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# Anomalous polarization in anisotropic media

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ABSTRACT

Elastic waves can, in principle, be classified according to their propagation velocity, e.g., through the values of the slowness or the group velocity surfaces along a given direction. We show that there are media with the same velocity distribution but drastically different polarization behaviour. Such media are kinematically identical but dynamically different. Therefore, classification according to wave velocity alone is not sufficient, and the identification of the wave type should be based on both velocity and the polarization distribution.

For transversely isotropic symmetry there are at most two and for orthorhombic symmetry at most four media that have the same velocity distribution. There is always one medium with a polarization distribution topologically similar to that of isotropy, which is called "normal polarization". All the other media are said to possess "anomalous polarization".

In orthorhombic media there is a different set of four media closely related to the above mentioned set. The two sets share all velocities in the symmetry planes, but the velocity distribution off the planes is different in the two sets. All members of this set possess anomalous polarization.

It is shown that anomalous companions also exist in media of monoclinic symmetry, and we find the corresponding conditions on the elastic constants. The stiffness matrix is that of an orthorhombic "root" medium with an anomalous companion, but with the symmetry broken through the addition of either $c_{14}$ or $c_{25}$ or $c_{36}$ (two of the three stiffnesses must vanish, with one of the three stiffnesses $c_{14}$, $c_{25}$ or $c_{36}$ added).

Note that this analysis does not imply that anomalous media could be found in nature, but only that they are not forbidden by the laws of physics.



## 1. Introduction

Slowness (and velocity) and polarization are together closely related to the "eigensystem" of the Kelvin–Christoffel matrix in a given direction (Helbig, 1994; Carcione, 2007); thus, they are closely related. The question arises whether this makes the polarization a simple consequence of the slowness in the sense that the polarization behavior could be derived uniquely from the slowness surface alone. In other words, are there distinct stiffness matrices that give rise to different polarization behavior, but identical propagation behavior from the kinematic point of view?

We show here that there are media with the same phase velocity or slowness surface that exhibit drastically different polarization behaviors. Such media are kinematically identical but dynamically different. Therefore, classification of the media according to velocity (or slowness) alone is not sufficient, and the identification of the wave type should be based on both velocity and polarization.

The slowness and wave surfaces of two members of an anomalous companion pair are identical, but the polarization differs. Examples of anomalous polarization have been discussed for transverse isotropy by Helbig and Schoenberg (1987), and for orthorhombic and monoclinic symmetries by Carcione and Helbig (2000) and/or Carcione (2007). We extend the latter work by studying in detail the stability constraints and providing a number of examples to show the physics in the different cases. We determine without prior restriction of the symmetry class under what conditions the phenomenon can occur. Since the three slownesses in a given direction are the square roots of the eigenvalues of the Kelvin–Christoffel matrix, while the polarizations are the corresponding eigenvectors, the condition for the existence of anomalous polarization can be formulated as:

Two media with different stiffness matrices are "anomalous companions" if the characteristic equations of their respective Kelvin–Christoffel matrices $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}^*$ are identical, i.e., if $\det(\boldsymbol{\Gamma} - \lambda\mathbf{I}) = \det(\boldsymbol{\Gamma}^* - \lambda\mathbf{I})$, $\lambda = \rho v_p^2$, where $v_p$ is the phase velocity, $\rho$ is the density, and $\mathbf{I}$ is the $3 \times 3$ identity matrix.

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The discussion is restricted to media with a distinct qP sheet, e.g., to media with $\{c_{11}, c_{22}, c_{33}\} > \{c_{44}, c_{55}, c_{66}\}$. A medium with a distinct qP sheet is called "normally polarized" if – with the continuous directional sense of the qP polarization vector $\mathbf{u}$ – the sign of the scalar product $\hat{\mathbf{n}} \cdot \mathbf{u}$, where $\hat{\mathbf{n}}$ is the unit propagation vector, is the same for all $\hat{\mathbf{n}}$. This allows deviations of the polarization between zero and $\pi/2$, but not $\geqslant \pi/2$. The anomalous medium has not restrictions on the polarizations.

We take the discussion beyond the realm of ray geometry by calculating finite-band wave fields (displayed as snapshots). This is important since it is not immediately obvious how kinematic and dynamic features that are derived under the ray-geometric approximation show up in observational data.

