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Propagation of Axially Symmetric Waves in Infinitely Long Composite Cylinder Systems *

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Abstract

In this work, we evaluate the general dispersion equation for axially symmetric harmonic waves propagating in an infinitely long composite cylindrical system. In particular, we study the velocity of the rotationally symmetric longitudinal and torsional modes travelling axially through the system, in terms of the material properties, and corresponding sizes (diameters) of the different shells. An application is the analysis of the waves propagating through a drill string tool since they provides useful information for the data processing of seismograms (recorded at the surface) produced by the roller cone bit. For torsional oscillations, the phase velocity decreases monotonically from infinity to the shear wave velocity as the frequency increases, and the group velocity increases monotonically from zero to the shear velocity. On the other hand, the phase velocity of the longitudinal modes are strongly dependent on the radial dimensions and the presence of fluid inside the drill string. In this case (a tube filled with mud), the vibration modes shift to the low frequencies.

1 Introduction

A fundamental problem in the classical theory of elasticity is the propagation of vibrations through bodies having cylindrical concentric boundaries. The most simple and complementary cases were investigated by Pochhammer [5] (propagation through a rod in vacuum) and Biot [1] (propagation through a hole in an infinite medium). In general, three types of vibrations take place; these are classed as longitudinal, torsional and flexural [3]. The study of the dynamic of more complicated multilayered shells has many applications, for instance, in guided-wave ultrasonic delay lines, shells used as components in aircrafts, missiles, solid-propellant rocket motors, etc.. In the exploration industry, the interest resides in the propagation of signals through a borehole system. For instance, there is a need of transmiting down-hole drilling information to the surface [2]. A simple model for the analysis of the vibrations of a drill string tool in a borehole considers a system of concentric cylindrical shells as illustrated in Figure 1. Then, the physics of wave propagation presents many similarities with the previous cases. Basically, the problem is solved by probing the elastic wave equation with a free harmonic wave and then determining the period (dispersion) equation by the requirement that an appropriate set of boundary conditions be fulfilled. The solution for the drill string problem requires the numerical solution of the period equation. In this work, we consider the propagation of axially symmetric fields, then, the possible solutions involve torsional and longitudinal waves exclusively.

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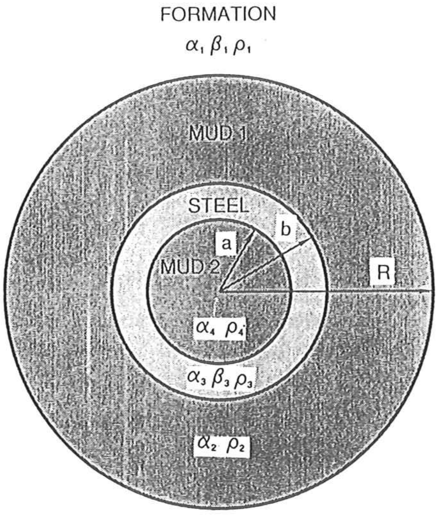

FIG. 1. *Cross-section of a borehole and drill string model*

2 Displacements, Stresses and General Solutions

We consider axially symmetric borehole, formation, drill string and source (a cross-section of the model is illustrated in Figure 1). Then, only the fundamental modes are excited ($n = 0$) and the wavefield does not depend on the azimuthal variable. In circular cylindrical coordinates (r, θ, z) with z directed along the axis of the borehole, the displacements are given in terms of the Helmholtz potentials ϕ, ψ and χ by [3]

$$u_r = \partial_r \phi + \partial_r \partial_z \chi, \tag{1}$$

$$u_\theta = -\partial_r \psi, \tag{2}$$

$$u_z = \partial_z \phi - \frac{1}{r}\partial_r (r \partial_r \chi) = \partial_z \phi - \partial_r^2 \chi - \frac{1}{r}\partial_r \chi, \tag{3}$$

where ∂ denotes differentiation. For isotropic materials with Lamé constants λ and μ, the relevant stress components read

$$\sigma_{rr} = \lambda \left(\frac{u_r}{r} + \partial_r u_r + \partial_z u_z \right) + 2\mu \partial_r u_r, \tag{4}$$

$$\sigma_{r\theta} = \mu r \partial_r \left(\frac{1}{r} u_\theta \right) = \mu \partial \left(\partial_r u_\theta - \frac{u_\theta}{r} \right), \tag{5}$$

$$\sigma_{\theta z} = \mu \partial_z u_\theta, \tag{6}$$

$$\sigma_{rz} = \mu (\partial_z u_r + \partial_r u_z). \tag{7}$$

Each potential obeys a wave equation of the type $(\Delta - c^{-2}\partial_{tt})f = 0$, where for $f = \phi$, $c = \alpha = \sqrt{(\lambda + 2\mu)/\rho}$, and for $f = \psi$ or $f = \chi$, $c = \beta = \sqrt{\mu/\rho}$. The operator Δ is the Laplacian in cylindrical coordinates and ρ is the density.

