TORSIONAL OSCILLATIONS OF ANISOTROPIC HOLLOW CIRCULAR CYLINDERS*

José M. Carcione and Géza Seriani
Osservatorio Geofisico Sperimentale, P.O. Box 2011 Opicina, 34016 Trieste, Italy.

Abstract

We obtain the dispersion equation for torsional axially symmetric harmonic waves propagating in an infinitely long anisotropic circular cylinder. The material is transversely isotropic, with its symmetry axis coincident with the axial axis of the cylinder. In particular, we study the phase and group velocity of the torsional modes in terms of the material properties, and corresponding sizes (radii). The fundamental mode is not dispersive and has the vertical velocity of shear body waves c. The phase velocity of the dispersive modes is infinite for infinite wavelengths, and for short wavelengths it is the vertical velocity c. On the other hand, the group velocity is always smaller than c. The differences between the isotropic and the anisotropic case can be substantial for tubes made of iron or zinc.

1 Introduction

The study of the dynamics of hollow cylinders and tubes has many applications, e.g., guided-wave ultrasonic delay lines, shells used as components in aircraft, missiles, solid-propellant rocket motors, etc. In the exploration industry, interest resides in the propagation of pulses through drill strings. These pulses are used as pilot signals for the data processing of seismograms (recorded at the surface) generated by the roller cone bit [1, 2]. Since the materials that make up the tubes (metals, in general) are anisotropic, the use of an isotropic constitutive equation may produce erroneous results.

In this letter, we compute the phase and group velocities of torsional oscillations propagating in an anisotropic hollow cylinder. The wave equation combines the equation of momentum conservation with the constitutive relations for transversely isotropic media. The problem is solved in cylindrical coordinates (r, φ, z) and an axially symmetric hollow cylinder of interior and exterior radii a and b is assumed. This implies that the symmetry axis of the medium coincides with the axial axis of the cylinder (z-axis). In this case, the wave field does not depend on the azimuthal variable φ.

In the absence of body forces, the equations describing the motion of torsional waves are

$$\rho \partial_{tt} u_\varphi = \partial_r \sigma_{r\varphi} + \partial_z \sigma_{\varphi z} + \frac{2}{r} \sigma_{r\varphi} , \tag{1}$$

$$\sigma_{r\varphi} = c_{66} \left(\partial_r u_\varphi - \frac{u_\varphi}{r} \right), \text{ and} \tag{2}$$

$$\sigma_{\varphi z} = c_{44} \partial_z u_\varphi , \tag{3}$$

where u_φ is the displacement component, $\sigma_{r\varphi}$ and $\sigma_{\varphi z}$ are the stress components, ρ is the density, and c_{44} and c_{66} are the elastic components. The symbol ∂ denotes partial differentiation with respect to the corresponding subindex, and t is the time variable.

Since the torsional waves are decoupled from the quasi-compressional and quasi-shear motions, they can be described, as in the isotropic case, by a potential function ψ,

$$u_\varphi = - \partial_r \psi . \tag{4}$$

Substituting the stresses into the conservation equation (1), and using (4), we obtain the following wave equation:

$$\rho \partial_{tt} \psi = c_{44} \partial_{zz} \psi + c_{66} \left(\partial_{rr} \psi + \frac{1}{r} \partial_r \psi \right). \tag{5}$$

Note that in the isotropic case $c_{44} = c_{66} = \mu$, the Lamè constant, equation (5) becomes $(\Delta - \beta^{-2} \partial_{tt})\psi = 0$, where $\beta = \sqrt{u/\rho}$, and

$$\Delta = \partial_{rr} + \frac{1}{r} \partial_r + \partial_{zz} \tag{6}$$

is the Laplacian operator.

