Numerical solutions of a poro-acoustic wave equation with generalized fractional integral operators

Andrzej Hanyga *   José M. Carcione †

Abstract

Numerical solutions of partial differential equations with the temporal evolution governed by an integro-differential Volterra operator with a weakly singular kernel are considered. Such equations appear in the theory of gas saturated porous media. The mathematical model of the medium is taken from a paper of Wilson but the method applies to many alternative acoustic models of porous media involving singular memory effects.

Introduction

...bolic integro-differential equations with singular time convolution operators fre...ly appear in problems of wave propagation in porous and viscoacoustic media. Such ...ns have some smoothness properties that set them apart. Most often the convolu... ...rnels have an algebraic singularity for the delay time tending to 0 and the solutions ...nitely smooth at the wavefront [15, 16, 18, 12, 13, 17]. From a practical point of ...is implies that signals propagate with a delay with respect to the wavefront and the ...an be ascertained by comparing the real travel time of a pulse with its travel time ...ed from high frequency propagation speed.

...-asymptotic solutions of such equations have been derived elsewhere [8, 11, 9]. ...me classes of equations and kernel singularities asymptotic solutions yield explicit ...e for the delay of the signal generated by a discontinuous signal or by a delta ...itted by the source [9, 11]. Besides, the signal shape can be calculated explicitly. ...otic methods based on real rays run into some difficulties at caustics [10]. This can ...ed by a recourse to complex ray tracing [6], which, at least in practical applications, ...that the model admits an analytic continuation.

...this reason, and for comparison with ray asymptotic solutions, development of ...al methods of the finite-difference type for this class of problems is highly desirable. ...otic analysis of the solutions [9, 8, 11] shows the primary importance of the singular ...the convolution kernels for the build-up of the signal after the passage of the ...t, while the regular part affects the tail and the rate of decay of the signal ...le.

...ointed out in [7], both wavefront and tail aspects of the wavefield can be fairly ...y represented in terms of generalized fractional derivative and fractional integral ...s of the form $A\,(\mathrm{D}+\gamma)^{\alpha}$, with $\gamma > 0$ and $-1 < \alpha < 1$. The order of the fractional .../integral and the coefficient A controls the initial build-up of the signal after the

...e of Solid Earth Physics, University of Bergen, Allégaten 41, 5007 Bergen, NORWAY
...Nazionale di Oceanografia e di Geofisica Sperimentale (OGS), Borgo Grotta Gigante 42c, 34010 ...) ITALY

passage of the wavefront while the parameter γ introduces attenuation of lo and controls the tail and the amplitude decay in the course of the propagation. hand it is fairly easy to construct finite-difference operators for the generaliz integrals and derivatives.

Mathematical models of wave propagation in poroelastic media considered enough flexibility to account for the most important observable effects in wave p pulse spreading, its delay with respect to the wavefront, the decay rate of the decay of the signal at the source. Moreover, various specific equations of thi been derived from physical models of porous media with idealized, complex or geometries [14, 2, 21, 4, 19]. It is therefore possible to study acoustic wave in realistic models of visco-porous media by numerical integration, combinin correct attenuation models with complicated geometry and inhomogeneity of t

For this presentation we have chosen a purely visco-acoustic model as well as model of a porous medium suggested by Wilson [20]. Virtually identical meth applied to alternative but mathematically similar models of acoustic waves in po e.g. [1].

2 Equations of motion

The generalized fractional integral $(\mathrm{D}+\gamma)^{-\alpha}$ is defined by the equation [7]:

(1) $$(\mathrm{D}+\gamma)^{-\alpha}\, u = \mathrm{e}^{-\gamma t} J^{\alpha}\left[\mathrm{e}^{\gamma t}\, u\right]$$

where

(2) $$J^{\alpha} u(t) = \int_0^t \tau^{\alpha-1}\, u(t-\tau)/\Gamma(\alpha)\, \mathrm{d}\tau$$

The Laplace transform of the generalized fractional integral operator $(\mathrm{D}+\gamma)^{-\alpha} u($ is given by the formula

(3) $$(s+\gamma)^{-\alpha}\, \tilde{u}(s),$$

where $\tilde{u}(s)$ denote the Laplace transform of $u(t)$.