The steady-state solution has the form

$$f = R(r)Z(z)\exp(\imath\omega t), \tag{8}$$

where t is the time and ω is the angular frequency. The axially symmetric general solutions are [3]

$$R(r) = AJ_0(\kappa r) + BY_0(\kappa r), \quad \text{and} \quad Z(z) = C\exp(\imath\gamma z) + D\exp(-\imath\gamma z), \tag{9}$$

where J_0 and Y_0 are the Bessel functions of the first and second kind, respectively, and $A, \ldots, D$ are arbitrary constants. The quantities κ and γ correspond to the radial and vertical wavenumbers, such that

$$\kappa^2 = \omega^2/c^2 - \gamma^2. \tag{10}$$

3 Torsional Oscillations

These type of vibrations are characterized by the conditions that $u_r = u_z = 0$ and u_θ is independent of θ. The motion is initiated by a torque. Different sections of the drill string rotate with respect to each other without distortion. Since the mud, outside and inside the string is considered as an ideal fluid, it does not affect the torsional oscillations. This is due to the fact that the stress component $\sigma_{r\theta}$ vanishes at the steel-fluid interface, so it is equivalent to a steel-vacuum interface.

The general solution for harmonic waves along the positive z-direction can be written as

$$\psi(r, z, t, \gamma, \omega) = [A_0 J_0(\xi r) + B_0 Y_0(\xi r)]\exp[\imath(\omega t - \gamma z)], \tag{11}$$

where

$$\xi^2 = \omega^2/\beta_3^2 - \gamma^2 \tag{12}$$

is the radial wavenumber. Substituting the potential into equations (2) and (5) we obtain

$$u_\theta = -[A_0 \partial_r J_0(\xi r) + B_0 \partial_r Y_0(\xi r)]\exp[\imath(\omega t - \gamma z)], \tag{13}$$

$$\sigma_{r\theta} = -\mu_3 \left[A_0\left(\partial_r^2 J_0(\xi r) - \frac{\partial_r J_0(\xi r)}{r}\right) + B_0\left(\partial_r^2 Y_0(\xi r) - \frac{\partial_r Y_0(\xi r)}{r}\right)\right]\exp[\imath(\omega t - \gamma z)]. \tag{14}$$

At the inner and outer surfaces of the drill string we have the following boundary conditions

$$\sigma_{r\theta}(r = a) = 0, \quad \text{and} \quad \sigma_{r\theta}(r = b) = 0. \tag{15}$$

This implies two equation with two unknowns, A_0 and B_0. Making the determinant equal to zero and using properties of the Bessel functions gives the following dispersion equation:

$$J_2(\xi a)Y_2(\xi a) - J_2(\xi b)Y_2(\xi b) = 0. \tag{16}$$

Equation (16) was obtained by Gazis [4], and is identical in form to the frequency equation of axially symmetric shear vibrations in plane strain (see [3]).

