SH-TM mathematical analogy for the two-layer case.
A magnetotellurics application

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Abstract

The same mathematical formalism of the wave equation can be used to describe anelastic and electromagnetic wave propagation. In this work, we obtain the mathematical analogy for the reflection/refraction (transmission) problem of two layers, considering the presence of anisotropy and attenuation – viscosity in the viscoelastic case and resistivity in the electromagnetic case. The analogy is illustrated for SH (shear-horizontally polarised) and TM (transverse-magnetic) waves. In particular, we illustrate examples related to the magnetotelluric method applied to geothermal systems and consider the effects of anisotropy. The solution is tested with the classical solution for stratified isotropic media.

1. Introduction

The role of mathematical analogies has been well illustrated and explained by Tonti [1]. Quoting Tonti: "Many physical theories show formal similarities due to the existence of a common mathematical structure. This structure is independent of the physical contents of the theory and can be found in classical, relativistic and quantum theories; for discrete and continuous systems". Carcione and Cavallini [2] found analogies between anelastic and electromagnetic vector wave fields, while Carcione et al. [3] relate the medium properties. Carcione and Cavallini [2] show that the 2-D Maxwell equations describing propagation of the transverse-magnetic mode in anisotropic media is mathematically equivalent to the SH wave equation in an anisotropic-viscoelastic solid where attenuation is described with the Maxwell mechanical model. Later, Carcione and Robinson [4] establish the analogy for the reflection-transmission problem at a single interface, showing that contrasts in compressibility yield the reflection coefficient for light polarized perpendicular to the plane of incidence (Fresnel's sine law – the electric vector perpendicular to the plane of incidence), and density contrasts yields the reflection coefficient for light polarized in the plane of incidence (Fresnel's tangent law). Carcione et al. [5] considered the reflection/transmission problem through an anisotropic and lossy layer. In particular, they obtained the analogy among P and SH elastic waves, TE and TM electromagnetic waves and wave mechanics in quantum theory.

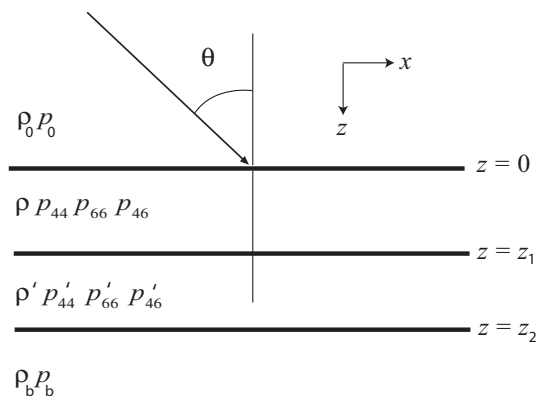Osella and Martinelli [6] have studied the effect of

Figure 1: Plane wave propagating through two layers. The viscoelastic properties are indicated.

anisotropic layers on the apparent resistivity curves, concluding that anisotropy cannot be detected with the TE mode and the TM response should be used. An extension of the technique used in that paper to 3D space can be found in Martinelli and Osella [7]. In this work, we solve the problem of horizontally polarized shear (SH) waves in a two-layer system and apply the analogy to obtain the TM solution. A geophysical application considering magnetotellurics in an anisotropic geothermal reservoir illustrates the use of the analogy. We analyze the apparent resistivity and phase angle for different orientations of the principal axis of anisotropy and angle of incidence of the plane wave.

2. Viscoelasticity. Propagation of SH waves

We start from the viscoelastic equations and then apply the analogy to obtain the equivalent electromagnetic expressions. Figure 1 shows two layers embedded between two isotropic half spaces with different properties.

In the following, we denote particle velocity by v, stress by σ, magnetic field by H, electric field by E, density by ρ, elasticity constant by c, viscosity by η, magnetic permeability by μ, dielectric permittivity by ϵ and electrical conductivity by σ (see below). Moreover, (x, y, z) indicates the spatial variables, ∂_x a partial derivative with respect to x and a dot above a variable denotes time differentiation. To distinguish between the stress and conductivity components, we use letters and numbers as subindices, respectively, e.g., σ_{xy} is a stress component and σ_{11} is a conduc-

tivity component.

The viscoelastic medium is characterized by the mass density ρ and elasticity, compliance and viscosity matrices

$$\mathbf{C} = \begin{pmatrix} c_{44} & c_{46} \\ c_{46} & c_{66} \end{pmatrix} = \mathbf{S}^{-1} = \begin{pmatrix} s_{44} & s_{46} \\ s_{46} & s_{66} \end{pmatrix}^{-1}$$

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_{44} & \eta_{46} \\ \eta_{46} & \eta_{66} \end{pmatrix} = \boldsymbol{\tau}^{-1} = \begin{pmatrix} \tau_{44} & \tau_{46} \\ \tau_{46} & \tau_{66} \end{pmatrix}^{-1}, \quad (1)$$

respectively [2, 4, 8]. Equations (1) correspond to the properties of cross-plane shear motion in the plane of symmetry of a monoclinic medium, and to a viscoelastic medium described by the Maxwell mechanical model. The subindices "44", "46" and "66" refer to the Voigt notation of the elasticity and viscosity tensors [8].

