ON THE ACOUSTIC-ELECTROMAGNETIC ANALOGY FOR THE REFLECTION-REFRACTION PROBLEM

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ABSTRACT

The same mathematical theory can be used to describe physical phenomena of different nature. For instance, the wave equation and the related mathematical developments can be used to describe elastic and electromagnetic wave propagation, and it is also extensively used in quantum mechanics. Fresnel's equations are a classical example of the analogy between shear waves and light waves. George Green in the nineteenth century, used analogies to obtain the reflection coefficients for sound waves and light waves, before the advent of the electromagnetic theory of light.

In this work, we investigate the mathematical analogy between elastic waves and electromagentic waves. We obtain a complete parallelism for the reflection and refraction problem, considering the most general situation, that is, the presence of anisotropy and attenuation – viscosity in the elastic case and conductivity in the electromagnetic case. The analogy is illustrated with Fresnel's equations, the Brewster and critical angles, the concept of reflectivity and transmissivity, and the corresponding duals fields. The analysis of the elastic-solid theory of reflection applied by Green to light waves, and a brief historical review of wave propagation through the ether, further illustrate the analogy.

1. INTRODUCTION

Many of the great scientists of the past have been occupied with the establishment of the theory of wave motion. Throughout this development there has been an interplay between the theory of light waves and the theory of material waves. As early as 1637 Rene Descartes (1596-1650) provided and explanation of the rainbow. His use of Snell's law led to further advances in the study of the reflection and refraction of light.

In 1660 Robert Hooke (1635-1703) formulated stress-strain relationships which established the elastic behavior of solid bodies. Hooke believed light to be a vibratory displacement of the medium, through which it propagates at finite speed. Significant experimental and mathematical advances came in the nineteenth century. Thomas Young

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(1775-1812) was the first to consider shear as an elastic strain, and defined the elastic modulus that was later named Young's modulus.

In 1809 Etienne Louis Malus (1775-1812) discovered polarization of light by reflection, which at the time David Brewster correctly described as "a memorable epoch in the history of optics". In 1815 Brewster discovered the law that regulated the polarization of light. Augustus Jean Fresnel (1778-1827) showed that if light were a transverse wave, then it would be possible to develop a theory accommodating the polarization of light.

Green (1838, 1842) makes extensive use of the analogy between elastic waves and light waves. Although some of his conclusions are erroneous, an analysis of his developments illustrates the power of the use of mathematical analogies.

Later in the nineteenth century, Maxwell and Kelvin made extensive use of physical and mathematical analogies to study wave phenomena in elastic theory and electromagnetism. In fact, the displacement current introduced by Maxwell into the electromagnetic equations arises from the analogy with elastic displacements. Maxwell assumed his equations were valid in an absolute system regarded as a medium (called the ether) that filled the whole of space. The ether was in a state of stress, and would only transmit transverse waves.

Of course, with advent of the theory of relativity the concept of the ether was abandoned. However the fact that electromagnetic waves are transverse waves is important. This situation is in contrast to a fluid, which can only transmit longitudinal waves. A visco-elastic body transmits both longitudinal waves and transverse waves. It is well known that the visco-elastic can be specialized to the form that hold for fluids. It is also possible to recast the visco-elastic equations into a form that closely parallels Maxwell's equations. In many cases this formal analogy becomes a complete mathematical equivalence such that the same analytic or numerical analogy can be used to solve problems in both disciplines.

Carcione and Cavallini (1995) showed that the 2-D Maxwell equations describing propagation of the TM mode in anisotropic media is mathematically equivalent to the SH wave equation in an anisotropic-viscoelastic solid where attenuation is described with the Maxwell model. We use this theory to obtain a complete mathematical analogy for the reflection-refraction problem.

2. THE ANALOGY

The analogy, as given by *Carcione and Cavallini (1995)*, is summarized in this section, in the time-space and wavenumber (or slowness)-frequency domains.

2.1 Equation of motion

Assume that the propagation is in the (x, z)-plane, and that the material properties are invariant in the y-direction. Then, E_x, E_z and H_y are decoupled from E_y, H_x and H_z. In the absence of electric source currents, the first three fields obey the TM (transverse magnetic field) differential equations:

$$\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = \mu_m \frac{\partial H_y}{\partial t} - M_y \ , \tag{1}$$

$$-\frac{\partial H_y}{\partial z} = \sigma_{11} E_x + \sigma_{13} E_z + \varepsilon_{11} \frac{\partial E_x}{\partial t} + \varepsilon_{13} \frac{\partial E_z}{\partial t} \ , \tag{2}$$

$$\frac{\partial H_y}{\partial x} = \sigma_{13} E_x + \sigma_{33} E_z + \varepsilon_{13} \frac{\partial E_x}{\partial t} + \varepsilon_{33} \frac{\partial E_z}{\partial t} \ , \tag{3}$$

where μ_m is the magnetic permeability, ε_{ij} and σ_{ij} are the components of the permittivity and conductivity tensors, respectively, and M_y is the magnetic source.

On the other hand, in the plane of mirror symmetry of a viscoelastic monoclinic medium, uniform properties in the y direction imply that one of the shear waves has its own (decoupled) differential equation, known in the literature as the SH wave equation. Propagation in this plane implies pure cross-plane strain motion, and is the most general situation for which pure shear waves exist at all propagation angles. Pure shear wave propagation in homogeneous hexagonal media occurs along all directions, since every plane containing the symmetry axis is a plane of symmetry. The differential equations describing the wave motion in the plane of symmetry of a monoclinic medium are

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} = \rho \frac{\partial v_y}{\partial t} - F_y \ , \tag{4}$$

$$-\frac{\partial v_y}{\partial z} = -\tau_{44} \sigma_{yz} - \tau_{46} \sigma_{xy} - s_{44} \frac{\partial \sigma_{yz}}{\partial t} - s_{46} \frac{\partial \sigma_{xy}}{\partial t} \ , \tag{5}$$

$$\frac{\partial v_y}{\partial x} = \tau_{46} \sigma_{yz} + \tau_{66} \sigma_{xy} + s_{46} \frac{\partial \sigma_{yz}}{\partial t} + s_{66} \frac{\partial \sigma_{xy}}{\partial t} \ , \tag{6}$$

where v_y is the particle velocity, σ denotes the stress, F_y is the body force,

$$\tau_{44} = \eta_{66}/\eta \,, \quad \tau_{66} = \eta_{44}/\eta \,, \quad \tau_{46} = -\eta_{46}/\eta \,, \quad \eta = \eta_{44}\eta_{66} - \eta_{46}^2 \ , \tag{7}$$

and

$$s_{44} = c_{66}/c \,, \quad s_{66} = c_{44}/c \,, \quad s_{46} = -c_{46}/c \,, \quad c = c_{44}c_{66} - c_{46}^2 \ , \tag{8}$$

with c_{IJ} the stiffness components, η_{IJ}, the viscosity components ($I, J = 4, 6$), and ρ is the density.

Note that equations (1) – (3) are converted into equations (4) – (6), and vice versa, under the following substitutions:

$$\begin{bmatrix} v_y \\ \sigma_{yz} \\ \sigma_{xy} \end{bmatrix} \Leftrightarrow \begin{bmatrix} H_y \\ -E_x \\ E_z \end{bmatrix} , \tag{9}$$

$$F_y \Leftrightarrow M_y , \tag{10}$$

$$\mathbf{S} \equiv \begin{bmatrix} s_{44} & s_{46} \\ s_{46} & s_{66} \end{bmatrix} \Leftrightarrow \begin{bmatrix} \varepsilon_{11} & -\varepsilon_{13} \\ -\varepsilon_{13} & \varepsilon_{33} \end{bmatrix} \equiv \boldsymbol{\varepsilon} , \tag{11}$$

$$\boldsymbol{\tau} \equiv \begin{bmatrix} \tau_{44} & \tau_{46} \\ \tau_{46} & \tau_{66} \end{bmatrix} \Leftrightarrow \begin{bmatrix} \sigma_{11} & -\sigma_{13} \\ -\sigma_{13} & \sigma_{33} \end{bmatrix} \equiv \boldsymbol{\sigma} , \tag{12}$$

$$\rho \Leftrightarrow \mu_m . \tag{13}$$

The 2 × 2 stiffness and viscosity matrices

$$\mathbf{C} = \begin{bmatrix} c_{44} & c_{46} \\ c_{46} & c_{66} \end{bmatrix} , \quad \boldsymbol{\eta} = \begin{bmatrix} \eta_{44} & \eta_{46} \\ \eta_{46} & \eta_{66} \end{bmatrix} \tag{14}$$

are respectively defined by 2-D identities $\mathbf{C} = \mathbf{S}^{-1}$ and $\boldsymbol{\eta} = \boldsymbol{\tau}^{-1}$. Then, the anisotropic SH wave equation based on a Maxwell rheology is mathematically equivalent to the anisotropic Maxwell equations whose forcing term is a magnetic current.