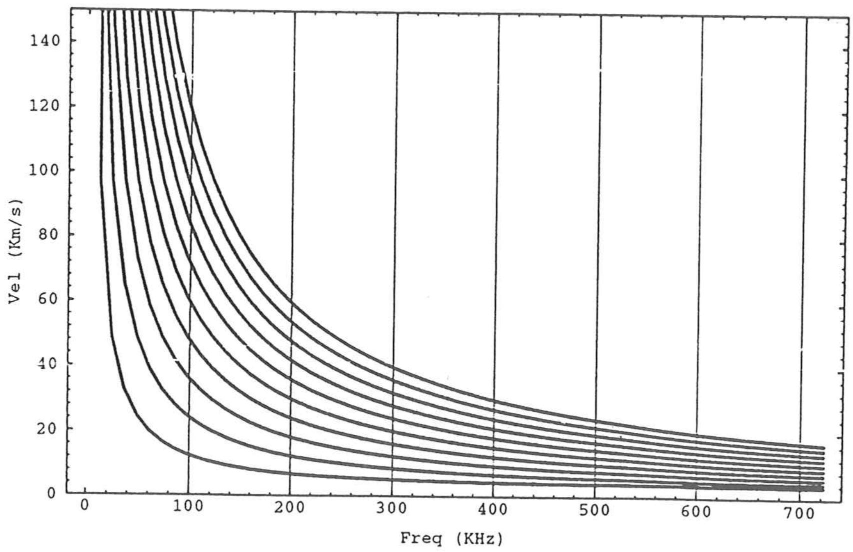

FIG. 2. *Phase velocity versus frequency for axially symmetric torsional waves*

Let us consider a shear velocity $\beta_3 = 3300$ m/s, and $a = 5.43$ cm and $b = 6.35$ cm.. The lowest torsional mode is not appropriately obtained from equation (16). This mode corresponds to $\xi^2 = 0$ and the displacement to a rotation of each transverse section of the cylinder as a whole about its center. This motion is not dispersive and both the phase and the group velocities are equal to β_3.

The higher modes are obtained from the roots of equation (16) that can be written as $F(c_p, \omega) = 0$ where $c_p = \omega/\gamma$ is the phase velocity. This is represented in Figure 2 for the first ten propagating modes. From equation (12) and the fact that $\xi^2 \geq 0$, the phase velocity of the torsional waves is always greater than or equal to β. Moreover, each mode has a cut-off frequency below which propagation cannot take place. In fact, let us assume that the roots of (16) are $\xi_1,\ \xi_2, \ldots, \xi_j, \ldots$. Then, the phase velocity corresponding to the j mode is

$$c_p = \frac{\omega}{\gamma} = \left(\frac{1}{\beta_3^2} - \frac{\xi_j^2}{\omega^2}\right)^{-1/2} . \tag{17}$$

Then, the cut-off frequency for this mode is $\omega_{0j}^2 = \beta_3^2 \xi_j^2$.

The group velocity is equal to $c_g = d\omega/d\gamma$, which can be easily calculated from equation (12). It yields

$$c_g = \frac{\beta_3^2}{c_p}. \tag{18}$$

Then, this velocity is always smaller than the body wave velocity β_3.

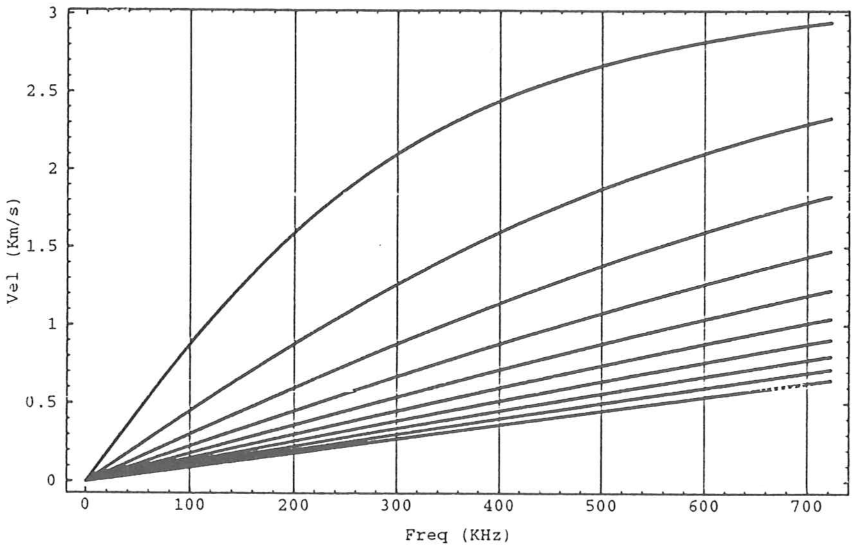

FIG. 3. *Group velocity versus frequency for axially symmetric torsional waves*

4 Longitudinal Oscillations

We consider longitudinal oscillations where $u_\theta = 0$ and u_r and u_z are independendent of θ. The motion is confined to planes perpendicular to the string axis. The potentials are the following:

(19) $$\text{Formation} \left\{ \begin{array}{l} \phi_1 = B_1 H_0^{(1)}(\kappa_1 r) \exp[\imath(\omega t - \gamma z)] \\ \chi_1 = D_1 H_0^{(1)}(\xi_1 r) \exp[\imath(\omega t - \gamma z)], \end{array} \right.$$

(20) $$\text{Mud 1} \left\{ \phi_2 = [A_2 J_0(\kappa_2 r) + B_2 Y_0(\kappa_2 r)] \exp[\imath(\omega t - \gamma z)], \right.$$

(21) $$\text{Steel} \left\{ \begin{array}{l} \phi_3 = [A_3 J_0(\kappa_3 r) + B_3 Y_0(\kappa_3 r)] \exp[\imath(\omega t - \gamma z)] \\ \chi_3 = [C_3 J_0(\xi_3 r) + D_3 Y_0(\xi_3 r)] \exp[\imath(\omega t - \gamma z)], \end{array} \right.$$

(22) $$\text{Mud 2} \left\{ \phi_4 = A_4 J_0(\kappa_4 r) \exp[\imath(\omega t - \gamma z)], \right.$$

where κ and ξ are the P and S radial wavenumbers.

From equations (1)-(3), the displacements are

(23) $$u_r^{(n)} = A_n U_{n1} + B_n U_{n2} + C_n U_{n3} + D_n U_{n4},$$

(24) $$u_z^{(n)} = A_n V_{n1} + B_n V_{n2} + C_n V_{n3} + D_n V_{n4},$$

where $n = 1, \ldots, 4$ denotes the medium, and the U_{nj} and V_{nj} different from zero are given by

$$U_{12} = -\kappa_1 H_1^{(1)}(\kappa_1 r), \quad U_{14} = \imath\gamma\xi_1 H_1^{(1)}(\xi_1 r), \quad V_{12} = -\imath\gamma H_0^{(1)}(\kappa_1 r), \quad V_{14} = \xi_1^2 H_1^{(1)}(\xi_1 r),$$

$$U_{21} = -\kappa_2 J_1(\kappa_2 r), \quad U_{22} = -\kappa_2 Y_1(\kappa_2 r), \quad V_{21} = -\imath\gamma J_0(\kappa_2 r), \quad V_{22} = -\imath\gamma Y_0(\kappa_2 r),$$

$$U_{31} = -\kappa_3 J_1(\kappa_3 r), \quad U_{32} = -\kappa_3 Y_1(\kappa_3 r), \quad U_{33} = \imath\gamma\xi_3 J_1(\xi_3 r), \quad U_{34} = \imath\gamma\xi_3 Y_1(\xi_3 r),$$

$$V_{31} = -\imath\gamma J_0(\kappa_3 r), \quad V_{32} = -\imath\gamma Y_0(\kappa_3 r), \quad V_{33} = \xi_3^2 J_0(\xi_3 r), \quad V_{34} = \xi_3^2 Y_0(\xi_3 r),$$

$$U_{41} = -\kappa_4 J_1(\kappa_4 r), \quad V_{41} = -\imath\gamma J_0(\kappa_4 r),$$

where the factor $\exp[\imath(\omega t - \gamma z)]$ has been omited for clarity. The stress components follow from equations (4)-(7),

$$\sigma_{rr}^{(n)} = A_n R_{n1} + B_n R_{n2} + C_n R_{n3} + D_n R_{n4}, \tag{25}$$

$$\sigma_{rz}^{(n)} = A_n S_{n1} + B_n S_{n2} + C_n S_{n3} + D_n S_{n4}, \tag{26}$$

where the coefficients different from zero are

$$R_{12} = -(\lambda_1\gamma^2 + E_1\kappa_1^2) H_0^{(1)}(\kappa_1 r) + 2\mu_1 \frac{\kappa_1}{r} H_1^{(1)}(\kappa_1 r),$$

$$R_{14} = -2\imath\mu_1\gamma\xi_1 \left[\frac{1}{r} H_1^{(1)}(\xi_1 r) - \xi_1 H_0^{(1)}(\xi_1 r)\right],$$

$$S_{12} = 2\imath\mu_1\gamma\kappa_1 H_1^{(1)}(\kappa_1 r), \quad S_{14} = \mu_1\xi_1(\gamma^2 - \xi_1^2) H_1^{(1)}(\xi_1 r),$$