The SH-wave differential equations corresponding to the Maxwell viscoelastic model are [8]

$$\partial_x \sigma_{xy} + \partial_z \sigma_{yz} = \rho \dot{v}_y,$$
$$-\partial_z v_y = -\tau_{44}\sigma_{yz} - \tau_{46}\sigma_{xy} - s_{44}\dot{\sigma}_{yz} - s_{46}\dot{\sigma}_{xy},$$
$$\tag{2}$$
$$\partial_x v_y = \tau_{46}\sigma_{yz} + \tau_{66}\sigma_{xy} + s_{46}\dot{\sigma}_{yz} + s_{66}\dot{\sigma}_{xy},$$

where

$$\tau_{44} = \eta_{66}/\bar{\eta}, \quad \tau_{66} = \eta_{44}/\bar{\eta}, \quad \tau_{46} = -\eta_{46}/\bar{\eta}$$

$$\bar{\eta} = \eta_{44}\eta_{66} - \eta_{46}^2 \tag{3}$$

and

$$s_{44} = c_{66}/c, \quad s_{66} = c_{44}/c, \quad s_{46} = -c_{46}/c$$

$$c = c_{44}c_{66} - c_{46}^2. \tag{4}$$

The displacement associated to a homogeneous viscoelastic SH plane wave has the form

$$\mathbf{u} = u_y\hat{\mathbf{e}}_2, \quad u_y = U_0 \exp[\mathrm{i}(\omega t - \mathbf{k} \cdot \mathbf{x})], \tag{5}$$

where $\mathbf{x} = (x, z)$ is the position vector, ω is the angular frequency, t is the time variable, $\mathrm{i} = \sqrt{-1}$ and

$$\mathbf{k} = (\kappa - \mathrm{i}\alpha)\hat{\boldsymbol{\kappa}} = k\hat{\boldsymbol{\kappa}} \tag{6}$$

defines the complex wavevector, with $\hat{\boldsymbol{\kappa}} = (l_1, l_3)^\top$, defining the propagation direction through the direction cosines l_1 and l_3. Replacing the plane wave (5) into equation (2) yields the dispersion relation

$$p_{66}l_1^2 + 2p_{46}l_1 l_3 + p_{44}l_3^2 - \rho\left(\frac{\omega}{k}\right)^2 = 0, \tag{7}$$

where the p_{IJ} are the components of $\mathbf{P}$ obtained as

$$\mathbf{P} = \left(\mathbf{S} - \frac{\mathrm{i}}{\omega}\boldsymbol{\tau}\right)^{-1} \equiv \mathbf{R}^{-1}. \tag{8}$$

The relation (7) defines the complex velocity,

$$v = \frac{\omega}{k} = \sqrt{\frac{p_{66}l_1^2 + 2p_{46}l_1 l_3 + p_{44}l_3^2}{\rho}}. \tag{9}$$

2.1. Reflection and transmission coefficients

The boundary conditions at the interfaces require continuity of [8]

$$\sigma_{yz} \quad \text{and} \quad v_y. \tag{10}$$

In the electromagnetic case, continuity of the tangential components of the electric and magnetic fields is required [9] (see below). Let us assume that the incident, reflected and refracted waves are identified by the subscripts and superscripts I, R and T.

For a single interface, say that at $z = z_1$, the particle velocities of the incident, reflected and refracted waves are given by

$$v_y^I = \exp[\mathrm{i}\omega(t - s_x x - s_z z)],$$
$$v_y^R = R \exp[\mathrm{i}\omega(t - s_x x - s_z^R z)], \tag{11}$$
$$v_y^T = T \exp[\mathrm{i}\omega(t - s_x x - s_z^T z)],$$

respectively, where $(s_x, s_z)^\top$ is the slowness vector, and R and T are the reflection and refraction (transmission) coefficients. The equations obtained below hold for incident inhomogeneous plane waves (non-uniform waves in electromagnetism), i.e., waves for which the wavenumber and attenuation vectors do not point in the same direction. In the special case where these two vector coincide, the wave is termed homogeneous (uniform in electromagnetism), and we have

$$(s_x, s_z)^\top = \frac{1}{v}(\sin\theta, \cos\theta)^\top, \tag{12}$$

where θ is the incidence angle.

In the general case, the reflection and transmission coefficients (TM case in electromagnetism) are given by

$$r = \frac{Z_I - Z_T'}{Z_I + Z_T'}, \quad \tau = \frac{2Z_I}{Z_I + Z_T'}, \tag{13}$$

where

$$Z_I = p_{46}s_x + p_{44}s_z, \quad Z_T' = p_{46}'s_x + p_{44}'s_z^{T'}, \tag{14}$$

with

$$s_x^R = s_x^T = s_x^I = s_x \ (\text{Snell's law}), \quad s_z^R = -s_z \tag{15}$$

and

$$s_z^{T'} = \frac{1}{p_{44}'}\left(-p_{46}'s_x + \mathrm{pv}\sqrt{\rho' p_{44}' - p'^2 s_x^2}\right),$$

$$p'^2 = p_{44}'p_{66}' - p_{46}'^2, \tag{16}$$

with "pv" denoting the principal value [4, 8].