2.2. Kinematics and energy

The particle velocity of an inhomogeneous plane wave, propagating in the symmetry plane (the (x, z)-plane) of an homogeneous monoclinic medium, can be expressed as $\mathbf{v} = v_y \hat{\mathbf{e}}_y$, where

$$v_y = v_y(x,z) = \iota\omega\upsilon_0 \exp[\iota\omega(t - s_x x - s_z z)] , \tag{15}$$

where s_x and s_z are the components of the complex slowness vector, υ_0 is a complex quantity and $\hat{\mathbf{e}}_y$ is the unit vector along the y-direction.

The complex slowness vector is

$$\mathbf{s} = \mathbf{r} - \iota(\boldsymbol{\alpha}/\omega) = [s_x, s_z]^{\mathsf{T}} , \tag{16}$$

where the symbol T denotes transpose. The real slowness vector is given by

$$\mathbf{r} = \Re(\mathbf{s}) = [\Re(s_x), \Re(s_z)]^{\mathsf{T}} \tag{17}$$

and the attenuation vector is

$$\boldsymbol{\alpha} = -\omega \Im(\mathbf{s}) = -\omega [\Im(s_x), \Im(s_z)]^\top \ , \tag{18}$$

where $\Re$ and $\Im$ denote real and imaginary parts, respectively.

According to the analogy, the complex stiffness matrix, defined by

$$\mathbf{P} = \left(\mathbf{S} - \frac{\iota}{\omega} \boldsymbol{\tau} \right)^{-1} = \begin{bmatrix} p_{44} & p_{46} \\ p_{46} & p_{66} \end{bmatrix} , \tag{19}$$

corresponds to the inverse of the complex permittivity matrix $\boldsymbol{\varepsilon}^* = (\boldsymbol{\varepsilon} - \iota \boldsymbol{\sigma}/\omega)$, namely:

$$\mathbf{P} \Leftrightarrow (\boldsymbol{\varepsilon}^*)^{-1} = \left(\boldsymbol{\varepsilon} - \frac{\iota}{\omega} \boldsymbol{\sigma} \right)^{-1} \ . \tag{20}$$

The Umov-Poynting vector is

$$\mathbf{p} = \frac{1}{2} \omega^2 |\upsilon_0|^2 \exp\{2\omega [\Im(s_x) x + \Im(s_z) z]\} (X \hat{\mathbf{e}}_x + Z \hat{\mathbf{e}}_z) \ , \tag{21}$$

where

$$X = p_{66} s_x + p_{46} s_z \ , \quad \text{and} \quad Z = p_{46} s_x + p_{44} s_z \ . \tag{22}$$

and $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_z$ are the unit vectors along the *x* and *z* directions, respectively.

The time-average potential and dissipated energy densities are

$$\langle V \rangle = \frac{1}{4} \omega^2 |\upsilon_0|^2 \exp\{2\omega [\Im(s_x) x + \Im(s_z) z]\} \Re(g) \ , \tag{23}$$

and

$$\langle D \rangle = \frac{1}{2} \omega^2 |\upsilon_0|^2 \exp\{2\omega [\Im(s_x) x + \Im(s_z) z]\} \Im(g) \ , \tag{24}$$

where

$$g = p_{44} |s_z|^2 + p_{66} |s_x|^2 + 2 p_{46} \Re(s_x^* s_z) \tag{25}$$

is a complex quantity and has the dimension of density (note the difference between *g* and the real-valued material density ρ).

The time-average kinetic energy density is simply

$$\langle T \rangle = \frac{1}{4} \rho \omega^2 |\upsilon_0|^2 \exp\{2\omega [\Im(s_x) x + \Im(s_z) z]\} \tag{26}$$

The velocity of the energy is defined as the average power flow density $(\langle \mathbf{p} \rangle = \Re(\mathbf{p}))$ divided by the average energy density

$$\mathbf{v}_e = \frac{\Re(\mathbf{p})}{\langle T+V \rangle} = \frac{2(\Re(X)\hat{\mathbf{e}}_x + \Re(Z)\hat{\mathbf{e}}_z)}{\rho + \Re(g)} \ , \tag{27}$$

The wave front is associated with the higher energy velocity. Since, in the lossless case, all the wave surfaces have the same velocity – there is no velocity dispersion – the concepts of wave front and wave surface are the same. In lossy media, the wave front is the wave surface associated with the unrelaxed energy velocity (*Carcione, 2001*).

The following two fundamental relations hold

$$\mathbf{v}_e \cdot \mathbf{r} = 1 \ , \quad \text{and} \quad \langle D \rangle = \frac{2}{\omega} \boldsymbol{\alpha} \cdot \langle \mathbf{p} \rangle \ . \tag{28}$$

If $\mathbf{r}$ and $\boldsymbol{\alpha}$ point in the same direction, the wave is called homogeneous. The real slowness and attenuation vectors for homogeneous plane waves can be expressed as

$$\mathbf{r} = r\,\hat{\mathbf{s}} = \Re\left(\frac{1}{v}\right)\hat{\mathbf{s}} \ , \tag{29}$$

and

$$\boldsymbol{\alpha} = -\omega \Im\left(\frac{1}{v}\right)\hat{\mathbf{s}} \ , \tag{30}$$

where

$$v \equiv \omega s = \sqrt{\frac{p_{66} l_x^2 + 2 p_{46} l_x l_z + p_{44} l_z^2}{\rho}} \tag{31}$$

is the complex velocity,

$$\hat{\mathbf{s}} = (l_x, l_z)^{\mathsf{T}} = (\sin\theta, \cos\theta)^{\mathsf{T}} \tag{32}$$

is the propagation direction.

The velocity of the energy is given by

$$\mathbf{v}_e = \frac{1}{r\Re(v)} \left\{ \Re\left(\frac{1}{\rho v} [(p_{66} l_x + p_{46} l_z)\,\hat{\mathbf{e}}_x + (p_{44} l_z + p_{46} l_x)\,\hat{\mathbf{e}}_z] \right) \right\} \ , \tag{33}$$

where r is the magnitude of the real slowness vector. The electromagnetic slowness, attenuation, and energy velocity can be calculated from equations (29), (30), and (33) by applying this equivalence, and that of the density with the permeability (13), to equation (31). The kinetic and strain energy densities are associated with the magnetic and electric energy densities. In terms of circuit elements, the kinetic, strain and dissipated energies represent the energies stored in inductances, capacitors and the dissipative ohmic losses, respectively.

