \rho v_0^2 \left[1 + \frac{2}{\pi Q_0} \exp(\mathrm{i}\bar{f})\mathrm{E}_1(\mathrm{i}\bar{f})\right]^{-1} \tag{44}$$

(Hanyga 2014, Eq. 57), that can be shown to give Eqs. (29) and (30) when computing the phase velocity and attenuation factor, respectively, using the property:

$$\mathrm{E}_1(\mathrm{i}x) = \mathrm{i}\left[\mathrm{Si}(x) - \frac{\pi}{2}\right] - \mathrm{Ci}(x) \tag{45}$$

(Amos 1990).

If $r = 1$, we obtain the Maxwell model and it can be shown that the corresponding complex modulus should be

$$M(\bar{f}) = \frac{\rho v_0^2 \bar{f}}{\bar{f} - \dfrac{2\mathrm{i}}{\pi Q_0}}, \tag{46}$$

equivalent to Eq. 2.167 in Carcione (2022), with $Q = \bar{f}(\pi Q_0/2)$.

5.8 Strick

In Strick (1984), the notation is such that $\Delta = 2/(\pi Q_0)$, $J_U = (\rho v_0^2)^{-1}$ and $s = 1 - r$. Strick (1984, Eqs. 13 and 15b) reports the following complex modulus for the Jeffreys model:

$$M(\bar{f}) = \rho v_0^2 \left[1 + \frac{2}{\pi Q_0}\mathrm{U}(1, 1 + r, \mathrm{i}\bar{f})\right]^{-1}, \tag{47}$$

which can be shown to be equivalent to Eq. (42), using the property of the Kummer function:

$$r\mathrm{U}(1, 1 + r, z) = z\mathrm{U}(1, 2 + r, z) - 1. \tag{48}$$

This expression can easily be obtained by taking the integral form of the Kummer function (e.g., Tricomi 1954, Eq. 21′), and integrating by parts, considering that the two functions in the argument of the integral over y are $u(y) = \exp(-yz)$ and $v(y) = (1 + y)^r/r$, with $v' = (1 + y)^{r-1}$.

5.9 Aki-Richards

Aki and Richards (2009, Eq. 5.88) report the following complex velocity regarding the Lomnitz model:

$$v_c = v_0 \left\{ 1 - \frac{2}{\pi Q_0} \left[\frac{1}{2} \ln(\gamma \bar{f}) + \frac{i\pi}{4} \right] \right\}^{-1}. \tag{49}$$

The sign of the imaginary part of the denominator has been reversed compared to the equation in Aki and Richards (2009), since the sign convention of the Fourier transform is the

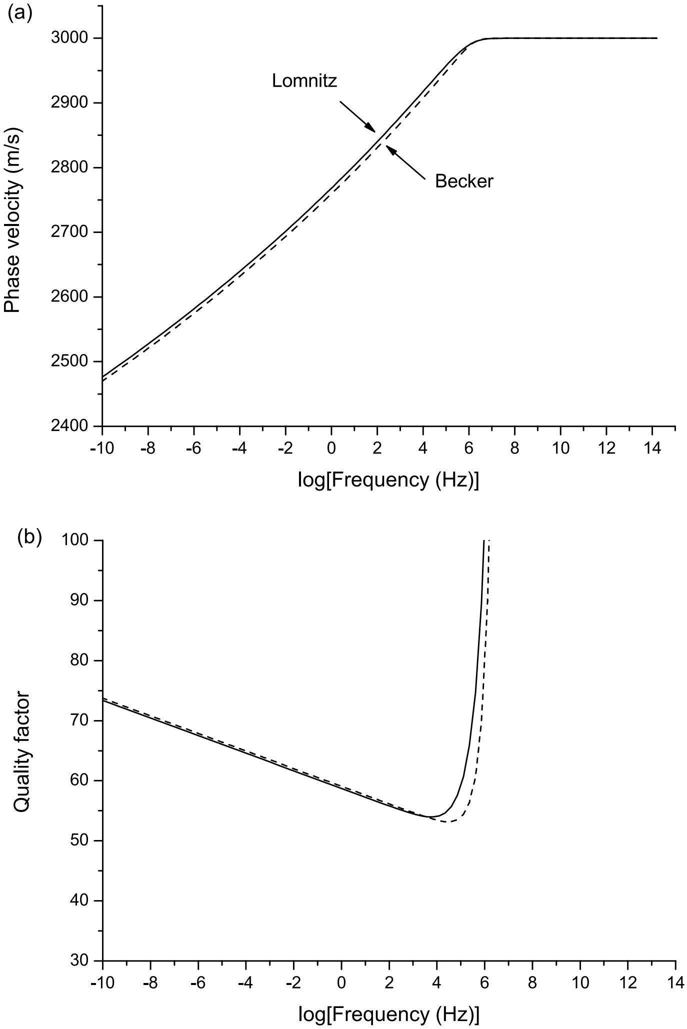

Fig. 1 Comparison between the phase velocity **a** and quality factor **b** of the Lomnitz (1957) model (solid line, Eqs. (29), (30) and (44)) and those of Becker (dashed line, see Eqs. (16) and (17))

opposite. Equations (37) and (49) differ, even for very high values of Q_0. We conclude that Eq. (49) does not correspond to Lomnitz model.