The coefficients for the interfaces at $z = 0$ and $z = z_2$ have similar forms but assuming isotropy for the upper and lower media, respectively, with $p_{44} = p_{66}$ and $p_{46} = 0$.

To obtain the reflection and transmission coefficients of the two layers, we follow the procedure indicated in Section 6.4 of Carcione [8]. At depth z in the second layer,

the particle-velocity field is a superposition of upgoing and downgoing waves of the form

$$v_y(z) =$$

$$[V^- \exp(\mathrm{i}\omega s_z^{T'} z) + V^+ \exp(-\mathrm{i}\omega s_z^{T'} z)] \exp[\mathrm{i}\omega(t - s_x x)], \quad (17)$$

where V^- and V^+ are upgoing- and downgoing-wave amplitudes.

From equation (2), the normal stress component is

$$\sigma_{yz}(z) = \frac{r_{66}' \partial_z v_y - r_{46}' \partial_x v_y}{\mathrm{i}\omega(r_{44}' r_{66}' - r_{46}'^2)} = \frac{1}{\mathrm{i}\omega}(p_{44}' \partial_z v_y + p_{46}' \partial_x v_y), \quad (18)$$

where $r_{IJ}' = s_{IJ}' - \mathrm{i}\tau_{IJ}'/\omega$ are the components of matrix $\mathbf{R}'$ defined in equation (8). Using equation (17), we obtain

$$\sigma_{yz}(z) = [V^-(-p_{46}' s_x + p_{44}' s_z^{T'}) \exp(\mathrm{i}\omega s_z^{T'} z)$$

$$-V^+ Z_T' \exp(-\mathrm{i}\omega s_z^{T'} z)] \exp[\mathrm{i}\omega(t - s_x x)]. \quad (19)$$

Omitting the phase $\exp[\mathrm{i}\omega(t - s_x x)]$, the particle-velocity/stress vector can be written as

$$\mathbf{t}(z) = \begin{pmatrix} v_y \\ \sigma_{yz} \end{pmatrix} =$$

$$\begin{pmatrix} \exp(\mathrm{i}\omega s_z^{T'} z) & \exp(-\mathrm{i}\omega s_z^{T'} z) \\ I_T' \exp(\mathrm{i}\omega s_z^{T'} z) & -Z_T' \exp(-\mathrm{i}\omega s_z^{T'} z) \end{pmatrix} \begin{pmatrix} V^- \\ V^+ \end{pmatrix}$$

$$\equiv \mathbf{T}(z) \begin{pmatrix} V^- \\ V^+ \end{pmatrix}, \quad (20)$$

where

$$I_T' = -p_{46}' s_x + p_{44}' s_z^{T'}. \quad (21)$$

Then, the fields at $z = z_1$ and $z = z_2$ are related by the following equation:

$$\mathbf{t}(z_1) = \mathbf{B}' \cdot \mathbf{t}(z_2), \quad (22)$$

where

$$\mathbf{B}' = \mathbf{T}(z_1) \cdot \mathbf{T}^{-1}(z_2) = \frac{1}{p_{44}' s_z^{T'}}$$

$$\begin{pmatrix} p_{44}' s_z^{T'} \cos\varphi' - \mathrm{i}p_{46}' s_x \sin\varphi' & -\mathrm{i}\sin\varphi' \\ -\mathrm{i}I_T' Z_T' \sin\varphi' & p_{44}' s_z^{T'} \cos\varphi' + \mathrm{i}p_{46}' s_x \sin\varphi' \end{pmatrix}, \quad (23)$$

where

$$\varphi' = \omega s_z^{T'}(z_2 - z_1). \quad (24)$$

Note that when $z_2 = z_1$, $\mathbf{B}'$ is the identity matrix.

Similarly, we have

$$\mathbf{t}(0) = \mathbf{B} \cdot \mathbf{t}(z_1), \quad (25)$$

where

$$\mathbf{B} = \mathbf{T}(0) \cdot \mathbf{T}^{-1}(z_1) = \frac{1}{p_{44} s_z^T}$$

$$\begin{pmatrix} p_{44} s_z^T \cos\varphi - \mathrm{i}p_{46} s_x \sin\varphi & -\mathrm{i}\sin\varphi \\ -\mathrm{i}I_T Z_T \sin\varphi & p_{44} s_z^T \cos\varphi + \mathrm{i}p_{46} s_x \sin\varphi \end{pmatrix}, \quad (26)$$

where

$$\varphi = \omega s_z^T z_1. \quad (27)$$

Combining equations (22) and (25), we finally obtain

$$\mathbf{t}(0) = \mathbf{B} \cdot \mathbf{B}' \cdot \mathbf{t}(z_2) \equiv \mathbf{A} \cdot \mathbf{t}(z_2). \quad (28)$$