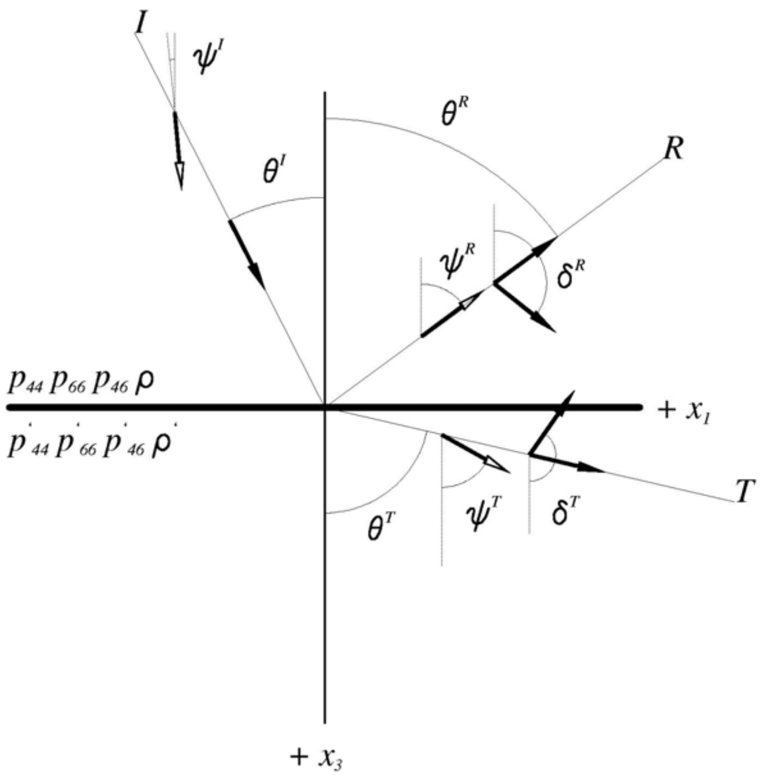

Fig. 1. Incident (*I*), reflected (*R*) and refracted (*T*) waves in the viscoelastic case. The angles θ, δ and ψ denote the propagation, attenuation and Umov-Poynting vector (energy) directions. The reflection angle is negative as shown.

2.3. The reflection-refraction problem

Let us assume that the positive *z*-axis points downwards, and that the incident, reflected and refracted waves are identified by the superscripts *I*, *R* and *T*. The upper medium is defined by the stiffnesses p_{IJ} and density ρ and the complex permittivities ε_{ij}^{*} and magnetic permeability μ_m. The lower medium is defined by the corresponding primed quantities. Figure 1 represents the incident (*I*), reflected (*R*) and refracted (*T*) waves at a boundary between two linear viscoelastic and monoclinic media. The angles θ, δ and ψ denote the propagation, attenuation and power-flow directions. Note that the propagation and energy directions do not necessarily coincide. Moreover, $|\theta - \delta|$ may exceed 90° in anisotropicviscoelastic media, while $|\theta - \delta|$ is strictly less than 90° in isotropic media (*Carcione, 1997*).

The analogy can be extended to the boundary conditions at a surface of discontinuity, because according to equation (9) continuity of

$$\sigma_{yz} \quad \text{and} \quad v_z \tag{34}$$

in the elastic case, is equivalent to continuity of

$$E_x \quad \text{and} \quad H_y \tag{35}$$

in the electromagnetic case. The variables in (35) are precisely the tangential components of the electric and magnetic fields. In the absence of surface current densities at the interface, the boundary conditions impose the continuity of those components (*Born and Wolf, 1964*).

The SH reflection-refraction problem was solved by *Carcione (1997)*. Carcione considered an incident homogeneous wave and a Zener model (e.g., *Carcione, 2001*) to describe the attenuation properties. In the case of an inhomogeneous incident wave and a general stiffness matrix **P**, the relevant equations can be summarized as follows:

2.3.1. Reflection and refraction coefficients

The particle velocities of the reflected and refracted waves are given by

$$v_y^R = \iota\omega\, R \exp\left[\iota\omega\left(t - s_x x - s_z^R z\right)\right] \tag{36}$$

and

$$v_y^T = \iota\omega\, T \exp\left[\iota\omega\left(t - s_x x - s_z^T z\right)\right] \tag{37}$$

respectively, and the reflection and refraction (transmission) coefficients are

$$R = \frac{Z^I - Z^T}{Z^I + Z^T}\ , \qquad T = \frac{2Z^I}{Z^I + Z^T}\ , \tag{38}$$

where

$$Z^I = p_{46} s_x + p_{44} s_z^I\ , \qquad Z^T = p'_{46} s_x + p'_{44} s_z^T\ , \tag{39}$$

with

$$s_x^R = s_x^T = s_x^I = s_x \quad \text{(Snell's law)}\ , \tag{40}$$

$$s_z^R = -\left(s_z^I + \frac{2p_{46}}{p_{44}} s_x \right)\ , \tag{41}$$

and

$$s_z^T = \frac{1}{p'_{44}}\left(-p'_{46} s_x + \mathrm{pv}\sqrt{\rho' p'_{44} - p'^2 s_x^2}\right)\ , \tag{42}$$

with

$$p'^2 = p'_{44} p'_{66} - p'^2_{46}\ . \tag{43}$$

For the principal value, the argument of the square root lies between $-\pi/2$ and $+\pi/2$. As indicated by *Krebes (1984)*, special care is needed when choosing the sign, since a wrong choice may lead to discontinuities of the vertical wavenumber as a function of the incidence angle.

2.3.2. Propagation, attenuation and ray angles

$$\tan\theta = \frac{\Re(s_x)}{\Re(s_z)} \qquad \tan\delta = \frac{\Im(s_x)}{\Im(s_z)} \qquad \tan\psi = \frac{\Re(X)}{\Re(Z)} \tag{44}$$

where

$$X^I = p_{66}s_x + p_{46}s_z^I$$

$$X^R = p_{66}s_x + p_{46}s_z^R$$

$$X^T = p'_{66}s_x + p'_{46}s_z^T \tag{45}$$

The ray angle denotes the direction of the power flow vector $\Re(\mathbf{p})$, as defined by *Auld (1990)*. A different definition is given in the context of complex ray theory (e.g., *Thomson, 1997*).

2.3.3. Energy flux balance

The balance of energy flux regards the continuity of the normal component of the Umov-Poynting vector across the interface. This is a consequence of the boundary conditions that impose continuity of normal stress σ_{zy} and particle velocity v_y. The balance of power flow at the interface, on a time-average basis, can be expressed as

$$\left\langle P^I \right\rangle + \left\langle P^R \right\rangle + \left\langle P^{IR} \right\rangle = \left\langle P^T \right\rangle , \tag{46}$$

where

$$\left\langle P^I \right\rangle = -\frac{1}{2}\Re\left(\sigma_{zy}^I v_y^{I^*}\right) = \frac{1}{2}\omega^2 \Re\left(Z^I\right) \exp\left[2\omega\Im(s_x)x\right] \tag{47}$$

is the incident flux,

$$\left\langle P^R \right\rangle = -\frac{1}{2}\Re\left(\sigma_{zy}^R v_y^{R^*}\right) = \frac{1}{2}\omega^2 |R|^2 \Re\left(Z^R\right) \exp\left[2\omega\Im(s_x)x\right] \tag{48}$$

is the incident flux,

$$\left\langle P^{IR} \right\rangle = -\frac{1}{2}\Re\left(\sigma_{zy}^I v_y^{R^*} + \sigma_{zy}^R v_y^{I^*}\right) = \omega^2 \Im(R)\Im\left(Z^I\right) \exp\left[2\omega\Im(s_x)x\right] \tag{49}$$

is the interference between the incident and reflected normal fluxes, and

$$\left\langle P^T \right\rangle = -\frac{1}{2}\Re\left(\sigma_{zy}^T v_y^{T^*}\right) = \frac{1}{2}\omega^2 |T|^2 \Re\left(Z^T\right) \exp\left[2\omega\Im(s_x)x\right] \tag{50}$$

is the refracted flux. In the elastic case, Z^I is real and the interference flux vanishes.

3. APPLICATION OF THE ANALOGY

On the basis of the solution of the elastic problem, we use the analogy to find the solution in the electromagnetic case. For every electromagnetic phenomenon – using the electromagnetic terminology – we analyze its corresponding mathematical and physical counterpart in the elastic case. *Maxwell (1891, p. 65)*, who used this approach, writes: *The analogy between the action of electromotive intensity in producing the displacement of an elastic body is so obvious that I have ventured to call the ratio of electromotive intensity to the corresponding electric displacement the coefficient of electric elasticity of the medium. ...*

The variations of electric displacements evidently constitute electric currents.