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2. The solution

The steady-state solution has the form

$$\psi = F(r, z)\exp(\iota\omega t)\,, \tag{7}$$

where ω is the angular velocity and $\iota = \sqrt{-1}$. Substitution of equation (7) into equation (5) gives the generalised Helmholtz equation

$$\frac{1}{\alpha^2 r}\partial_r(r\partial_r F) + \partial_{zz}F + \frac{\omega^2}{c^2}F = 0\,, \tag{8}$$

where

$$\alpha = \sqrt{\frac{c_{44}}{c_{66}}} \tag{9}$$

and $c = \sqrt{c_{44}/\rho}$ is the body wave phase velocity along the symmetry axis of the medium. Assuming $F(r, z) = R(r)Z(z)$, we separate variables and obtain the differential equations

$$r^2\partial_{rr}R + r\partial_r R + (\kappa r)^2 R = 0\,, \tag{10}$$

$$\partial_{zz}Z + \gamma^2 Z = 0\,, \tag{11}$$

where

$$\kappa = \alpha\left(\frac{\omega^2}{c^2} - \gamma^2\right)^{1/2}. \tag{12}$$

The quantities κ and γ are separation constants, and correspond to the radial and vertical wavenumbers. The axially symmetric general solutions of equations (10) and (11) are

$$R(r) = AJ_0(\kappa r) + BY_0(\kappa r) \tag{13}$$

and

$$Z(z) = C\exp(\iota\gamma z) + D\exp(-\iota\gamma z)\,, \tag{14}$$

respectively, where J_0 and Y_0 are Bessel functions of the first and second kinds respectively, and $A,\ldots,D$ are arbitrary constants.

The general solution for harmonic waves along the positive z-direction can be written as

$$\psi(r, z, t; \gamma, \omega) = \left[A_0 J_0(\kappa r) + B_0 Y_0(\kappa r)\right]\exp\left[\iota(\omega t - \gamma z)\right]. \tag{15}$$

Substituting this potential into equations (2) and (4) we obtain

$$u_\varphi = -\left[A_0\partial_r J_0(\kappa r) + B_0\partial_r Y_0(\kappa r)\right]\exp\left[\iota(\omega t - \gamma z)\right] \tag{16}$$

and

$$\sigma_{r\varphi} = -c_{44}\left[A_0\left(\partial_r^2 J_0(\kappa r) - \frac{\partial_r J_0(\kappa r)}{r}\right) + B_0\left(\partial_r^2 Y_0(\kappa r) - \frac{\partial_r Y_0(\kappa r)}{r}\right)\right]\exp\left[\iota(\omega t - \gamma z)\right]. \tag{17}$$

At the inner and outer surfaces of the cylinder we have the following boundary conditions

$$\sigma_{r\varphi}(r = a) = 0 \quad\text{and}\quad \sigma_{r\varphi}(r = b) = 0\,, \tag{18}$$

respectively. This implies that

$$A_0\left[\partial_{rr}J_0(\kappa a) - \frac{\partial_r J_0(\kappa a)}{a}\right] + B_0\left[\partial_{rr}Y_0(\kappa a) - \frac{\partial_r Y_0(\kappa a)}{a}\right] = 0 \tag{19}$$

and

$$A_0\left[\partial_{rr}J_0(\kappa b) - \frac{\partial_r J_0(\kappa b)}{b}\right] + B_0\left[\partial_{rr}Y_0(\kappa b) - \frac{\partial_r Y_0(\kappa b)}{b}\right] = 0\,. \tag{20}$$

The period equation follows by making zero the determinant of the linear system. Moreover, using the properties of the cylinder functions (e.g., [3], p.694), we obtain

$$\det\begin{bmatrix} \frac{2}{a}J_1(\kappa a) - \kappa J_0(\kappa a) & \frac{2}{a}Y_1(\kappa a) - \kappa Y_0(\kappa a) \\ \frac{2}{b}J_1(\kappa b) - \kappa J_0(\kappa b) & \frac{2}{b}Y_1(\kappa b) - \kappa Y_0(\kappa b) \end{bmatrix} = 0\,, \tag{21}$$

or

$$J_2(\kappa a)Y_2(\kappa b) - J_2(\kappa b)Y_2(\kappa a) = 0\,. \tag{22}$$

A similar period equation was obtained by Gazis [4] for an isotropic hollow cylinder, for which $\alpha = 1$. This result and equation (12) indicate that the period equation of an anisotropic cylinder of interior and exterior radii a and b is the same as the period equation of an isotropic cylinder of interior and exterior radii αa and αb, and rigidity $\mu = c_{44}$.

The velocity of the lowest torsional mode is not appropriately obtained from equation (22). This mode corresponds to $\kappa = 0$, and the displacement to a rotation of each transverse section of the cylinder as a whole about its center. This motion is not dispersive and both phase and group velocities are equal to c, the body wave phase velocity along the vertical direction.