6 Results

We consider the following values: $\rho = 2.5$ g/cm^3, $v_0 = 3000$ m/s, $Q_0 = 50$ and $\omega_0 = 10^7$ rad/s ($f_0 \approx 1.6$ MHz). The following plots show the phase velocity and quality factor, unless otherwise indicated, and are given as a function of the logarithm (base 10) of the frequency $f = \omega/(2\pi)$.

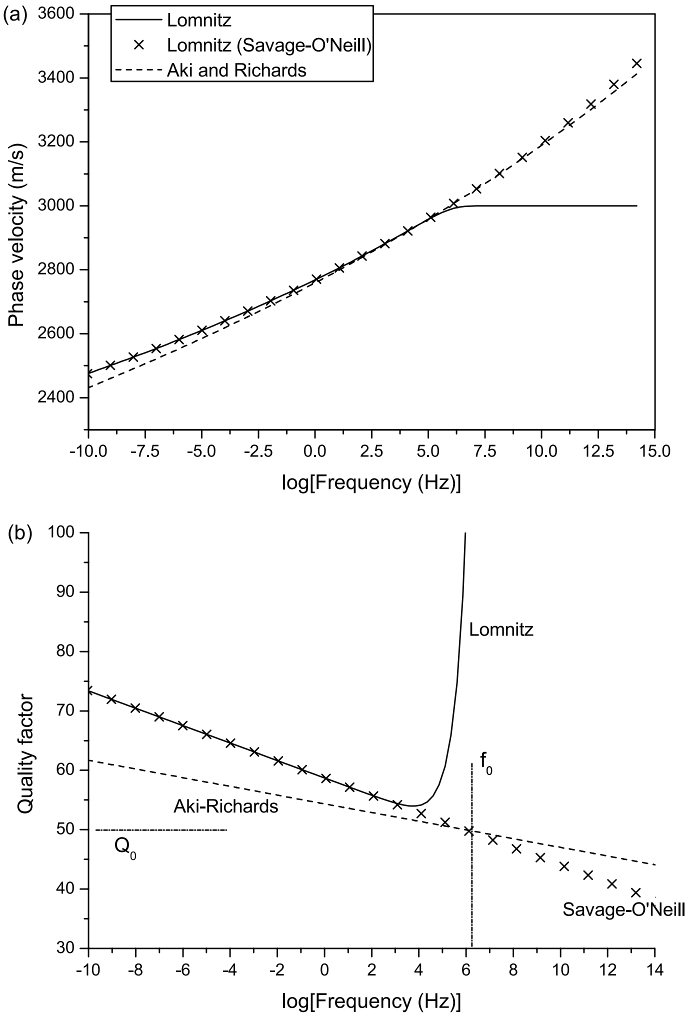

Fig. 2 Comparison between the phase velocity **a** and quality factor **b** for the Lomnitz (1957) model (solid line, Eqs. (29), (30) and (44)), approximations of the Lomnitz model by Savage and O'Neill (1975) (crosses, Eqs. (33), (35) and (38)) and those of Aki and Richards (2009) (dashed line, Eq. (49))