On the other hand, using equations (11) and (14), the particle-velocity/stress field at $z = 0$ and $z = z_2$ can be expressed as

$$\mathbf{t}(0) = (1 + R, Z_0(R - 1))^\top,$$

$$\mathbf{t}(z_2) = T(1, -Z_b)^\top, \quad (29)$$

where R and T are here the reflection and transmission coefficients of the two-layer system and

$$Z_0 = \sqrt{\rho_0 p_0} \cos\theta \quad \text{and} \quad Z_b = \sqrt{(\rho_b - p_b s_x^2)p_b} \quad (30)$$

for an incident homogeneous plane wave. Substituting equation (29) into (28), we have

$$R = \frac{\alpha + 1}{\alpha - 1}, \quad \alpha = \left(\frac{a_{11} - a_{12}Z_b}{a_{21} - a_{22}Z_b}\right) Z_0, \quad (31)$$

where a_{ij} are the components of matrix $\mathbf{A}$.

If the two layers have the same properties and the "46" stiffness components are zero, we obtain the equations given in Carcione et al. [5],

$$R = \frac{r_{12} + r_{23} \exp(-2\mathrm{i}\varphi)}{1 + r_{12}r_{23} \exp(-2\mathrm{i}\varphi)}, \quad (32)$$

where $\varphi = -\omega s_z^T z_2$ and

$$r_{12} = \frac{Z_0 - Z_T}{Z_0 + Z_T} \quad \text{and} \quad r_{23} = \frac{Z_T - Z_b}{Z_T + Z_b}. \quad (33)$$

Equation (32) is similar to equation (5.22) of Born and Wolf [9] if the layers are isotropic and lossless. In the case in which the media above and below the layer have the same properties, i.e., when $r_{23} = -r_{12}$, equation (33) becomes

$$R = \frac{r_{12}[1 - \exp(-2\mathrm{i}\varphi)]}{1 - r_{12}^2 \exp(-2\mathrm{i}\varphi)}. \quad (34)$$

On the other hand, the transmission coefficient is

$$T = \frac{2Z_0}{(a_{11} - a_{12}Z_b)Z_0 - (a_{21} - a_{22}Z_b)}. \quad (35)$$

2.2. Surface impedance and apparent viscosity

We define the impedance in the upper half-space as

$$-\frac{\sigma_{yz}}{v_y} \equiv Z_s, \quad (36)$$

where

$$v_y = [\exp(-\mathrm{i}\omega s_z z) + R \exp(\mathrm{i}\omega s_z z)] \exp[\mathrm{i}\omega(t - s_x x)],$$

$$\sigma_{yz} = s_z p_0[-\exp(-\mathrm{i}\omega s_z z) + R \exp(\mathrm{i}\omega s_z z)] \exp[\mathrm{i}\omega(t - s_x x)], \quad (37)$$

where $s_z = \cos\theta/v$ and we have used equations (11), (12) and (19). Substituting equation (37) into (36) at $z = 0$, we obtain the surface impedance

$$Z_s(z=0) = -\left(\frac{1-R}{1+R}\right)Z_0 = -\frac{a_{21} - a_{22}Z_b}{a_{11} - a_{12}Z_b}, \quad (38)$$

where we have used equations (30) and (31).

On the other hand, the surface impedance can be obtained from the fields of the first layer at $z = 0$. From equation (29) we have

$$v_y(0) = (a_{11} - a_{12}Z_b)T, \text{ and } \sigma_{yz}(0) = (a_{21} - a_{22}Z_b)T, \quad (39)$$

which, using (36), yields equation (38).

We define the apparent surface viscosity as

$$p_a = \frac{1}{\omega\rho_0}\left|\frac{\sigma_{yz}}{v_y}\right|^2 = \frac{1}{\omega\rho_0}|Z_s|^2, \quad (40)$$

where we have used equation (38). A phase angle can be defined as

$$\phi = \tan^{-1}\left[\frac{\text{Im}(Z_s)}{\text{Re}(Z_s)}\right]. \quad (41)$$

Note that Z_s, p_a and ϕ do not depend on Z_0.

Let us assume $z_1 = z_2 = 0$. Then, $\mathbf{B}$, $\mathbf{B}'$ and $\mathbf{A}$ are identity matrices and $\alpha = -Z_0/Z_b$. We obtain

$$p_a = \frac{|Z_b|^2}{\omega\rho_0} = \frac{|\rho_b p_b - p_b^2 s_x^2|}{\omega\rho_0}, \quad (42)$$

where we used (30). At normal incidence $s_x = 0$, and

$$p_a = \frac{\rho_b|p_b|}{\omega\rho_0}, \quad (43)$$

Since, from equation (8), it is $p_b^{-1} = 1/c_b - i/(\omega\eta_b)$, where c_b and η_b are the elastic constant and viscosity of the lower half space, respectively, we obtain

$$p_a = \frac{\rho_b|p_b|}{\omega\rho_0} = \frac{\rho_b}{\omega\rho_0}\left|\frac{1}{c_b} - \frac{i}{\omega\eta_b}\right|^{-1} =$$

$$\frac{\rho_b}{\omega\rho_0}\left(\frac{1}{c_b^2} - \frac{1}{\omega^2\eta_b^2}\right)^{-1/2}. \quad (44)$$