The velocities of the dispersive modes are obtained from the roots of equation (22) that can be written as $P(c_p, \omega) = 0$ where $c_p = \omega/\gamma$ is the phase velocity. Assume that the roots of (22) are $\kappa_1, \kappa_2,\ldots, \kappa_j,\ldots$. Then, the phase velocity corresponding to the j mode is

$$c_p = \frac{\omega}{\gamma} = c\left[1+\left(\frac{\kappa_j\lambda}{2\pi\alpha}\right)^2\right]^{1/2}, \tag{23}$$

where $\lambda = 2\pi/\gamma$ is the wavelength. The group velocity is equal to $c_g = d\omega/d\gamma$, which can easily be calculated from equation (12). It yields

$$c_g = \frac{c^2}{c_p}. \tag{24}$$

The phase velocity is infinite for infinite wavelengths, and for short wavelengths it is the body wave velocity c. On the other hand, the group velocity is always smaller than c.

We consider two materials: iron, with $c_{44} = 116$GPa, $\rho = 7.87$g/cm^3 and $\alpha = 1.55$; and zinc, with $c_{44} = 39.6$GPa, $\rho = 7.14$g/cm^3 and $\alpha = 0.77$. The calculations were carried out for a thickness to mean-radius ratio $h/R = 1/4$, where $h = b-a$ and $R = (b+a)/2$. Figures 1 and 2 represent the phase (a) and group (b) velocities versus wavelength for torsional oscillations. The first and third modes are shown, with the broken lines corresponding to the isotropic case ($\mu = c_{44}$).

The differences between the isotropic and the anisotropic case can be substantial. For instance, consider the group velocity of the first mode for iron (Figure 1b), and a wavelength equal to twice the tube thickness ($h/\lambda = 1/2$). Since $c = 3839$m/s, the velocity

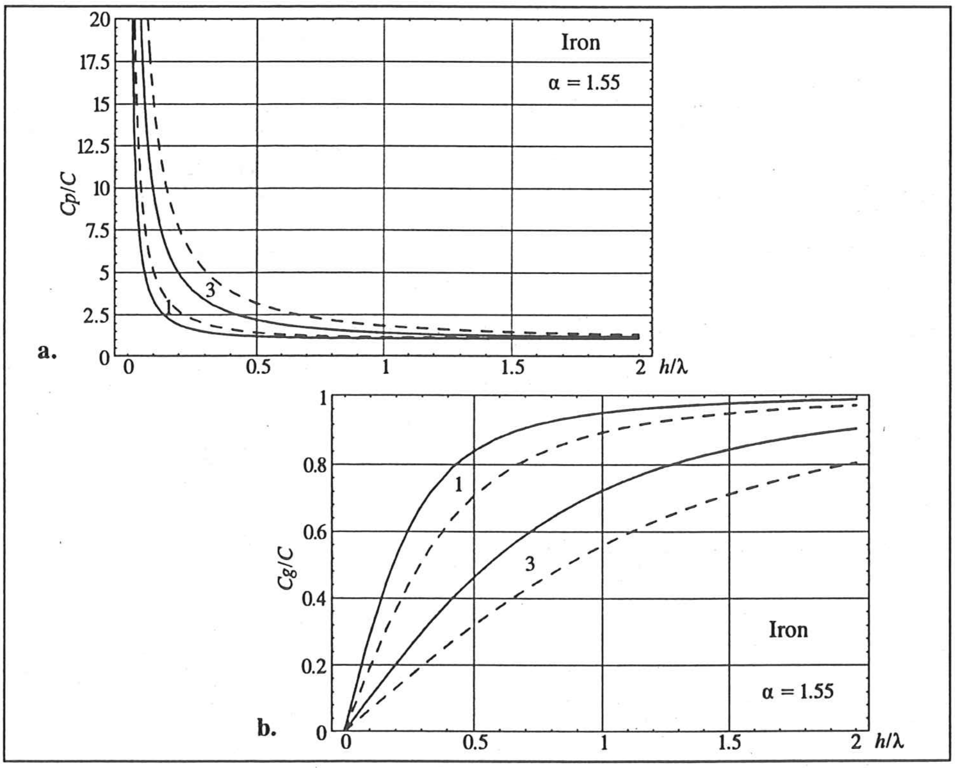

Figure 1. Phase **a.** and group **b.** velocities of torsional waves in an iron tube (first and third modes). The broken lines correspond to the isotropic case.