3.1. Refraction index and Fresnel formulae

Let us assume a lossless, isotropic medium. Isotropy implies $c_{44} = c_{66} = \mu$ and $c_{46} = 0$, and $\varepsilon_{11} = \varepsilon_{33} = \varepsilon$, and $\varepsilon_{13} = 0$. It is easy to show that the reflection and refraction coefficients reduce to

$$R = \frac{\sqrt{\rho\mu}\cos\theta^I - \sqrt{\rho'\mu'}\cos\theta^T}{\sqrt{\rho\mu}\cos\theta^I + \sqrt{\rho'\mu'}\cos\theta^T} \ , \quad \text{and} \quad T = \frac{2\sqrt{\rho'\mu'}\cos\theta^I}{\sqrt{\rho\mu}\cos\theta^I + \sqrt{\rho'\mu'}\cos\theta^T} \ , \tag{51}$$

respectively. From the analogy (equation (11)) and equation (8) we have

$$\mu^{-1} \Leftrightarrow \varepsilon \ , \tag{52}$$

The refraction index is defined as the velocity of light in vacuum, c_0, divided by the phase velocity in the medium, where the phase velocity is the reciprocal of the real slowness. For lossless, isotropic media the refraction index is

$$n = rc_0 = \sqrt{\frac{\mu_m \varepsilon}{\mu_0 \varepsilon_0}} \ , \tag{53}$$

where $c_0 = 1/\sqrt{\mu_0\varepsilon_0}$, and $\varepsilon_0 = 8.85\times10^{-12}$ F/m; $\mu_0 = 4\pi\times10^{-7}$ Hm. In elastic media there is not a limit velocity, but using the analogy we can define

$$n_e = k\sqrt{\frac{\rho}{\mu}} \ , \tag{54}$$

where k is a constant with the dimensions of velocity.

Assuming $\rho = \rho'$ in (51), the electromagnetic coefficients are

$$R = \frac{\sqrt{\varepsilon'}\cos\theta^I - \sqrt{\varepsilon}\cos\theta^T}{\sqrt{\varepsilon'}\cos\theta^I + \sqrt{\varepsilon}\cos\theta^T} \ , \quad \text{and} \quad T = \frac{2\sqrt{\varepsilon}\cos\theta^I}{\sqrt{\varepsilon'}\cos\theta^I + \sqrt{\varepsilon}\cos\theta^T} \ . \tag{55}$$

In terms of the refraction index we have

$$R = \frac{n' \cos\theta^I - n\cos\theta^T}{n' \cos\theta^I + n\cos\theta^T} \ , \quad \text{and} \quad T = \frac{2n\cos\theta^I}{n' \cos\theta^I + n\cos\theta^T} \ . \tag{56}$$

Equations (56) are Fresnel formulae (derived by Fresnel in 1923) corresponding to the electric field vector in the plane of incidence (*Born and Wolf, 1964, p. 40*). He assumed a magnetic permeability equal to one and zero conductivity.

3.2. Brewster (polarization) angle. No reflection

Fresnel formulae can be written in an alternative form, which may be obtained from (56) by using Snell's law

$$\frac{\sin\theta^I}{\sin\theta^T} = \sqrt{\frac{\mu}{\mu'}} = \frac{n'_e}{n_e} = \sqrt{\frac{\varepsilon'}{\varepsilon}} = \frac{n'}{n} \ . \tag{57}$$

It yields

$$R = \frac{\tan\left(\theta^I - \theta^T\right)}{\tan\left(\theta^I + \theta^T\right)} \ , \qquad \text{and} \qquad T = \frac{2\sin\theta^T\cos\theta^I}{\sin\left(\theta^I + \theta^T\right)\cos\left(\theta^I - \theta^T\right)} \ . \tag{58}$$

The denominators in (58) are finite, except when $\theta^I + \theta^T = \pi/2$. In this case the reflected and refracted rays are perpendicular to each other and $R = 0$. It follows from Snell's law that the incidence angle, $\theta^I = \theta_B$, satisfies

$$\tan\theta_B = \cot\theta^T = \sqrt{\frac{\mu}{\mu'}} = \frac{n'_e}{n_e} = \sqrt{\frac{\varepsilon'}{\varepsilon}} = \frac{n'}{n} \ . \tag{59}$$

The angle θ_B is called the Brewster angle, first noted by Étienne Malus and David Brewster (*Brewster, 1815*). It follows that the Brewster angle in elasticity can be obtained when the medium is lossless and isotropic, and the density is constant across the interface. This angle is also called polarization angle, because, as Brewster states, *When a polarised ray is incident at any angle upon a transparent body, in a plane at right angles to the plane of its primitive polarisation, a portion of the ray will lose its property of being reflected, and will entirely penetrate the transparent body. This portion of light, which has lost its reflexibility, increases as the angle of incidence approaches to the polarising angle, when it becomes a maximum..* Thus, at the polarizing angle, the electric vector of the reflected wave has no components in the plane of incidence. In the elastic case, we cannot define the same concept, since a plane wave whose particle velocity is not perpendicular to the plane of incidence (the symmetry plane) will undergo mode conversion.

The restriction about the density can be removed and the Brewster angle is

$$\tan\theta_B = \sqrt{\frac{\rho\mu/\mu' - \rho'}{\rho' - \rho\mu'/\mu}} \ , \tag{60}$$

but $\theta^I + \theta^T \neq \pi/2$, in this case (*Carcione, 1997*). The analogies (12) and (13) imply

$$\tan\theta_B = \sqrt{\frac{\mu_m'\varepsilon'/\varepsilon - \mu_m}{\mu_m - \mu_m'\varepsilon/\varepsilon'}} \tag{61}$$

in the electromagnetic case.

In the anisotropic and lossless case the angle is obtained from

$$\cot\theta_B = \left(-b \pm \sqrt{b^2 - 4ac}\right)\Big/(2a) \ , \tag{62}$$

where

$$a = c_{44}\left(\rho c_{44} - \rho' c_{44}'\right)/\rho \ , \qquad b = 2c_{46}a/c_{44} \ , \tag{63}$$

and

$$c = c_{46}^2 - c_{46}'^2 - c_{44}'\left(\rho' c_{66} - \rho c_{66}'\right)/\rho \ . \tag{64}$$

If $c_{46} = c_{46}' = 0$, we obtain

$$\tan\theta_B = \sqrt{\frac{c_{44}\left(\rho c_{44} - \rho' c_{44}'\right)}{c_{44}'\left(\rho' c_{66} - \rho c_{66}'\right)}} \ , \tag{65}$$

or, using the analogy,

$$c_{44}^{-1} \Leftrightarrow \varepsilon_{11} \ ,$$

$$c_{66}^{-1} \Leftrightarrow \varepsilon_{33} \ ,$$

$$\rho \Leftrightarrow \mu_m \ , \tag{66}$$

the electromagnetic Brewster angle is

$$\tan\theta_B = \frac{1}{\varepsilon_{11}}\sqrt{\frac{\varepsilon_{33}\varepsilon_{33}'\left(\mu_m\varepsilon_{11}' - \mu_m'\varepsilon_{11}\right)}{\mu_m'\varepsilon_{33}' - \mu_m\varepsilon_{33}}} \ . \tag{67}$$

In the viscoelastic case, tan θ_B is complex, in general, and there is no Brewster angle. However let us consider equation (61) and incident homogeneous waves. According to the analogy (19), its extension to the lossy case is

$$\tan\theta_B = \sqrt{\frac{\mu_m'\varepsilon^{*\prime}/\varepsilon^* - \mu_m}{\mu_m - \mu_m'\varepsilon^*/\varepsilon^{*\prime}}} \ . \tag{68}$$

The Brewster angle exists if $\varepsilon^{*\prime}$ is proportional to ε^*, for instance, if the conductivity of the refraction medium satisfies $\sigma' = (\varepsilon'/\varepsilon)\sigma$ ($\eta' = (\mu'/\mu)\eta$ in the elastic case). This situation is unlikely to occur in reality, unless the contact is designed for this purpose.