3. Analogy with electromagnetism

3.1. SH-TM analogy

Let us consider Maxwell equations and assume that the propagation is in the (x,z)-plane, and that the material properties are invariant in the y-direction. Then, E_x, E_z and H_y are decoupled from E_y, H_x and H_z. In the absence of electric-source currents, the first three fields obey the TM (transverse-magnetic) differential equations:

$$\partial_x E_z - \partial_z E_x = \mu\partial_t H_y,$$
$$-\partial_z H_y = \sigma_{11}E_x + \sigma_{13}E_z + \epsilon_{11}\partial_t E_x + \epsilon_{13}\partial_t E_z, \quad (45)$$
$$\partial_x H_y = \sigma_{13}E_x + \sigma_{33}E_z + \epsilon_{13}\partial_t E_x + \epsilon_{33}\partial_t E_z,$$

where E_i and H_i denote the electric and magnetic field components, μ is the magnetic permeability, and ϵ_{ij} and σ_{ij} are the permittivity and electrical conductivity components, respectively [8].

Equations (2) and (45) are mathematically equivalent if

$$\begin{pmatrix} v_y \\ \sigma_{yz} \\ \sigma_{xy} \end{pmatrix} \Leftrightarrow \begin{pmatrix} H_y \\ -E_x \\ E_z \end{pmatrix}, \quad (46)$$

$$\mathbf{S} \equiv \begin{pmatrix} s_{44} & s_{46} \\ s_{46} & s_{66} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \epsilon_{11} & -\epsilon_{13} \\ -\epsilon_{13} & \epsilon_{33} \end{pmatrix} \equiv \tilde{\boldsymbol{\epsilon}}, \quad (47)$$

$$\boldsymbol{\tau} \equiv \begin{pmatrix} \tau_{44} & \tau_{46} \\ \tau_{46} & \tau_{66} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \sigma_{11} & -\sigma_{13}, \\ -\sigma_{13} & \sigma_{33} \end{pmatrix} \equiv \tilde{\boldsymbol{\sigma}} \quad (48)$$

$$\rho \Leftrightarrow \mu. \quad (49)$$

From equation (8), in virtue of the acoustic-electromagnetic equivalence (46)-(49), it follows that $\mathbf{P}$ corresponds to the inverse of the complex dielectric-permittivity matrix $\bar{\epsilon}$, namely:

$$\mathbf{P}^{-1} \Leftrightarrow \bar{\epsilon} \equiv \tilde{\epsilon} - \frac{i}{\omega}\tilde{\boldsymbol{\sigma}}. \quad (50)$$

Therefore, all the equations obtained in the previous section can be used to obtain the electromagnetic properties using the mathematical analogies (46)-(50).

3.2. Surface impedance and apparent resistivity

In particular, the surface impedance (36) is

$$Z_s = -\frac{\sigma_{yz}}{v_y} = \frac{E_x}{H_y}, \quad (51)$$

and is given by equation (38). It is $Z_0 = \sqrt{\mu_0/\bar{\epsilon}_0}\cos\theta$, $\bar{\epsilon}_0 = \epsilon_0 - i\sigma_0/\omega$, where μ_0, ϵ_0 and σ_0 are the magnetic permeability, dielectric permittivity and electric conductivity of the upper half-space, but, as above, this quantity disappear from the calculations.

The equivalent of the surface apparent viscosity (42) is the apparent resistivity

$$\hat{\rho}_a = \frac{1}{\omega\rho_0}\left|\frac{\sigma_{yz}}{v_y}\right|^2 = \frac{1}{\omega\mu_0}\left|\frac{E_x}{H_y}\right|^2. \quad (52)$$

In the case $z_1 = z_2 = 0$, we have from equation (44),

$$\hat{\rho}_a = \frac{\mu_b}{\omega\mu_0}\left(\epsilon_b^2 - \frac{\sigma_b^2}{\omega^2}\right)^{-1/2}, \quad (53)$$

where μ_b, ϵ_b and σ_b are the magnetic permeability, dielectric permittivity and electric conductivity of the lower half-space. In magnetotellurics, the upper space is air and the magnetic permeability is assumed to the constant, i.e., $\mu_b = \mu_0$. Moreover, displacement currents are neglected ($\epsilon_b \ll \sigma_b/\omega$). Hence, we obtain

$$\hat{\rho}_a = \frac{1}{\sigma_b} = \hat{\rho}_b, \quad (54)$$

where $\hat{\rho}_b$ is the resistivity of the lower half-space, as expected.