$$R_{21} = -\lambda_2(\gamma^2 + \kappa_2^2) J_0(\kappa_2 r), \quad R_{22} = -\lambda_2(\gamma^2 + \kappa_2^2) Y_0(\kappa_2 r),$$

$$R_{31} = -(\lambda_3\gamma^2 + E_3\kappa_3^2) J_0(\kappa_3 r) + 2\mu_3 \frac{\kappa_3}{r} J_1(\kappa_3 r),$$

$$R_{32} = -(\lambda_3\gamma^2 + E_3\kappa_3^2) Y_0(\kappa_3 r) + 2\mu_3 \frac{\kappa_3}{r} Y_1(\kappa_3 r),$$

$$R_{33} = -2\imath\mu_3\gamma\xi_3 \left[\frac{1}{r} J_1(\xi_3 r) - \xi_3 J_0(\xi_3 r)\right],$$

$$R_{34} = -2\imath\mu_3\gamma\xi_3 \left[\frac{1}{r} Y_1(\xi_3 r) - \xi_3 Y_0(\xi_3 r)\right],$$

$$S_{31} = 2\imath\mu_3\gamma\kappa_3 J_1(\kappa_3 r), \quad S_{32} = 2\imath\mu_3\gamma\kappa_3 Y_1(\kappa_3 r),$$

$$S_{33} = \mu_3\xi_3(\gamma^2 - \xi_3^2) J_1(\xi_3 r), \quad S_{34} = \mu_3\xi_3(\gamma^2 - \xi_3^2) Y_1(\xi_3 r),$$

$$R_{41} = -\lambda_4(\gamma^2 + \kappa_4^2) J_0(\kappa_4 r),$$

where $E_n = \lambda_n + 2\mu_n$, and the relations

$$\kappa_n^2 = k_{\alpha n}^2 - \gamma^2, \quad k_{\alpha n} = \frac{\omega}{\alpha_n}, \qquad \xi_n^2 = k_{\beta n}^2 - \gamma^2, \quad k_{\beta n} = \frac{\omega}{\beta_n} \tag{27}$$

have been used.

The field must satisfy the following boundary conditions:

$$\begin{aligned} &Formation - Mud\ 1\ (r = R) \quad u_r^{(1)} = u_r^{(2)}, \quad \sigma_{rr}^{(1)} = \sigma_{rr}^{(2)}, \quad \sigma_{rz}^{(1)} = 0, \\ &Mud\ 1 - Steel\ (r = b) \quad u_r^{(2)} = u_r^{(3)}, \quad \sigma_{rr}^{(2)} = \sigma_{rr}^{(3)}, \quad \sigma_{rz}^{(3)} = 0, \\ &Steel - Mud\ 2\ (r = a) \quad u_r^{(3)} = u_r^{(4)}, \quad \sigma_{rr}^{(3)} = \sigma_{rr}^{(4)}, \quad \sigma_{rz}^{(3)} = 0, \end{aligned} \tag{28}$$

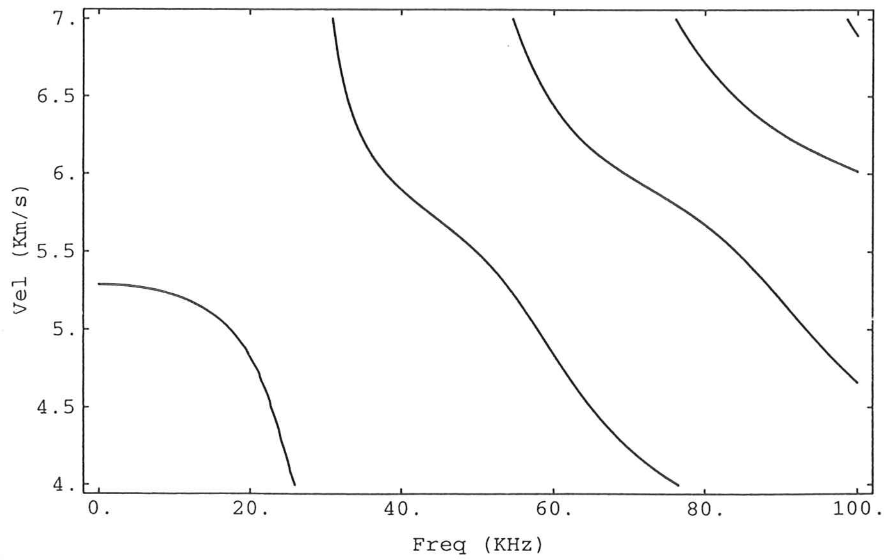

FIG. 4. *Phase velocity versus frequency for longitudinal waves in a rod made of steel*