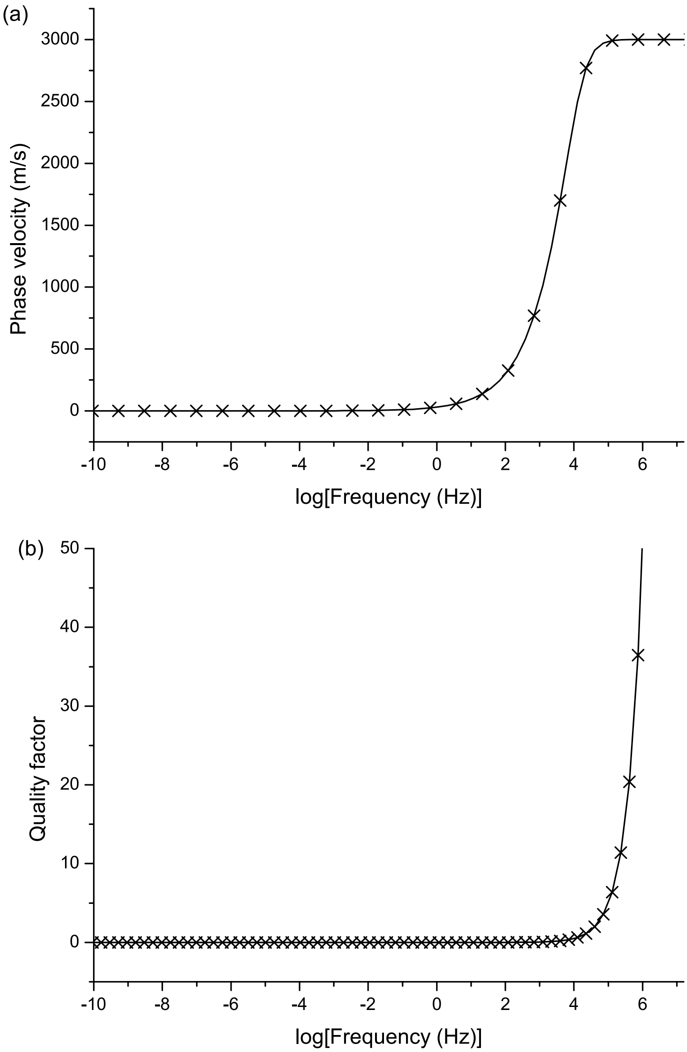

Fig. 3 Phase velocity **a** and quality factor **b** for the Maxwell model ($r = 1$). Comparison between the numerical calculation with Eq. (42) (crosses) and the analytical expressions using Eq. (46) (solid line)

Figure 1 compares the Lomnitz (1957) model with that of Becker (dashed line). As can be seen, the two results differ but from a practical point of view they are indistinguishble. The Q factors tend to infinity at high frequencies.

Figure 2 compares the Lomnitz (1957) model (solid line), the approximation of the Lomnitz model by Savage and O'Neill (1975) (crosses) and that of Aki and Richards (2009) (dashed line). As can be seen, Lomnitz Q tends to ∞ at high frequencies and the phase velocity to v_0. The Savage-O'Neill approximation is excellent for frequencies much less than f_0, while the Aki-Richards model (reported as Lomnitz in the book) is not that of Lomnitz. Q_0 is obtained at a frequency slightly less than f_0, but the exact Lomnitz model has values greater than Q_0 at all frequencies.

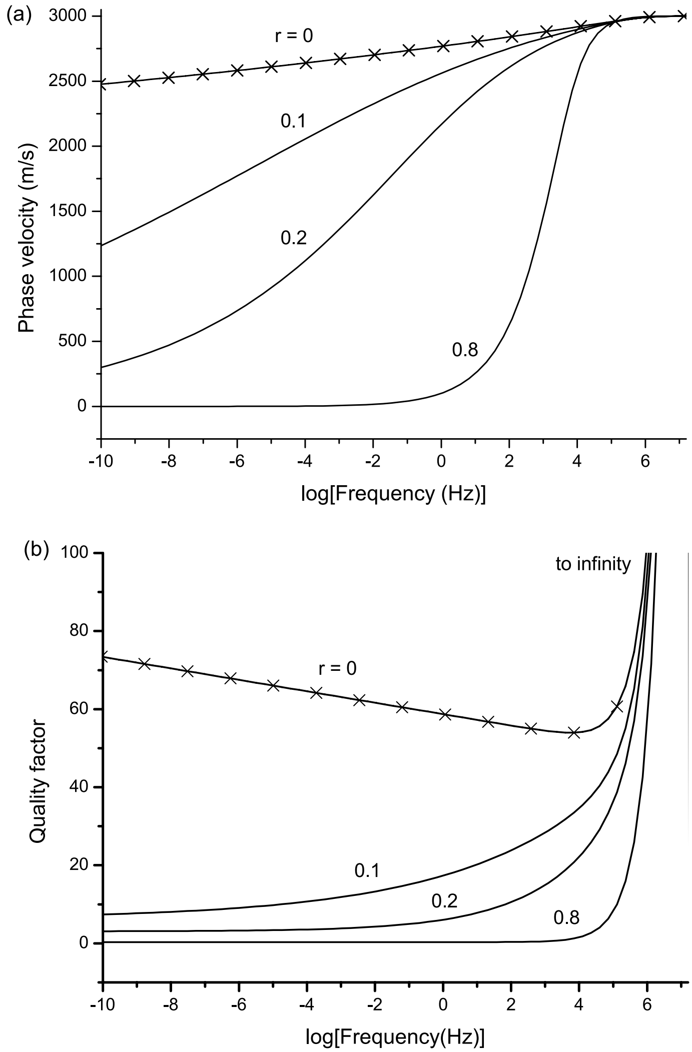

Fig. 4 Phase velocity **a** and quality factor **b** for the Jeffreys model for different values of r based on Eq. (42) (solid lines). When $r = 0$, we obtain the Lomnitz model. The crosses correspond to a numerical calculation with Eq. (45)