Using eq. (3), Wilson's frequency-domain equations [20] are equivalent to the hy system of fractional PDEs

(4) $$\left[1+\Gamma_\rho\left(\mathrm{D}+\frac{1}{\tau_\rho}\right)^{-\alpha}\right] \operatorname{div} \mathbf{v} = \rho_\infty \mathrm{D}^2 \theta,$$

(5) $$\left[1+\Gamma_K\left(\mathrm{D}+\frac{1}{\tau_K}\right)^{-\alpha}\right] \mathbf{v} = K_\infty \operatorname{grad} \theta$$

where θ denotes the dilatation.

The two different relaxation mechanisms appearing in the above equations are rel viscous relaxation (relaxation time τ_K) and thermal relaxation (relaxation time τ_ρ)

The condition of non-negative dissipation in the high-frequency limit implie $\Gamma_K - \Gamma_\rho > 0$. It is however clear that positive values of Γ_ρ correspond to a non-p relaxation which partly offsets the dissipative effect of $\Gamma_K > 0$. We shall therefore our considerations to the case $\Gamma_K \geq 0, \Gamma_\rho \leq 0$. Eqs (4–5) will be solved for the initial data $\theta(t) = v_i(t) \equiv 0$ for $t < 0$, $\theta(0) = \theta_0$, $\mathrm{D}\theta(0) = \theta_1$.

In Wilson's model [20] the values of $\Gamma_\rho < 0$ and τ_ρ are such that in the limit of $\omega \to 0$ the density $\hat{\rho}(\omega)$ tends to zero. This feature has little effect on the propagation of a signal with a typical seismic frequency as considered below.

As shown in [8, 9] for a problem with identical asymptotic properties, the pulse spreads out and arrives with a delay with respect to the wavefront. The delay is proportional to t^2. For a point source in a homogeneous medium the speed of the wavefront is constant and given by $c_\infty := (K_\infty/\rho_\infty)^{1/2}$. Consequently, the rate of growth of the diameter of the circle on which the pulse is located varies in time even though the medium is homogeneous.

Discretized equations

Fractional integrals can be approximated by a formula which is a direct extension of the Grünwald-Letnikow [5]. From (1) the following FD approximation of the generalized fractional integral operator is immediately derived

$$\left(\mathrm{D} + \frac{1}{\tau}\right)^{-\alpha} f(nh) = h^\alpha \sum_{j=1}^{J} (-1)^j \binom{-\alpha}{j} f^{n-j} \exp(-jh/\tau). \tag{6}$$

where the upper limit $J = [t/h]$ (the entire part of t/h) because of the assumed initial condition.

The finite-difference approximation of eqs (4–5) is obtained by converting them to the form

$$\left[1 + \Gamma_\rho \mathrm{D}^{-\alpha}\right] \left(\mathrm{e}^{t/\tau_\rho} \operatorname{div} \mathbf{v}\right) = \rho_\infty \mathrm{e}^{t/\tau_\rho} \mathrm{D}^2 \theta \tag{7}$$

$$\left[1 + \Gamma_K \mathrm{D}^{-\alpha}\right] \left(\mathrm{e}^{t/\tau_K} \mathbf{v}\right) = K_\infty \mathrm{e}^{t/\tau_K} \operatorname{grad} \theta \tag{8}$$

Substituting the backward Euler derivatives for D^2 and eq. (6) for $\mathrm{D}^{-\alpha}$ and solving for the values at nh we have For eqs (4–5) we get

$$v_i^n = (1 + \Gamma_K h^\alpha)^{-1} \left[K_\infty \left(\frac{\partial \theta}{\partial x_i}\right)^n - \Gamma_K h^\alpha \sum_{j=1}^{J} (-1)^j \binom{-\alpha}{j} v_i^{n-j} \exp(-jh/\tau_K) \right] \tag{9}$$

for $i = 1, 2$, and

$$\theta^n = \frac{h^2}{\rho_\infty} \left[w^{n-1} + \Gamma_\rho h^\alpha \sum_{j=0}^{J} (-1)^j \binom{-\alpha}{j} w^{n-1-j} \exp(-jh/\tau_\rho) \right] + 2\theta^{n-1} - \theta^{n-2}, \tag{10}$$

where $w = \operatorname{div} \mathbf{v}$. The last equation must be solved before equations (9). Fourier domain representation is applied to the spatial derivatives.