3.3. Critical angle. Total reflection

In isotropic, lossless media, total reflection occurs when Snell's law

$$\sin\theta^T = \sqrt{\frac{\rho\mu'}{\rho'\mu}}\sin\theta^I = \sqrt{\frac{\mu_m\varepsilon}{\mu'_m\varepsilon'}}\sin\theta^I \tag{69}$$

does not give a real value for the refraction angle θ^T. When the angle of incidence exceeds the critical angle θ_C defined by

$$\sin\theta^I = \sin\theta_C = \sqrt{\frac{\rho'\mu}{\rho\mu'}} = \frac{n'_e}{n_e} = \sqrt{\frac{\mu'_m\varepsilon'}{\mu_m\varepsilon}} = \frac{n'}{n} \ , \tag{70}$$

all the incident wave is reflected back into the incidence medium (*Born and Wolf, 1964, p. 47*). Note from equations (59) and (70) that $\tan\theta_B = \sin\theta_C$ if $\rho' = \rho$ and $\mu'_m = \mu_m$.

The critical angle is defined as the angle(s) of incidence beyond which the refracted Umov-Poynting vector is parallel to the interface. The condition $\mathrm{Re}(Z^T) = 0$ (see equation (21)) yields the critical angle θ_C. For the anisotropic, lossless case, with $c_{46} = c'_{46} = 0$, we obtain

$$\tan\theta_C = \sqrt{\frac{\rho' c_{44}}{\rho c'_{66} - c_{66}\rho'}} = \sqrt{\frac{\rho'_m\varepsilon_{33}\varepsilon'_{33}}{\varepsilon_{11}(\rho_m\varepsilon_{33} - \rho'_m\varepsilon'_{33})}} \tag{71}$$

where we have used the analogy.

In the isotropic and lossy case we have

$$\tan\theta_C = \sqrt{\frac{\rho'_m\varepsilon^{*\prime}}{\rho_m\varepsilon^* - \rho'_m\varepsilon^{*\prime}}} \ . \tag{72}$$

The critical angle exists if $\varepsilon^{*\prime}$ is proportional to ε^*, i.e., when the conductivity of the refraction medium satisfies $\sigma' = (\varepsilon'/\varepsilon)\sigma$.

3.3.1. Example

The acoustic properties of the incidence and refraction media are

$$c_{44} = 9.68 \text{ GPa}\ , \quad c_{66} = 12.5 \text{ GPa}\ , \quad \eta_{44} = 20\, c_{44}/\omega\ , \quad \eta_{66} = \eta_{44}\ , \quad \rho = 2000 \text{ kg/m}^3$$

and

$$c'_{44} = 25.6 \text{ GPa}\ , \quad c'_{66} = c'_{44}\ , \quad \eta'_{44} = \eta'_{66} = \infty\ , \quad \rho = 2500 \text{ kg/m}^3\ ,$$

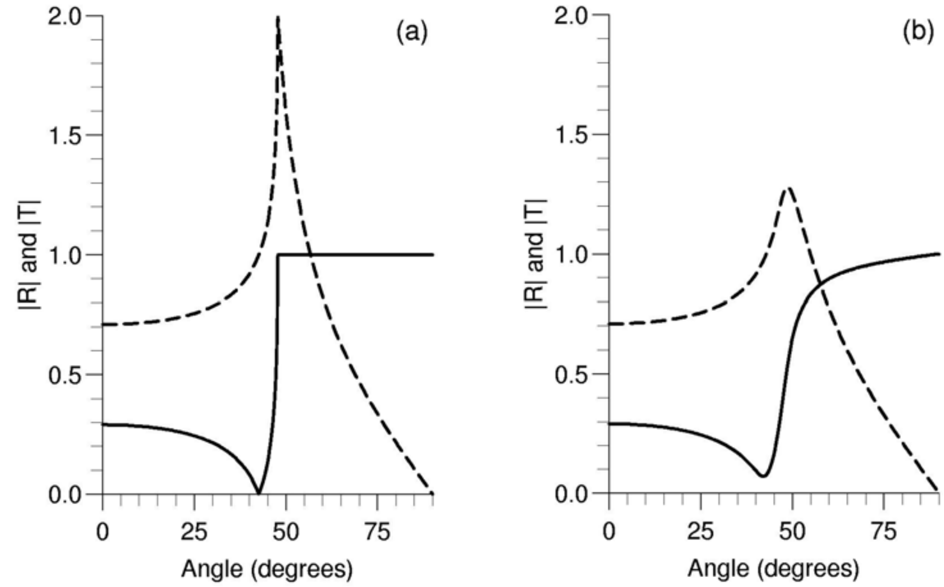

Fig. 2. Reflection and refraction coefficients (solid and dashed lines) for acoustic media – lossless (a) and (b) lossy cases.

respectively, where $\omega = 2\pi f$, with $f = 25$ Hz. The refraction medium is isotropic and lossless. The absolute value of the acoustic reflection and refraction coefficients – solid and dashed lines – are shown in Figures 2a and 2b for the lossless and lossy cases, respectively. The Brewster and critical angles are $\theta_B = 42.61°$ and $\theta_C = 47.76°$ (see Figure 2a), which can be verified from equations (65) and (71).

The electromagnetic properties of the incidence and refraction media are

$$\varepsilon_{11} = 3\,\varepsilon_0\,, \quad \varepsilon_{33} = 7\,\varepsilon_0\,, \quad \sigma_{11} = \sigma_{33} = 0.15 \text{ S/m}\,, \quad \mu_m = 2\,\mu_0$$

and

$$\varepsilon_{11} = \varepsilon_{33} = \varepsilon_0\,, \quad \sigma_{11} = \sigma_{33} = 0\,, \quad \mu_m = \mu_0\,,$$

respectively, where we consider a frequency f of 1 GHz. The refraction medium is vacuum. We apply the analogy

$$c_{44}^{-1} \Leftrightarrow \varepsilon_{11}\ ,$$

$$c_{66}^{-1} \Leftrightarrow \varepsilon_{33}\ ,$$

$$\eta_{44}^{-1} \Leftrightarrow \sigma_{11}\ ,$$

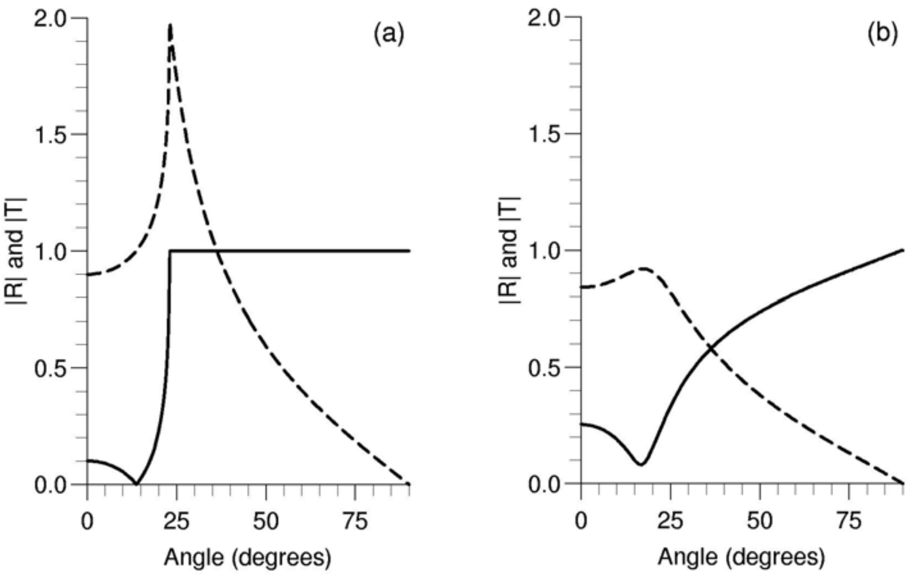

Fig. 3. Reflection and refraction coefficients (solid and dashed lines) for electromagnetic media – lossless (a) and (b) lossy cases.

$$\eta_{66}^{-1} \Leftrightarrow \sigma_{33} \ ,$$

$$\rho \Leftrightarrow \mu_m \ , \tag{73}$$

and use the same computer code to obtain the acoustic reflection and refraction coefficients. The absolute value of the electromagnetic reflection and refraction coefficients – solid and dashed lines – are shown in Figures 3a and 3b for the lossless and lossy cases, respectively. The Brewster and critical angles are $\theta_B = 13.75°$ and $\theta_C = 22.96°$ (see Figure 3a), which can be verified from equations (67) and (71).