The problem has 9 unknowns, B_1, D_1, A_2, B_2, A_3, B_3, C_3, D_3 and A_4, with 9 boundary conditions. The determinant of the set of nine homogeneous linear equations must vanish, giving the following dispersion equation:

$$\begin{vmatrix} U_{12}(R) & U_{14}(R) & -U_{21}(R) & -U_{22}(R) & 0 & 0 & 0 & 0 & 0 \\ R_{12}(R) & R_{14}(R) & -R_{21}(R) & -R_{22}(R) & 0 & 0 & 0 & 0 & 0 \\ S_{12}(R) & S_{14}(R) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & U_{21}(b) & U_{22}(b) & -U_{31}(b) & -U_{32}(b) & -U_{33}(b) & -U_{34}(b) & 0 \\ 0 & 0 & R_{21}(b) & R_{22}(b) & -R_{31}(b) & -R_{32}(b) & -R_{33}(b) & -R_{34}(b) & 0 \\ 0 & 0 & 0 & 0 & S_{31}(b) & S_{32}(b) & S_{33}(b) & S_{34}(b) & 0 \\ 0 & 0 & 0 & 0 & U_{31}(a) & U_{32}(a) & U_{33}(a) & U_{34}(a) & -U_{41}(a) \\ 0 & 0 & 0 & 0 & R_{31}(a) & R_{32}(a) & R_{33}(a) & R_{34}(a) & -R_{41}(a) \\ 0 & 0 & 0 & 0 & S_{31}(a) & S_{32}(a) & S_{33}(a) & S_{34}(a) & 0 \end{vmatrix} = 0$$

(29)

We are interested in the wave traveling through the steel. In particular the fundamental mode, that in a rod of radius b ($a = 0$) and at the low-frequency limit, has the velocity $c_{\mathrm{rod}} = \sqrt{Y/\rho_3}$, where $Y = \mu_3(3\lambda_3 + 2\mu_3)/(\lambda_3 + \mu_3)$ is the Young modulus. The dispersion relation corresponding to the rod is

$$\begin{vmatrix} -R_{31}(b) & -R_{33}(b) \\ S_{31}(b) & S_{33}(b) \end{vmatrix} = 0. \tag{30}$$

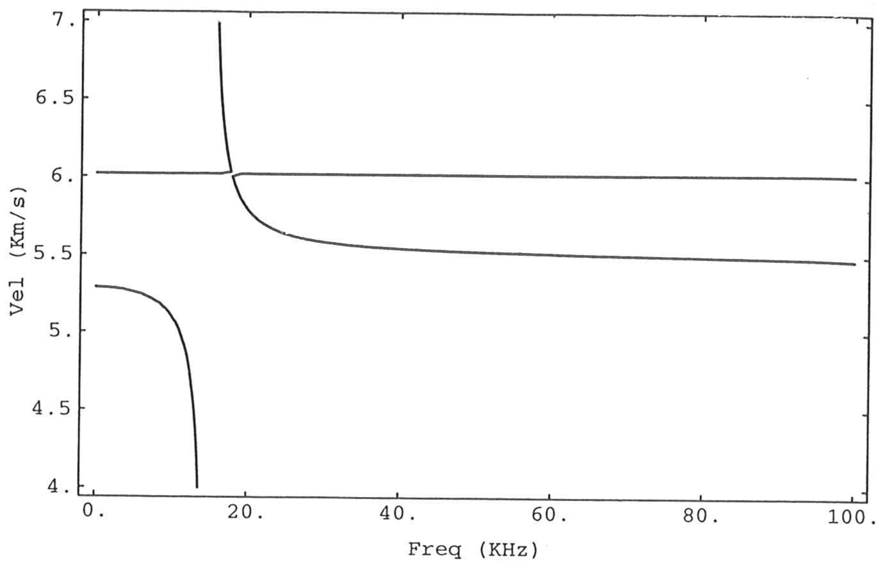

FIG. 5. *Phase velocity versus frequency for longitudinal waves in a tube made of steel.*