4. Example: Magnetotellurics

Electrical anisotropy in the Earth can be due to preferred orientation of fracture porosity, fluidised, melt-bearing or graphitised shear zones, lithologic layering, oriented heterogeneity, or hydrous defects within shear aligned olivine crystals [10]. Magnetotellurics is a technique to measure the apparent resistivity of the subsoil to interpret the nature of the geological formations. The method operates at low frequencies, where the displacement term is neglected in Maxwell equations [11], such that the apparent resistivity is given by equation (52), with

$$\mathbf{P}^{-1} = -\frac{i}{\omega}\boldsymbol{\tau} \quad \Leftrightarrow \quad \bar{\boldsymbol{\epsilon}} = -\frac{i}{\omega}\tilde{\boldsymbol{\sigma}}, \tag{55}$$

in order to apply the analogy. Specifically,

$$\mathbf{P}^{-1} \quad \Leftrightarrow \quad -\frac{i}{\omega}\begin{pmatrix} \tilde{\sigma}_{11} & \tilde{\sigma}_{13} \\ \tilde{\sigma}_{13} & \tilde{\sigma}_{33} \end{pmatrix} = -\frac{i}{\omega}\begin{pmatrix} \sigma_{11} & -\sigma_{13} \\ -\sigma_{13} & \sigma_{33} \end{pmatrix} \tag{56}$$

or

$$\mathbf{P} = \begin{pmatrix} p_{44} & p_{46} \\ p_{46} & p_{66} \end{pmatrix} \quad \Leftrightarrow \quad \frac{i\omega}{\sigma_{11}\sigma_{33} - \sigma_{13}^2}\begin{pmatrix} \sigma_{33} & \sigma_{13} \\ \sigma_{13} & \sigma_{11} \end{pmatrix}. \tag{57}$$

Moreover, magnetotellurics assumes that plane waves are normally incident on the surface of the earth (z-direction). In this case, equation (45) simplify to

$$\begin{aligned} \partial_z E_x &= \mu\partial_t H_y, \\ -\partial_z H_y &= \sigma_{11}E_x + \sigma_{13}E_z, \\ 0 &= \sigma_{13}E_x + \sigma_{33}E_z, \end{aligned} \tag{58}$$

Eliminating the magnetic field, we obtain

$$\partial_z^2 E_x = \mu\left(\sigma_{11} - \frac{\sigma_{13}^2}{\sigma_{33}}\right)\partial_t E_x. \tag{59}$$

The quantity between round parentheses can be seen as an effective conductivity. The standard magnetotelluric method cannot distinguish changes in the conductivity components if that quantity is kept constant [12, 10]. On the other hand, for a plane wave travelling along the x-direction and based on the vertical component E_z, the effective conductivity $\sigma_{33} - \sigma_{13}^2/\sigma_{11}$ can be obtained. Another measurement is required with a plane wave incident at an intermediate angle to obtain the three conductivity components. Moreover, note that if $\sigma_{13} = 0$, variations in σ_{33} have no effects on the result when the plane wave is normally incident.

Let us consider the model shown in Figure 2 and analyze the apparent resistivity as the properties of the formation vary for plane waves incident at different angles.

The conductivity components of the geothermal zone are obtained from a clockwise rotation by an angle β about the y-axis of the conductivity matrix from the principal sys-
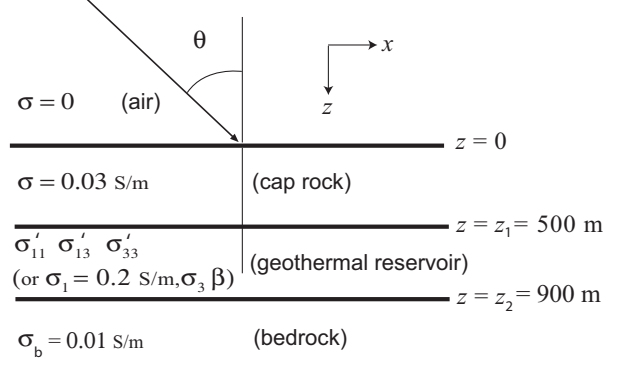

Figure 2: Model of a geothermal reservoir. The conductivity tensor in its principal system has components σ_1 and σ_3. A rotation of the tensor by an angle β gives the components σ_{ij} in the system (x, z).

tem with components σ_1 and σ_3,

$$\begin{aligned} \sigma'_{11} &= \sigma_1\cos^2\beta + \sigma_3\sin^2\beta, \\ \sigma'_{13} &= \tfrac{1}{2}(\sigma_3 - \sigma_1)\sin 2\beta, \\ \sigma'_{33} &= \sigma_3\cos^2\beta + \sigma_1\sin^2\beta. \end{aligned} \tag{60}$$

Here we study the apparent resistivity as a function of σ_3 and β and keep constant the other properties. In particular $\sigma_1 = 0.2$ S/m. The different cases are shown in Table 1, with Case 1 the standard magnetotelluric technique.

Table 1: Values of the rotation angle and conductivity component (in S/m).