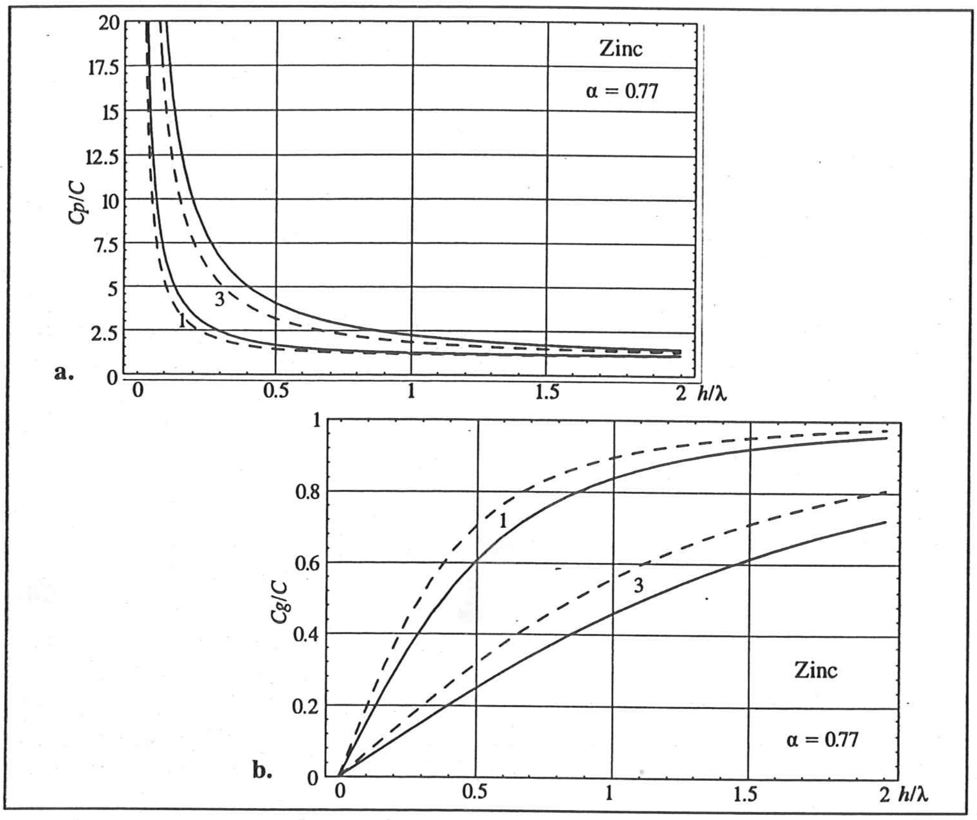

Figure 2. Phase **a.** and group **b.** velocities of torsional waves in an zinc tube (first and third modes). The broken lines correspond to the isotropic case.

for isotropic torsional waves is c_g = 2687m/s, and that for anisotropic waves is c_g = 3225m/s. On the contrary, for zinc, the isotropic velocity is greater than the anisotropic velocity. The maximum difference between the isotropic and anisotropic group velocities of the first mode is obtained for $h/\lambda \approx 0.3$ for iron tubes, and $h/\lambda \approx 0.4$ for zinc tubes.

3 Conclusions

The calculations show that, for wavelengths comparable to the tube thickness, the difference between the isotropic and anisotropic group velocities can exceed 500m/s (e.g., in a cylinder made of iron). On the other hand, the period equation of an anisotropic cylinder of interior and exterior radii a and b is the same as the period equation of an isotropic cylinder of interior and exterior radii αa and αb and shear velocity $\sqrt{c_{44}/\rho}$, where $\alpha = \sqrt{c_{44}/c_{66}}$. This result indicates that experiments interpreted with the isotropic period equation will either underestimate or overestimate the radii, depending on whether $\alpha < 1$ or $\alpha > 1$, respectively. Moreover, if the interior and exterior radii are known, and the axial shear velocity $\sqrt{c_{44}/\rho}$ is obtained from measurements on the fundamental mode, the elastic constant c_{66} can be estimated from the velocity of the dispersive modes, by using α as a fitting parameter.

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