Figure 3 shows comparisons for the Maxwell model obtained with Eqs. (42) (symbols) and (46) (solid line). It provides a verification of the calculation with the hypergeometric function. Figure 4 shows results for the Jeffreys model and different values of r based on Eq. (42) (solid lines). The crosses correspond to a numerical calculation with Eq. (45). The agreement provides another test of the mathematics and computations. For $r = 1$, the Maxwell results are obtained.

Figure 5 shows results for the Jeffreys model and negative values of r. When $r = 0$, we obtain the Lomnitz model and $r \rightarrow -\infty$ gives the Hooke law, i.e., the lossless case, with the

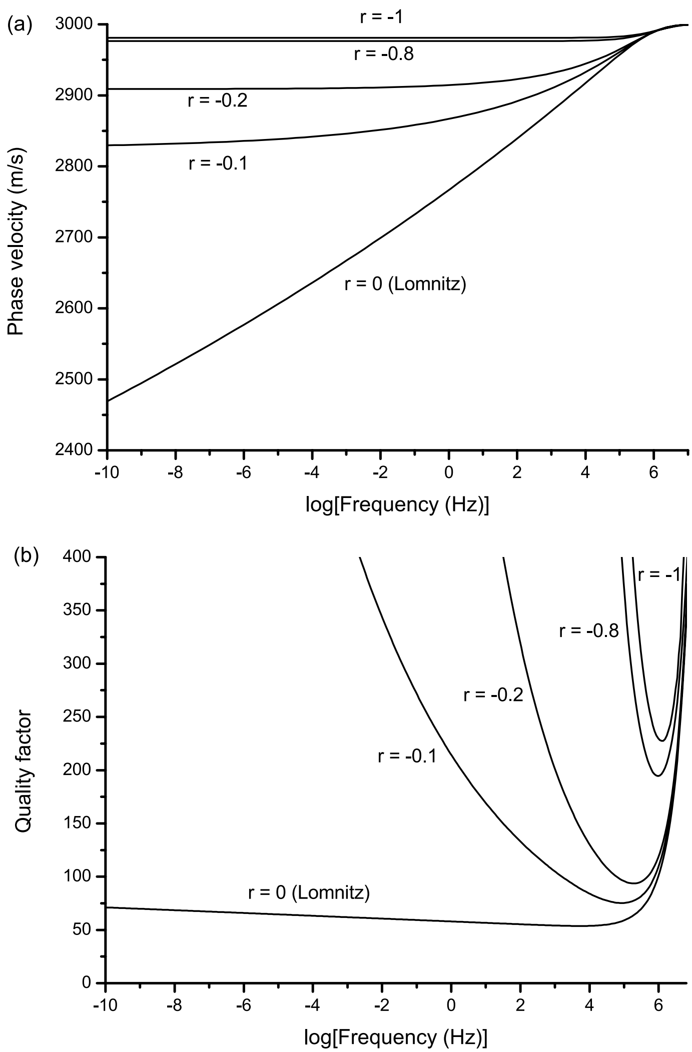

Fig. 5 Phase velocity **a** and quality factor **b** for the Jeffreys model for different values of negative r based on Eq. (42). When $r = 0$, we obtain the Lomnitz model and $r \rightarrow -\infty$ gives Hooke law, i.e., the lossless case

velocity reaching the value v_0 and $Q \rightarrow \infty$. Strick (1984) finds that $r < 0$ for sedimentary rocks and $r > 0$ for igneous rocks.

Figure 6 compares the Lomnitz model (solid line), approximation of the Lomnitz model by Savage and O'Neill (1975) (dots) and that of Scott Blair (dashed line, based on Eq. (19)). As can be seen, the Scott Blair model provides a perfectly constant Q (Q_0) intersecting the Savage-O'Neill Q at approximately f_0 $[\omega_0/(2\pi)]$.