Case	β	σ_3
1	–	σ_1
2	$\pi/4$	0.1
3	$\pi/4$	0.5
4	$\pi/2$	0.1
5	$\pi/2$	0.5

As indicated above, the conductivity component σ_{33} has no effect on the results if $\sigma_{13} = 0$. Hence, in this case anisotropy cannot be detected. Figure 3 shows the apparent resistivity (a) and phase (b) for $\sigma_3 = \sigma_1$ (Case 1 in Table 1), where $T = 1/(2\pi\omega)$ is the period.

It can be shown that the results coincide with the classical magnetotelluric solution (65) given in Appendix A. Figures 4 and 5 compare the isotropic and anisotropic cases for incidence angles of $\pi/4$ and $\pi/2$, respectively. Cases 2 and 4 have more apparent resistivity, since the σ_3 component has a lower value. The differences due to anisotropy, and the fact that wave is not normally incident, are significant.

However, as mentioned above, the classical magnetotelluric method can only determine effective conductivities. Assume $\sigma_{11} = 0.2$ S/m, $\sigma_{33} = 0.1$ S/m and $\sigma_{13} = 0.05$ S/m. The effective component is $\sigma_{11} - \sigma_{13}^2/\sigma_{33} = 0.175$ S/m. An isotropic medium with $\sigma_{11} = \sigma_{33} = 0.175$ S/m has the same response than the anisotropic medium.
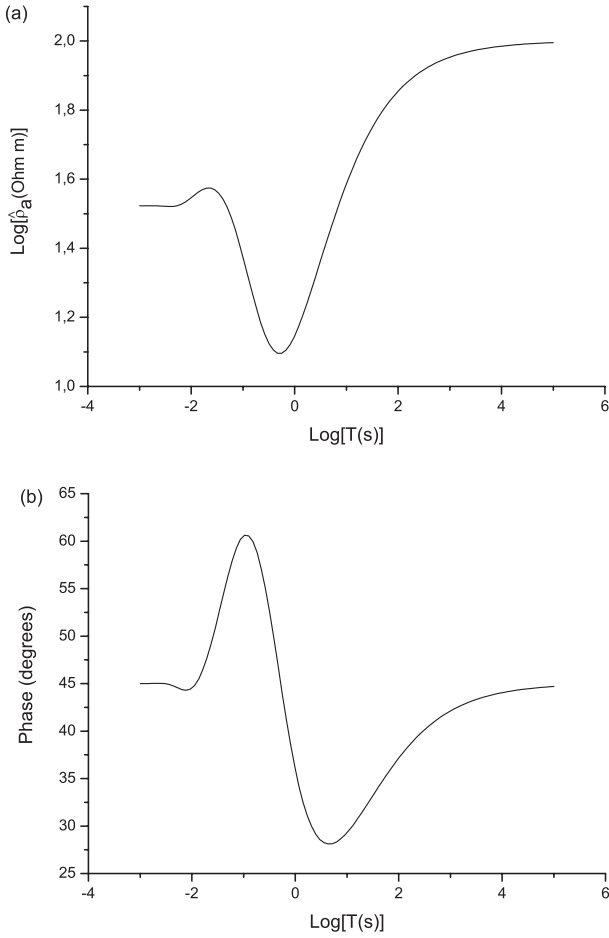
Figure 3: Apparent resistivity (a) and phase (b) for $\sigma_3 = \sigma_1$ (Case 1 in Table 1).

Figure 4: Apparent resistivity (a) and phase (b) for Cases 1, 2 and 3.

5. Conclusions

Theories describing wave propagation and field diffusion in different fields of physics consist in partial differential equations, which have identical or similar mathematical expressions. Here, we have considered the reflection/transmission problem of SH waves through a two-layer anisotropic and lossy system. We have shown that the same mathematical equations can be used in electromagnetism to describe the propagation and diffusion of TM waves. An example shows how the SH-wave equation reduce to the differential equation describing the magnetotelluric technique.

The analogy can be useful in the space-time domain using numerical simulations. In this case, the same computer code, with appropriate input variables can be used to solve the different physical problems.