Results of numerical simulation and discussion

The model parameters are chosen as follows: $\alpha = 0.5$; $\rho_\infty = 2600$ kg/m^3; phase velocity of the wavefront $c_\infty = \sqrt{K_\infty/\rho_\infty} = 2000$ m/s. The frequency of the transition zone $\nu_B = 144$ Hz; $\tau_K = 1/\nu_B$; $\Gamma_K = 2\sqrt{\nu_B}$. For the visco-acoustic model we assumed $\Gamma_\rho = 0$. For the poro-acoustic model we additionally set $\tau_\rho = \tau_K$; $\Gamma_\rho = -0.28\sqrt{\nu_B}$.

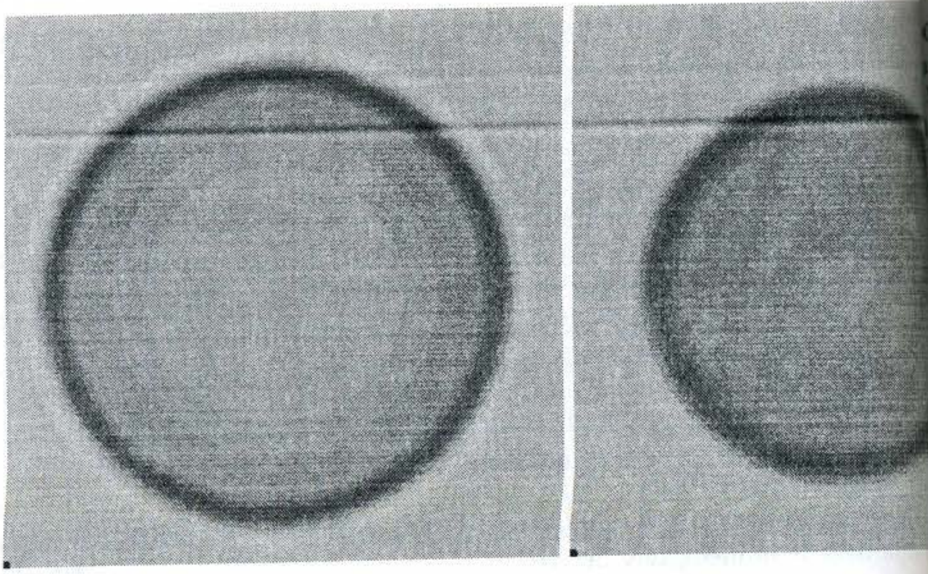

FIG. 1. *A snapshot of the wavefield at 600 ms, for* $\Gamma_\rho = 0$ *(left) and for* $\Gamma_\rho <$

The wavefield is excited by a point source emitting a Ricker wavelet wi... frequency 25 Hz and a cut-off frequency of 50 Hz. The value of ν_B is close... frequency of the source signal.

The time step is $h = 1$ ms. The spatial grid parameters are $\mathrm{d}x = \mathrm{d}z = 20$m... grid points.

Snapshots of the wavefield at $t = 600$ ms for $\Gamma_\rho = 0$ and $\Gamma_\rho < 0$ are s... 1. The dimensions of the snapshots are 1540 m × 1540 m. For comparison, ... corresponding diameter of the wavefront is 1200 m. For the viscoelastic case Γ_ρ... there is a considerable pulse delay with respect to the wavefront and pulse spr... diameter of the visco-elastic pulse is ca 0.52 of the diameter of the elastic pulse... velocity. Additional relaxation effects associated with the density ρ, for $\Gamma_\rho <$... the rate of amplitude decay, pulse spreading and the pulse delay.

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