The attenuation factor gives a better physical insight of the damping as a function of distance. Figure 7 shows the attenuation factors corresponding to different models for high (a) and low (b) frequencies. As can be seen, at low frequencies (say, the geophysical

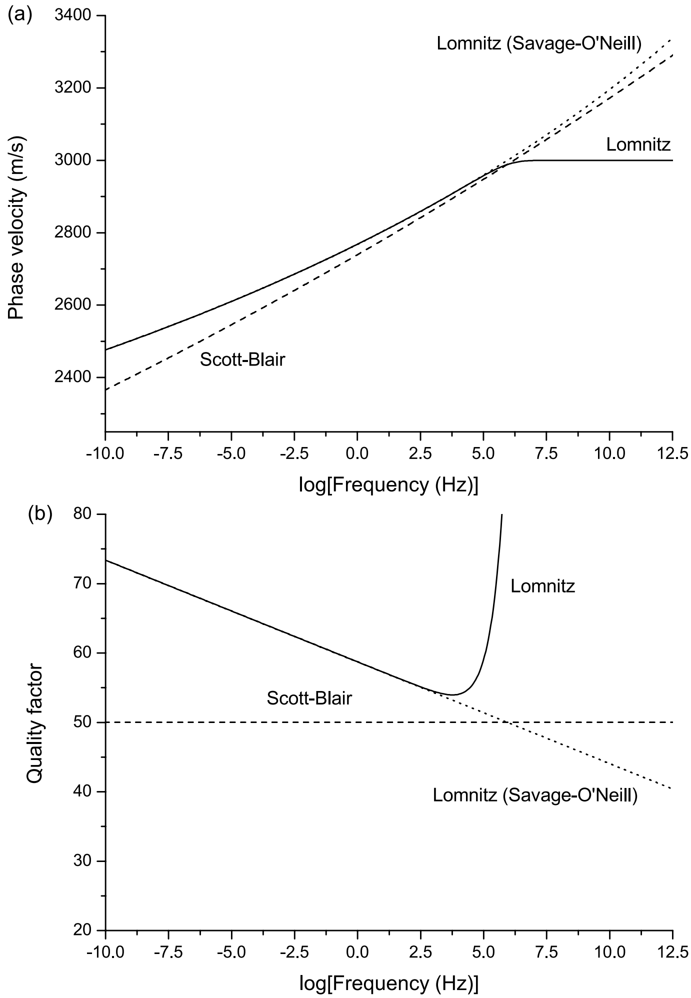

Fig. 6 Comparison between the phase velocity **a** and quality factor for the Lomnitz model (solid line), approximations of the Lomnitz model by Savage and O'Neill (1975) (dots) and those of Scott Blair (dashed line, based on Eq. (19))

prospecting band) the Lomnitz, Savage and O'Neill, and Becker attenuation factors are nearly the same. A log-log plot is shown in Fig. 8, where it is clear that there is a linear trend of the attenuation factor over a wide frequency range.

Next, we show the Lomnitz phase velocity and quality factor for two values of ω_0, with $Q_0 = 50$ (Fig. 9) and for two values of Q_0 (Fig. 10), with $\omega_0 = 10^7$ rad/s. The modeled Q increases compared to Q_0 with increasing ω_0, while the velocity decreases (Fig. 9). Since $\alpha \propto 1/(v_p Q)$, the attenuation factor remains almost unchanged. The intersection of the dashed lines corresponds approximately to $f_0 = \omega_0/(2\pi)$.

Finally, Fig. 11 compares Kolsky's models with those of Savage and O'Neill and Scott Blair. The Kolsky 1 model is the original one introduced by Kolsky (1964) [Eqs. (25)–(27)], while Kolsky 2 corresponds to versions used by other authors [Eqs. (22)–(24)] (e.g., Ursin and Toverud 2002). The properties differ, while the attenuation