3.4. Reflectivity and transmissivity

Equation (46) is the balance of energy flux across the interface. After substitution of the fluxes (47) – (50), we obtain

$$\Re\left(Z^I\right) = -\Re\left(Z^R\right)|R|^2 + \Re\left(Z^T\right)|T|^2 - 2\Im\left(Z^I\right)\Im(R) \ . \tag{74}$$

Let us consider the isotropic lossy case and an incident homogeneous wave. Thus, $p_{46} = 0$, $p_{44} = p_{66} = \mu^*$, and equations (22) and (31) imply $Z = \sqrt{\rho\mu^*}\cos\theta$. Then, the balance (74) becomes

$$\Re\left(\sqrt{\rho\mu^*}\right)\cos\theta^I = |R|^2\,\Re\left(\sqrt{\rho\mu^*}\right)\cos\theta^I + |T|^2\,\Re\left[\mathrm{pv}\left(\sqrt{\rho'\mu'^*}\sqrt{1-\frac{\rho\mu'^*}{\rho'\mu^*}\sin^2\theta^I}\right)\right]$$

$$-2\Im(R)\,\Im\left(\sqrt{\rho\mu^*}\right)\cos\theta^I \qquad (75)$$

where we have used that $Z^R = -Z^I$. For lossless media the interference flux – the last term on the right-hand side – vanishes, because μ^* is real. Moreover, using Snell's law (69) we obtain

$$1 = \mathcal{R} + \mathcal{T} \qquad (76)$$

where

$$\mathcal{R} = |R|^2 \ , \qquad \text{and} \qquad \mathcal{T} = \sqrt{\frac{\rho'\mu'}{\rho\mu}}\,\frac{\cos\theta^T}{\cos\theta^I}\,|T|^2 \qquad (77)$$

are called the reflectivity and transmissivity, respectively. Using the analogy (52) and assuming $\rho = \rho'$ and $\mu'_m = \mu_m$, we obtain

$$\mathcal{T} = \frac{n'_e}{n_e}\,\frac{\cos\theta^T}{\cos\theta^I}\,|T|^2 = \frac{n'}{n}\,\frac{\cos\theta^T}{\cos\theta^I}\,|T|^2 \qquad (78)$$

(*Born and Wolf, 1964, p. 41*).

3.5. Dual fields

The reflection and refraction coefficients that we have obtained above correspond to the particle-velocity field or, to be more precise, to the displacement field (due to the factor $\iota\omega$ in equation (15)). In order to obtain the reflection coefficients for the stress components, we should make use of the constitutive equations, which for the plane wave (15) are

$$\sigma_{xy} = -Xv_y \ , \qquad \text{and} \qquad \sigma_{yz} = -Zv_y \ , \qquad (79)$$

where X and Z are defined in equation (22). Let us consider the reflected wave. Then, combining equation (79) and (36) we obtain

$$\sigma_{xy}^R = R_{xy}\exp\left[\iota\omega\left(t - s_x x - s_z^R z\right)\right] \ ,$$

$$\sigma_{zy}^R = R_{yz}\exp\left[\iota\omega\left(t - s_x x - s_z^R z\right)\right] \ , \qquad (80)$$

where

$$R_{xy} = -\iota\omega X^R R \ , \qquad \text{and} \qquad R_{yz} = -\iota\omega Z^R R \ , \qquad (81)$$

are the stress reflection coefficients.

In isotropic and lossless media we have

$$R_{xy} = -\iota\omega\sqrt{\rho\mu}\sin\theta^I R \ , \qquad \text{and} \qquad R_{yz} = \iota\omega\sqrt{\rho\mu}\cos\theta^I R \ . \tag{82}$$

The analogies (9), (13) and (52) imply

$$E_z = -\iota\omega\sqrt{\mu_m\varepsilon}\sin\theta^I R \ , \qquad \text{and} \qquad E_x = -\iota\omega\sqrt{\mu_m\varepsilon}\cos\theta^I R \tag{83}$$

(*Born and Wolf, 1964, p. 39*).

3.6. Back to acoustics. Sound waves

There is a mathematical analogy between the TM equations (11) and a modified version of the so-called acoustic wave equation for fluids. Denoting the pressure field by *p*, the modified acoustic equations can be written as

$$\frac{\partial v_z}{\partial z} + \frac{\partial v_x}{\partial x} = -\kappa\frac{\partial p}{\partial t} \ , \tag{84}$$

$$-\frac{\partial p}{\partial z} = \gamma v_z + \rho\frac{\partial v_z}{\partial t} \ , \tag{85}$$

$$-\frac{\partial p}{\partial x} = \gamma v_x + \rho\frac{\partial v_x}{\partial t} \ , \tag{86}$$

where κ is the fluid compressibility, and $\gamma = 0$ yields the standard acoustic equations of motion. Equations (84) – (86) correspond to a generalized density of the form

$$\hat{\rho}(t) = \gamma I(t) + \rho H(t) \ , \tag{87}$$

where $H(t)$ is the Heaviside function and $I(t)$ is the integral operator. The acceleration term for, say, the *x*-component is

$$\frac{\partial\hat{\rho}(t)}{\partial t} * \frac{\partial v_x}{\partial t} = \gamma v_x + \rho\frac{\partial v_x}{\partial t} \ . \tag{88}$$

Equations (84) – (86) are mathematically analogous to the isotropic-medium electromagnetic equations (1) – (3) for the following correspondence

$$H_y \Leftrightarrow -p$$

$$v_x \Leftrightarrow E_z$$

$$v_z \Leftrightarrow -E_x$$

$$\varepsilon \Leftrightarrow \rho$$

$$\sigma \Leftrightarrow \gamma$$

$$\mu_m \Leftrightarrow \kappa \ , \tag{89}$$

where $M_y = 0$ has been assumed. Let us assume a lossless electromagnetic medium, and consider Snell's law (69) and the analogy between the SH case and the TM case. That is, transform equation (51) to the TM equations by using the analogies $\mu^{-1} \Leftrightarrow \varepsilon$ and $\rho \Leftrightarrow \mu_m$. In order to apply the mathematical analogies correctly, we need to recast the reflection coefficients only as a function of the material properties and incidence angle. We get

$$R = \frac{\sqrt{\frac{\mu_m}{\varepsilon}} \cos\theta^I - \sqrt{\frac{\mu'_m}{\varepsilon'}} \sqrt{1 - \frac{\mu_m \varepsilon}{\mu'_m \varepsilon'} \sin^2\theta^I}}{\sqrt{\frac{\mu_m}{\varepsilon}} \cos\theta^I + \sqrt{\frac{\mu'_m}{\varepsilon'}} \sqrt{1 - \frac{\mu_m \varepsilon}{\mu'_m \varepsilon'} \sin^2\theta^I}} \ . \tag{90}$$

If $\kappa^{-1} = \rho c^2$, where c is the sound wave velocity, application of the analogy (89) to equation (90) implies

$$R = \frac{\rho' c' \cos\theta^I - \rho c \cos\theta^T}{\rho' c' \cos\theta^I + \rho c \cos\theta^T} \ , \tag{91}$$

where we have used the acoustic Snell's law

$$\frac{\sin\theta^I}{c} = \frac{\sin\theta^T}{c'} \ . \tag{92}$$

If we assume $\rho = \rho'$ and use Snell's law, we obtain

$$R = \frac{\sin\left(\theta^T - \theta^I\right)}{\sin\left(\theta^T + \theta^I\right)} \ , \tag{93}$$

which is the reflection coefficient for light polarized perpendicular to the plane of incidence (the electric vector perpendicular to the plane of incidence), as we shall see in the next section. However, note that we started from the TM equation, corresponding to the electric vector lying in the plane of incidence.