In the case of a tube ($a \neq 0$), the dispersion relation is

$$\tag{31} \begin{vmatrix} -R_{31}(b) & -R_{32}(b) & -R_{33}(b) & -R_{34}(b) \\ S_{31}(b) & S_{32}(b) & S_{33}(b) & S_{34}(b) \\ R_{31}(a) & R_{32}(a) & R_{33}(a) & R_{34}(a) \\ S_{31}(a) & S_{32}(a) & S_{33}(a) & S_{34}(a) \end{vmatrix} = 0.$$

If we take into consideration the mud inside the tube, the equation is

$$\tag{32} \begin{vmatrix} -R_{31}(b) & -R_{32}(b) & -R_{33}(b) & -R_{34}(b) & 0 \\ S_{31}(b) & S_{32}(b) & S_{33}(b) & S_{34}(b) & 0 \\ U_{31}(a) & U_{32}(a) & U_{33}(a) & U_{34}(a) & -U_{41}(a) \\ R_{31}(a) & R_{32}(a) & R_{33}(a) & R_{34}(a) & -R_{41}(a) \\ S_{31}(a) & S_{32}(a) & S_{33}(a) & S_{34}(a) & 0 \end{vmatrix} = 0.$$

The dispersion relation of the system *Mud 1-Steel-Mud 2* corresponds to the lower 6×6 matrix (at the right side) in equation (29). We consider the following material properties: steel shear velocity, $\beta_3 = 3300$ m/s; steel compressional velocity, $\alpha_3 = 6000$ m/s; steel density, $\rho_3 = 8$ g/cm^3; mud velocity, $\alpha_4 = 1500$ m/s; mud density, $\rho_4 = 2$ g/cm^3; tube inner an outer radii, $a = 5.43$ cm and $b = 6.35$ cm, respectively. These values give a rod velocity of $c_{\text{rod}} = 5286$ m/s. The phase velocity versus frequency for the rod is shown in Figure 4, whence it is clear that the fundamental mode travels with velocity c_R at very low frequencies. Also illustrated in the Figure are the higher modes of vibration. The phase velocity corresponding to the empty tube is represented in Figure 5. As can be seen, the

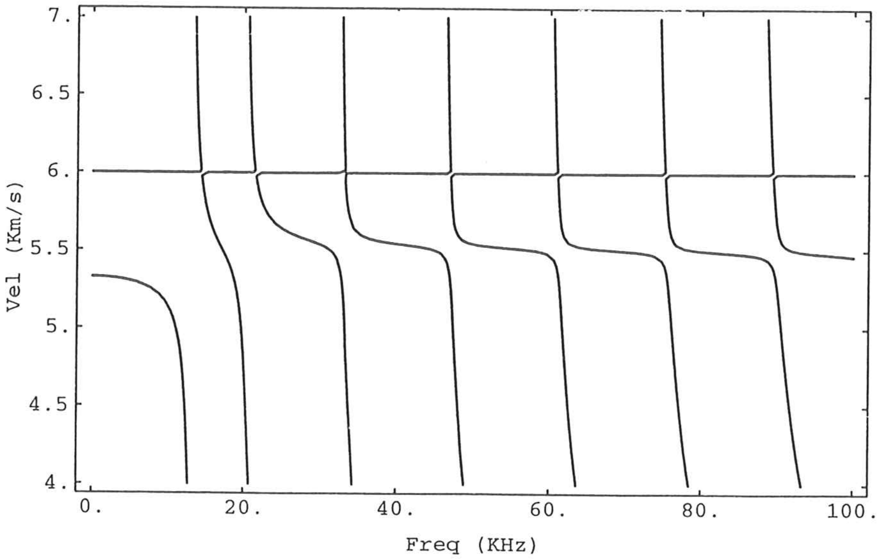

FIG. 6. *Phase velocity versus frequency for longitudinal waves in a tube filled with mud.*

velocity of the fundamental mode is the same as in the rod at very low frequencies, but it decreases more rapidly for higher frequencies. The horizontal line (no dispersion) is a solution of the period equation corresponding to the steel phase velocity. The inclusion of the mud inside the string [equation (32)] produces a shift of the higher vibration modes towards the low frequencies, as can be appreciated in Figure 5.

The continuation of this research involves the calculation of the longitudinal phase and group velocities corresponding to the different composite systems (including that of Figure 1) and, in particular, the influence that the different material properties and the radial dimensions exert on the fundamental mode traveling through the drill string.

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