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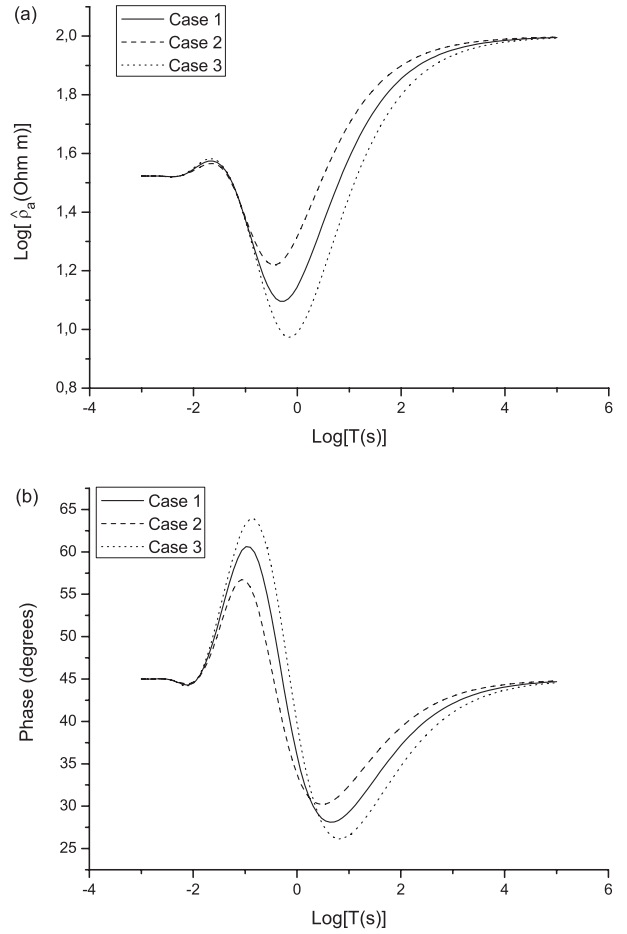
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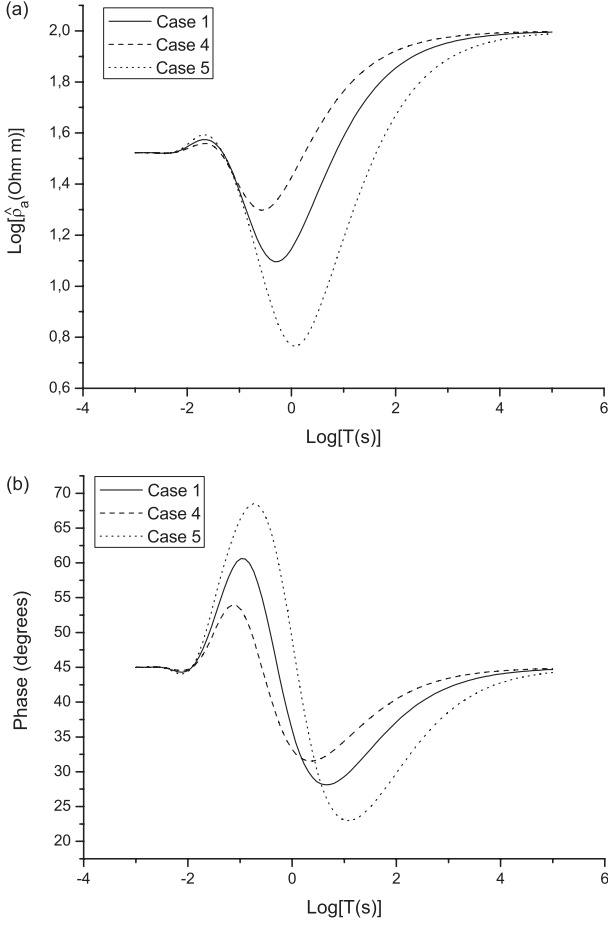
Figure 5: Apparent resistivity (a) and phase (b) for Cases 1, 4 and 5.

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A. Classical magnetotelluric solution for an isotropic Earth

Wait [13, 14] obtained the surface impedance for n layers by a recurrence approach to solve the Riccati equation

$$\partial_z Z_s - \frac{1}{\hat{\rho}} Z_s^2 = i\omega\mu_0, \qquad (61)$$

where $\hat{\rho}$ is the resistivity and $\mu_0 = 4\,\pi\,10^{-7}$ H/m is the magnetic permeability of free space. For n layers with resistivity $\hat{\rho}_j = 1/\sigma_j$ and thickness h_j, the surface impedance at $z = d_{j-1}$ is

$$Z_{s(j-1)} = \left[\frac{Z_{sj} + Z_j \tanh(ik_j h_j)}{Z_j + Z_{sj} \tanh(ik_j h_j)}\right] Z_j, \qquad (62)$$

where

$$k_j = \sqrt{-i\omega\mu_0\sigma_j} \qquad (63)$$

is the complex wavenumber and

$$Z_j = \frac{\omega\mu_0}{k_j} = \sqrt{i\omega\mu_0\hat{\rho}_j} \qquad (64)$$

is the impedance of each layer, $d_1 = h_1$ and $d_j = d_{j-1} + h_j$ ($d_0 = 0$ is the surface). The iteration starts with $Z_{s(n-1)} = Z_n$ and ends with

$$Z_s = \left[\frac{Z_{s1} + Z_1 \tanh(ik_1 h_1)}{Z_1 + Z_{s1} \tanh(ik_1 h_1)}\right] Z_1. \qquad (65)$$

For $n = 3$, $Z_{s2} = Z_3$, we then compute

$$Z_{s1} = \left\{\frac{Z_3 + Z_2 \tanh[ik_2(z_2 - z_1)]}{Z_2 + Z_3 \tanh[ik_2(z_2 - z_1)]}\right\} Z_2. \qquad (66)$$

and $Z_s = Z_{s0}$ using equation (65) with $h_1 = z_1$. The preceding equations hold for $\theta = 0$, i.e., a normally incident plane wave.