3.7. The TM-TE analogy

The lossless TE (transverse electric field) differential equations for an isotropic material are

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \varepsilon \frac{\partial E_y}{\partial t} \ , \tag{94}$$

$$\frac{\partial E_y}{\partial z} = \mu_m \frac{\partial H_x}{\partial t} \ , \tag{95}$$

$$-\frac{\partial E_x}{\partial z} = \mu_m \frac{\partial H_z}{\partial t} \tag{96}$$

(a more general analogy can be obtained by assuming electric sources and magnetic relaxation, but we do not consider these terms for clarity). The isotropic version of equations (1) – (3) and (94) – (96) are mathematically analogous for the following correspondence

$$\mathrm{TM} \Leftrightarrow \mathrm{TE}$$

$$H_y \Leftrightarrow -E_y$$

$$E_x \Leftrightarrow H_x$$

$$E_z \Leftrightarrow H_z$$

$$\varepsilon \Leftrightarrow \mu_m$$

$$\mu_m \Leftrightarrow \varepsilon \ . \tag{97}$$

From equation (90), and using the analogy (97) and Snell's law (69), the TE reflection coefficient is

$$R = \frac{\sqrt{\dfrac{\varepsilon}{\mu_m}}\cos\theta^I - \sqrt{\dfrac{\varepsilon'}{\mu'_m}}\cos\theta^T}{\sqrt{\dfrac{\varepsilon}{\mu_m}}\cos\theta^I + \sqrt{\dfrac{\varepsilon'}{\mu'_m}}\cos\theta^T} \ . \tag{98}$$

Assuming $\mu'_m = \mu_m$ and using again Snell's law, we obtain

$$R = \frac{\sin\left(\theta^T - \theta^I\right)}{\sin\left(\theta^T + \theta^I\right)} \ . \tag{99}$$

This is the reflection coefficient for the electric field component E_y, i.e., light polarized perpendicular to the plane of incidence. Note that R for H_y (equation (58)) and R for E_y have different functional dependences in terms of the incidence and refraction angles.

3.8. Green's analogies

On December 11, 1837 Green read two papers to the Cambridge Philosophical Society. The first paper (*Green, 1838*) makes the analogy between sound waves and light waves polarized in the plane of incidence. In order to obtain his analogy we establish the following correspondence between the acoustic equations (84) – (86) and the TE equations (94) – (96):

$$E_y \Leftrightarrow -p$$

$$H_x \Leftrightarrow v_z$$

$$H_z \Leftrightarrow -v_x$$

$$\varepsilon \Leftrightarrow \kappa$$

$$\mu_m \Leftrightarrow \rho \ . \tag{100}$$

Using Snell's law (69), the TE reflection coefficient (98) can be rewritten as

$$R = \frac{\sqrt{\dfrac{\varepsilon}{\mu_m}}\cos\theta^I - \sqrt{\dfrac{\varepsilon'}{\mu'}}\sqrt{1-\dfrac{\mu_m\varepsilon}{\mu'_m\varepsilon'}\sin^2\theta^I}}{\sqrt{\dfrac{\mu_m}{\varepsilon}}\cos\theta^I + \sqrt{\dfrac{\varepsilon'}{\mu'}}\sqrt{1-\dfrac{\mu_m\varepsilon}{\mu'_m\varepsilon'}\sin^2\theta^I}} \tag{101}$$

If we apply the analogy (100) to this equation and Snell's law (92), we obtain equation (91). Green obtains the reflection coefficient for the potential field, and assumes $\kappa = \kappa'$ or

$$\frac{\rho c}{\rho' c'} = \frac{c'}{c} \ . \tag{102}$$

Using this condition and Snell's law (92) to equation (91), we obtain

$$R = \frac{\sin\theta^I \cos\theta^I - \sin\theta^T \cos\theta^T}{\sin\theta^I \cos\theta^I + \sin\theta^T \cos\theta^T} = \frac{\tan\left(\theta^I - \theta^T\right)}{\tan\left(\theta^I + \theta^T\right)} \ , \tag{103}$$

which is the same ratio as for light polarized in the plane of incidence. *Green (1838)* has the opposite convention for describing the polarization direction. i.e., his convention is to denote R as given by equation (103) as the reflection coefficient for light polarized perpendicular to the plane of incidence. Conversely, he consider the reflection coefficient (93) to correspond to light polarized in the plane of incidence. This is a convention dictated probably by the experiments performed, for instance, by Malus, *Brewster (1815)* and Faraday, since Green did not know that light is an electromagnetic phenomenon related to the electric and magnetic fields – this relation was discovered by Maxwell nearly 30 years later (*Maxwell, 1865*). Note that different assumptions lead to the different electromagnetic reflection coefficients. Assuming $\rho = \rho'$, we obtain the reflection coefficient for light polarized perpendicular to the plane of incidence (equation (93)), and assuming $\kappa = \kappa'$, we obtain the reflection coefficient light polarized in the plane of incidence (equation (103)), Green's second paper (*Green, 1942*), read to the Cambridge Philosophical Society in December 1837, is an attempt to obtain the electromagnetic reflection coefficients by using the equations of elasticity (isotropic case). Firstly, he considers the SH wave equation (Green's equations (7) and (8)) and the boundary conditions for the case $\mu = \mu'$ (his equation (9)). He obtains equation (51) for the displacement reflection coefficient. If we use the condition (102) and Snell's law (92), we obtain precisely equation (93). i.e., the reflection coefficient for light polarized perpendicular to the plane of incidence – in the plane of incidence according to Green.

Secondly, Green considers the P-SV equation of motion in terms of the potential fields (Green's equations (14) and (16)), and makes the following assumptions

$$\rho c_P^2 = \rho' c_P'^2 \ , \qquad \rho c_S^2 = \rho' c_S'^2 \ , \tag{104}$$

that is, the plane-wave and shear moduli are the same for both media. This condition implies

$$\frac{c_P}{c_S} = \frac{c_P'}{c_S'} \ , \tag{105}$$

which means that both media have the same Poisson's ratio. Conversely, relation (105) implies that the P-wave and the S-wave velocity contrasts are similar:

$$\frac{c_P}{c_P'} = \frac{c_S}{c_S'} \equiv \alpha \ . \tag{106}$$

Green is aware – on the basis of experiments – that waves with polarization perpendicular to the wave front were not observed experimentally. He writes: *But in the transmission of light through a prism, though the wave which is propagated by normal vibrations were incapable itself of affecting the eye, yet it would be capable of giving rise to an ordinary wave of light propagated by transverse vibrations....* He is then constrained to assume that $c_P >> c_S$, that is, according to his own words, *that in the luminiferous ether, the velocity of transmission of waves propagated by normal vibrations, is very great compared with that of ordinary light*. The implications of this constraint will be clear below.