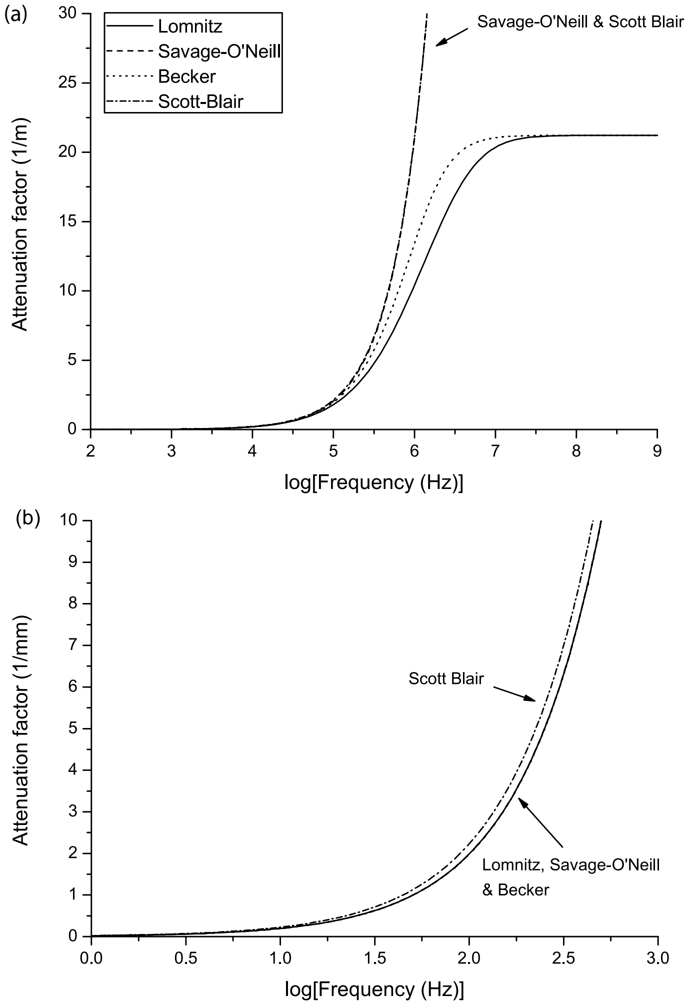

Fig. 7 Attenuation factors corresponding to different models: **a** high frequencies; **b** low frequencies

factors (not shown) are very similar. The Q factors agree approximately at the reference frequency f_0, but unlike that of Scott Blair, they are far from being constant.

7 Conclusions

We have given a historical overview of the development of the models of Lomnitz and Jeffreys and clarified the relationship between the different versions of these models in terms of the mathematical expressions of the complex and frequency-dependent modulus based on hypergeometric functions.

The main conclusions are as follows. The exact Lomnitz Q (1956) is infinite at the high-frequency limit (lossless medium), while its Savage and O'Neill approximation tends to zero (diffusion limit). The Becker model, developed much earlier (1925), gives

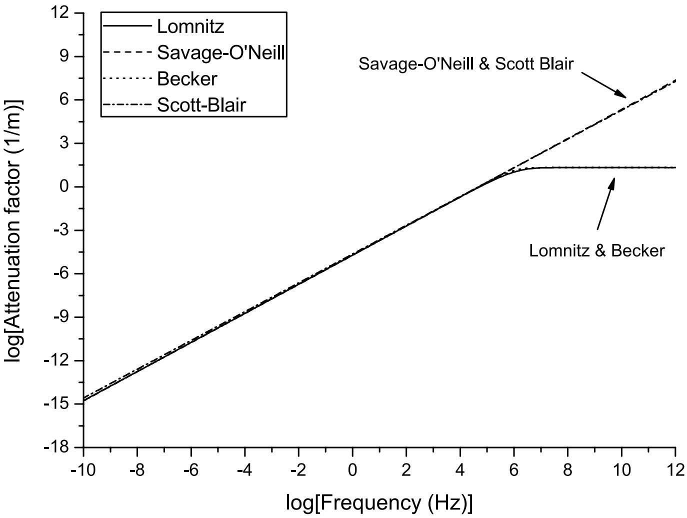

Fig. 8 Log-log plot of the attenuation factors

results (velocity and Q) that are indistinguishable from the Lomnitz model from a practical point of view. The Lomnitz model never assumes the value Q_0, unlike its approximation (Savage and O'Neill model, 1975) which assumes this value approximately at the reference frequency. Jeffrey's power law gives Lomnitz, Maxwell, and Hooke limits for the exponent equal to 0, 1, and ∞. The Lomnitz Q factor is not "almost constant" and exhibits measurable differences, while the attenuation factor of the Savage and O'Neill approximation is virtually the same as that of Scott Blair at high and low frequencies and shows a linear behavior. Because of these differences and similarities, different fits of real data could be obtained with Q and the attenuation factor. Moreover, increasing the reference frequency in the Lomnitz model implies a higher Q and lower velocity, compensating effects that yield a similar almost linear attenuation factor. Finally, the version of the Lomnitz model published in the book by Aki and Richards (2009) differs from the Lomnitz model and its approximation by Savage and O'Neill, and there are two versions of the Kolsky model in the literature. Even though these models could be approximation of exact formulas, they differ and lead to different results. Considering the similarities and differences, we conclude that it is more correct to speak of a "nearly linear attenuation factor" than of a "nearly constant quality factor."

Appendix: Computation of the Kummer Function

The Kummer function is a special case of the hypergeometric function, which is based on hypergeometric series, a term introduced by John Wallis in his 1655 book *Arithmetica Infinitorum*.

We have coded the hypergeometric function ${}_2\mathrm{F}_1(a, b, c, z)$ in Fortran 77 (Abramowitz 1970; Press et al. 1997, p. 263), such that the Kummer function is

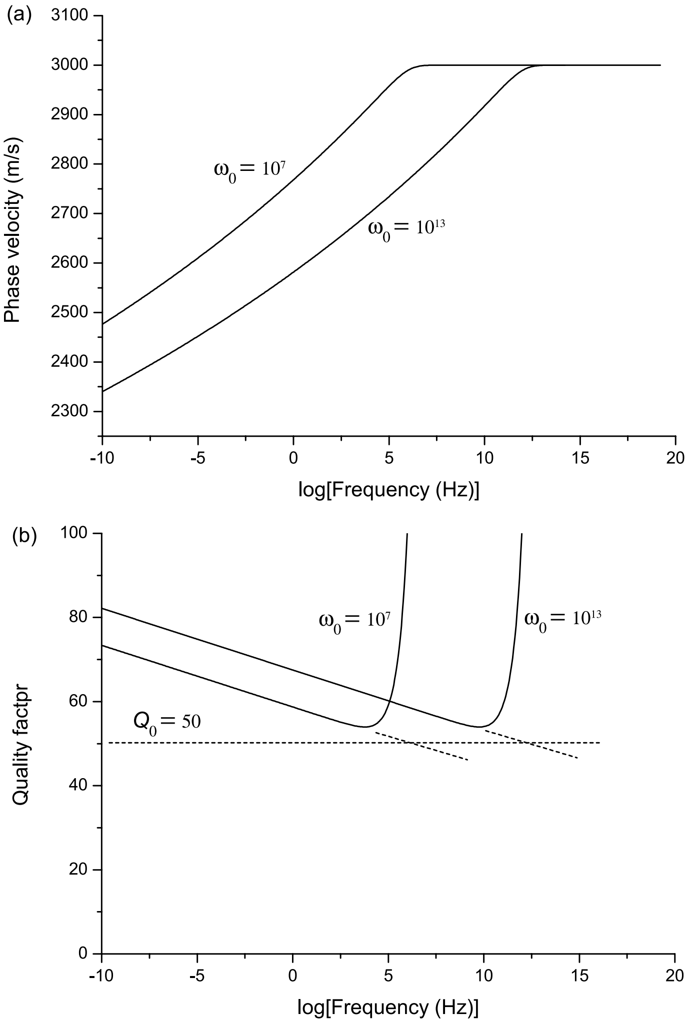

Fig. 9 Lomnitz phase velocity **a** and quality factor **b** for two values of ω_0 (in rad/s), with $Q_0 = 50$

$$\mathrm{U}(a, c, z) = \frac{\Gamma(1-c)}{\Gamma(a+1-c)} \mathcal{M}(a, c, z) + \frac{\Gamma(c-1)}{\Gamma(a)} z^{1-c} \mathcal{M}(a+1-c, 2-c, z), \quad z \text{ complex}, \tag{50}$$

where

$$\mathcal{M}(a, c, z) = \lim_{b \to \infty} {}_2\mathrm{F}_1\left(a, b, c, \frac{z}{b}\right) \tag{51}$$

is the confluent hypergeometric function. Then,

$$\mathrm{U}(1, 2+r, \mathrm{i}\bar{f}) = \frac{\Gamma(-1-r)}{\Gamma(-r)} \mathcal{M}(1, 2+r, \mathrm{i}\bar{f}) + \frac{\Gamma(1+r)}{\Gamma(1)} (\mathrm{i}\bar{f})^{-1-r} \mathcal{M}(-r, -r, \mathrm{i}\bar{f}) \tag{52}$$