The reflection coefficient obtained by *Green (1842)*, for the shear potential and an incident shear wave, has the following expression using our notation:

$$R^2 = \frac{\left(\alpha^2+1\right)^2 \left(\alpha^2 - \dfrac{s_{zS}^T}{s_{zS}^I}\right)^2 + \left(\alpha^2-1\right)^4 \dfrac{s_x^2}{s_{zS}^{I2}}}{\left(\alpha^2+1\right)^2 \left(\alpha^2 + \dfrac{s_{zS}^T}{s_{zS}^I}\right)^2 + \left(\alpha^2-1\right)^4 \dfrac{s_x^2}{s_{zS}^{I2}}} \tag{107}$$

(Green's equation (26)), where s_{zS}^I and s_{zS}^T are the vertical components of the slowness vector corresponding to the S wave. On the basis of the condition $c_P >> c_S$, Green assumed that the vertical components of the slowness vector corresponding to the incident, reflected and refracted P waves satisfy

$$\imath s_{zP}^I = -\imath s_{zP}^R = \imath s_{zP}^T = s_x \ . \tag{108}$$

These relations can be obtained from the dispersion relation $s_x^2 + s_z^2 = \omega / c_P^2$ of each wave assuming $c_P \to \infty$. This assumption gives an incompressible medium and inhomogeneous P waves confined at the interface. The complete expression for the SS reflection coefficients are given, for instance, in *Pilant (1979, p. 137)*. He defines $a = c_S / c_P$ and $c = c_S / c'_P$. Green's solution (107) is obtained for $a = c = 0$. Note a mistake in Pilant's equation (12-21): the (43) coefficient of matrix Δ_s should be $-2\sin\theta_{S1}\sqrt{c^2 - \sin^2\theta_{S1}} \Big/ \left(b^2 d\right)$ instead of $-2\sin\theta_{S1}\sqrt{a^2 - \sin^2\theta_{S1}} \Big/ \left(b^2 d\right)$. However, this mistake does not affect the approximate solution.

The vertical components of the shear slowness vector are given by

$$s_{zS}^I = \sqrt{\frac{1}{c_S^2} - s_x^2} \ , \qquad s_{zS}^T = \sqrt{\frac{1}{c_S'^2} - s_x^2} \ . \tag{109}$$

However, equation (107) is not Fresnel's equation. To obtain this equation, Green assumes that $\alpha \approx 1$; in his own words: *When the refractive power in passing from the upper to the lower medium is not very great*, α (μ using his notation) *does not differ much from 1*. The result of applying this approximation to equation (107) is

$$R = \frac{\alpha^2 - \dfrac{s_{zS}^T}{s_{zS}^I}}{\alpha^2 + \dfrac{s_{zS}^T}{s_{zS}^I}} \ . \tag{110}$$

If θ^I is the incidence angle of the shear wave and θ^T is the angle of the refracted shear wave, equation (106), Snell's law and the relation

$$\frac{s_{zS}^T}{s_{zS}^I} = \frac{\cot\theta^T}{\cot\theta^I} \tag{111}$$

(which can be obtained by using equation (109) and Snell's law), yield

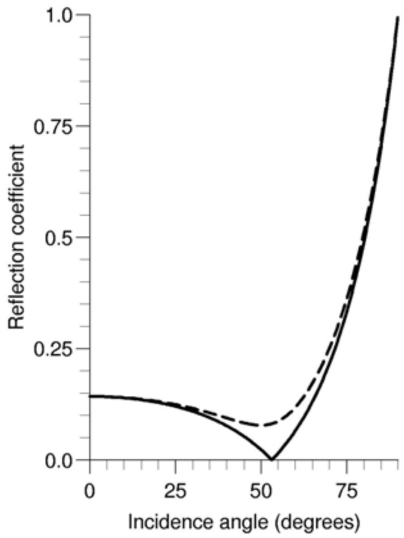

Fig. 4. Green's reflection coefficient for light polarized in the plane of incidence (dashed line) and corresponding Fresnel's reflection coefficient (continuous line).

$$R = \frac{\dfrac{\sin^2\theta^I}{\sin^2\theta^T} - \dfrac{\cot\theta^T}{\cot\theta^I}}{\dfrac{\sin^2\theta^I}{\sin^2\theta^T} + \dfrac{\cot\theta^T}{\cot\theta^I}} = \frac{\sin 2\theta^I - \sin 2\theta^T}{\sin 2\theta^I + \sin 2\theta^T} = \frac{\tan\left(\theta^I - \theta^T\right)}{\tan\left(\theta^I + \theta^T\right)} , \tag{112}$$

which is the reflection coefficient for light polarized in the plane of incidence. Green considers that equation (112) is an approximation of the observed reflection coefficients. He claims, on the basis of experimental data, that *the intensity of the reflected light never becomes absolutely null, but attains a minimum value*. Moreover, he calculates the minimum value of the reflection coefficient and obtains

$$R_{\min}^2 = \frac{\left(\alpha^2 - 1\right)^4}{4\alpha^2\left(\alpha^2 + 1\right)^2 + \left(\alpha^2 - 1\right)^4} , \tag{113}$$

which using the approximation $\alpha \approx 1$ gives zero reflection coefficient. This minimum value corresponds to the Brewster angle when using the Fresnel equation (112). Green assumes $\alpha = 4/3$ for air-water interface. The absolute values of the reflection coefficient R given by equations (107) and (112) are shown in Figure 4. The dashed line correspond to equation (107). We have assumed $c_S = 30$ cm/ns and $c'_S = c_S/\alpha$. At the Brewster angle $\theta = \text{atan}(\alpha)$, Green obtains a minimum value $R_{min} = 0.08138$.

The non-existence of the Brewster angle (zero reflection coefficient), can be explained by the presence of dissipation (ionic conductivity effects), as can be seen in Figure 3b. Green attributes this to the fact that the refraction medium is highly refracting. Quoting him: *This minimum value [R_{min}] increases rapidly, as the index of refraction increases, and thus the quantity of light reflected at the polarizing [Brestwer] angle, becomes considerable for highly refracting substances, a fact which has been long known to experimental philosophers (Green, 1842)*. For instance, fresh water is almost lossless and is a less refracting medium than salt water, which has a higher conductivity.

3.9. Brief historical review

We have seen in the previous section that Green's theory of refraction does not provide an exact parallel with the phenomenon of light propagation. MacCullagh (Trans. Roy. Irish. Acad., xxi, 1848; *Whittaker, 1987, p. 141*) presented an alternative approach to the Royal Irisih Academy in 1839. He devised an isotropic medium, whose potential energy is only based on rotation of the volume elements, thus ignoring pure dilatations from the beginning. The result is a rotationally elastic ether and the wave equation for shear waves. The corresponding reflection and refraction coefficients coincide with Fresnel's equations.

Green (1842) assumed the P-wave velocity to be infinite and dismissed a zero P-wave velocity on the basis that the medium would be unstable (the potential energy must be positive). Cauchy (Comptes Rendus, ix (25 Nov. 1839), p. 676, and (2 Dec. 1839), p. 726; *Whittaker, 1987, p. 145*), neglecting this fact, considered that P-waves have zero velocity, and obtained the sine law and tangent law of Fresnel. He assumed the shear modulus to be the same for both media. Cauchy's ether is known as the *contractile or labile ether*. It corresponds to an elastic medium of negative compressibility. The P-wave dispersion relation for this medium is $s_x^2 + s_z^2 = 0$, which leads to an infinite vertical slowness. This condition confines the propagation direction of the compressional waves to be normal to the interface. The energy carried away by the P waves is negligible, since no work is required to generate a dilatational displacement, due to the negative value of the compressibility. If we assume the shear modulus of both media to be the same (the differences depend on density contrasts only), we obtain Fresnel's equations. The advantage of the labile ether is that it overcomes the difficulty of requiring continuity of the normal component of the displacement at the interface. Light waves do not satisfy this condition, but light waves plus dilatational vibrations, taken together, do satisfy the condition.

4. CONCLUSIONS

Scientists of the 19th century made frequent use of analogies to solve problems of different physical nature. In many cases, this practice lead to important discoveries. For instance, Fresnel's equations and Maxwell's equations were obtained from mathematical analogies – and physical analogies to a lesser degree – with shear wave propagation and Hooke's law.

We have solved the electromagentic reflection-refraction problem by using the analogy between cross-plane shear waves in the symmetry plane of a monoclinic medium and transverse-magnetic waves. A mathematical analogy exists also between electromagentic waves and sound waves (in the isotropic case), and between TM and TE electromagnetic waves. Illustrative examples are given in the papers by George Green, who used the analogy between sound waves and light waves, and elastic waves and light waves.

The analogy constitutes a mathematical equivalence that allows the acoustic and electromagnetic problems to be solved with the same analytical methodology. The most powerful application of the analogy is the use of the same computer code to solve acoustic and electromagnetic propagation problems in general inhomogeneous media.

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