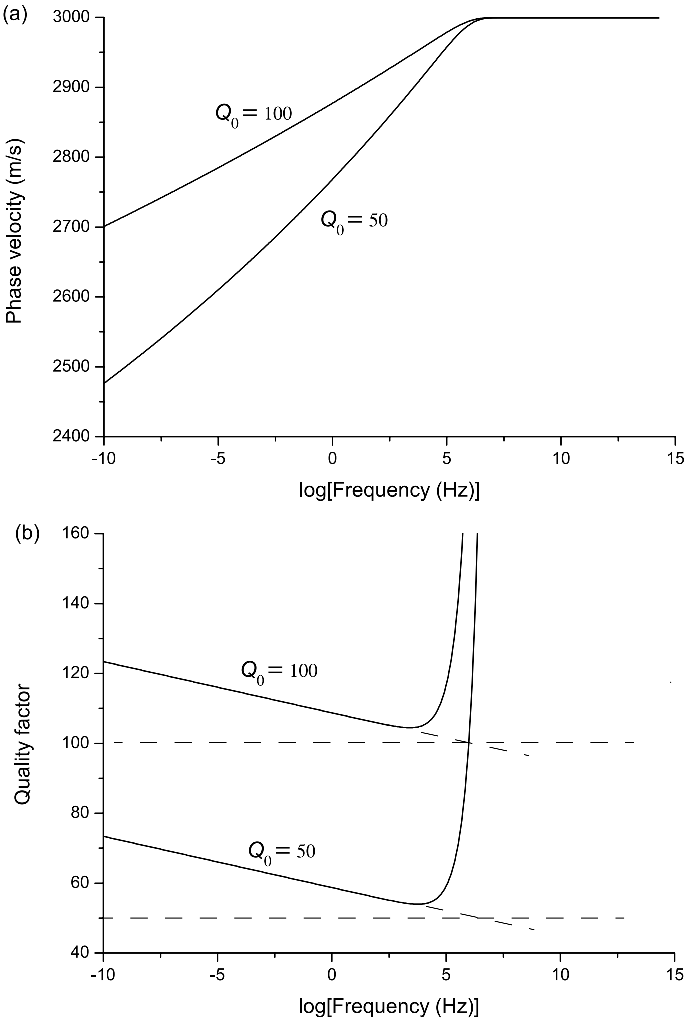

Fig. 10 Lomnitz phase velocity **a** and quality factor **b** for two values of Q_0, with $\omega_0 = 10^7$

or

$$\mathrm{U}(1, 2 + r, \mathrm{i}\bar{f}) = \frac{\Gamma(-1 - r)}{\Gamma(-r)} \, {}_2\mathrm{F}_1\left(1, b, 2 + r, \frac{\mathrm{i}\bar{f}}{b}\right) + \frac{\Gamma(1 + r)}{\Gamma(1)} (\mathrm{i}\bar{f})^{-1-r} \, {}_2\mathrm{F}_1\left(-r, b, -r, \frac{\mathrm{i}\bar{f}}{b}\right), \tag{53}$$

provided that b is very large.

The Kummer function can also be used to compute the sine and cosine integrals, namely Eq. (43), and

$$\mathrm{Si}(z) = \frac{\pi}{2} + \mathrm{Im}[\mathrm{E}_1(z)], \quad \text{and} \quad \mathrm{Ci}(z) = -\mathrm{Re}[\mathrm{E}_1(z)], \tag{54}$$

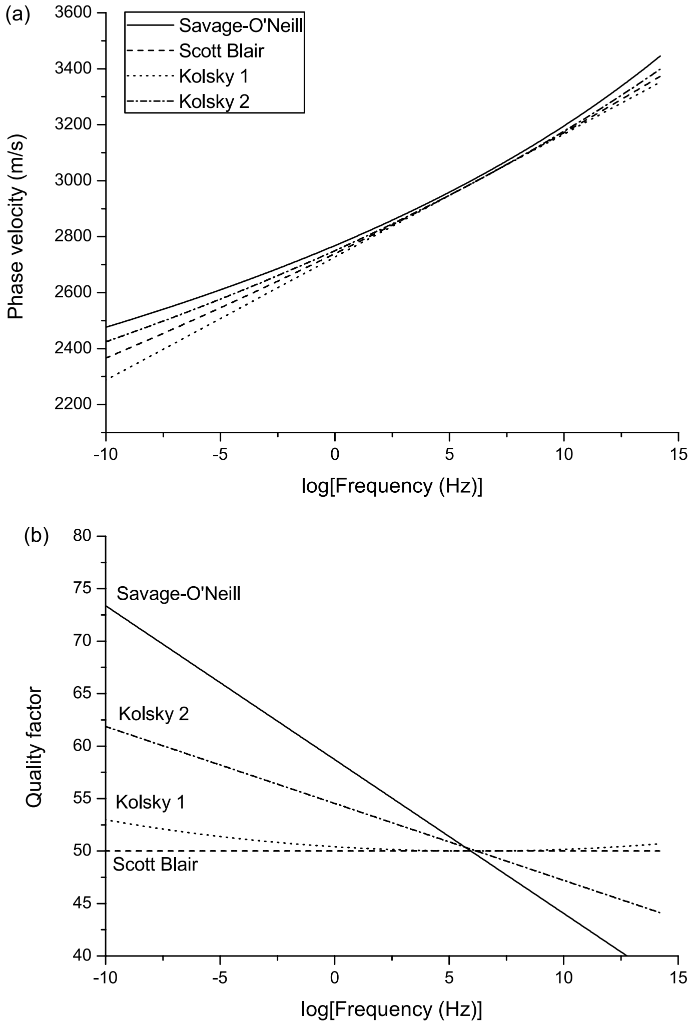

Fig. 11 Comparison of the Kolsky models with those of Savage and O'Neill and Scott Blair: **a** phase velocity and **b** Q factor. The Kolsky 1 model is the original one introduced by Kolsky (1964) while Kolsky 2 corresponds to versions used by other authors (e.g., Ursin and Toverud 2002)

where $\mathrm{E}_1(z) = \exp(-z)\mathrm{U}(1, 1, z) = \Gamma(0, z)$ (Amos 1990).

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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