## 2. Conditions for the existence of anomalous polarization

Without loss of generality we assume that the elastic 4th-rank stiffness tensors (and the corresponding 6 × 6 stiffness matrices) are referred to a natural coordinate system of the media. The Kelvin–Christoffel dispersion relation is

$$\det(\boldsymbol{\Gamma} - \lambda \mathbf{I}) = \lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0, \tag{1}$$

where

$$\begin{aligned} I_1 &= \Gamma_{11} + \Gamma_{22} + \Gamma_{33}, \\ I_2 &= \Gamma_{22}\Gamma_{33} - \Gamma_{23}^2 + \Gamma_{11}\Gamma_{33} - \Gamma_{13}^2 + \Gamma_{11}\Gamma_{22} - \Gamma_{12}^2, \\ I_3 &= \Gamma_{11}\Gamma_{22}\Gamma_{33} + 2\Gamma_{23}\Gamma_{13}\Gamma_{12} - \Gamma_{11}\Gamma_{23}^2 - \Gamma_{22}\Gamma_{13}^2 - \Gamma_{33}\Gamma_{12}^2 \end{aligned} \tag{2}$$

are the principal invariants of the Kelvin–Christoffel matrix (Helbig, 1994). Two elastic media with different elasticity tensors are said to be "anomalous companions" if for each propagation direction their Kelvin–Christoffel tensors have the same principal invariants. As we shall see below, one particular class of companions which restricts the symmetry to monoclinic media is given by the following three conditions:

1. The diagonal terms of the Kelvin–Christoffel matrices $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}^*$ are identical.
2. The squares of their off-diagonal terms are identical.
3. The products of their three off-diagonal terms are identical.

The second and third conditions can be satisfied simultaneously if all corresponding off-diagonal terms have the same magnitude, and precisely two corresponding terms have opposite sign. The conditions are sufficient but do not exclude that anomalous companions can exist for media of lower symmetry than monoclinic, such as triclinic media. In fact, the equality of the invariants of two companions do not necessarily imply the three conditions, which restrict the media to monoclinic symmetry.

In explicit form, the components of the Kelvin–Christoffel matrix are

$$\begin{aligned} \Gamma_{11} &= c_{11}l_1^2 + c_{66}l_2^2 + c_{55}l_3^2 + 2c_{56}l_2l_3 + 2c_{15}l_3l_1 + 2c_{16}l_1l_2, \\ \Gamma_{22} &= c_{66}l_1^2 + c_{22}l_2^2 + c_{44}l_3^2 + 2c_{24}l_2l_3 + 2c_{46}l_3l_1 + 2c_{26}l_1l_2, \\ \Gamma_{33} &= c_{55}l_1^2 + c_{44}l_2^2 + c_{33}l_3^2 + 2c_{34}l_2l_3 + 2c_{35}l_3l_1 + 2c_{45}l_1l_2, \\ \Gamma_{12} &= c_{16}l_1^2 + c_{26}l_2^2 + c_{45}l_3^2 + (c_{46} + c_{25})l_2l_3 \\ &\quad + (c_{14} + c_{56})l_3l_1 + (c_{12} + c_{66})l_1l_2, \\ \Gamma_{13} &= c_{15}l_1^2 + c_{46}l_2^2 + c_{35}l_3^2 + (c_{45} + c_{36})l_2l_3 \\ &\quad + (c_{13} + c_{55})l_3l_1 + (c_{14} + c_{56})l_1l_2, \\ \Gamma_{23} &= c_{56}l_1^2 + c_{24}l_2^2 + c_{34}l_3^2 + (c_{44} + c_{23})l_2l_3 \\ &\quad + (c_{36} + c_{45})l_3l_1 + (c_{25} + c_{46})l_1l_2, \end{aligned} \tag{3}$$

where $c_{IJ}$ are the elements of the stiffness matrix, and $(l_1, l_2, l_3)$ are the direction cosines defining the wavevector direction.

Let us consider the three conditions.

1. The diagonal terms of the Kelvin–Christoffel matrices of an anomalous companion pair are equal for all propagation directions if they share the following 15 stiffnesses:

$$\begin{aligned} &c_{11},\ c_{22},\ c_{33},\ c_{44},\ c_{55},\ c_{66},\ c_{15},\ c_{16},\ c_{56}, \\ &\quad c_{24},\ c_{26},\ c_{46},\ c_{34},\ c_{35}\ \text{and}\ c_{45}. \end{aligned} \tag{4}$$

Two anomalous companion matrices can thus differ only in

$$c_{23},\ c_{13},\ c_{12},\ c_{14},\ c_{25}\ \text{and}\ c_{36}. \tag{5}$$

The position of these stiffnesses in the stiffness matrix relating the stress and strain vectors is

| | | | | | |
|---|---|---|---|---|---|
| | $c_{12}$ | $c_{13}$ | $c_{14}$ | | |
| $c_{12}$ | | $c_{23}$ | | $c_{25}$ | |
| $c_{13}$ | $c_{23}$ | | | | $c_{36}$ |
| $c_{14}$ | | | | | |
| | $c_{25}$ | | | | |
| | | $c_{36}$ | | | |

2. Two of the three off-diagonal terms of the Kelvin–Christoffel matrices for an anomalous companion pair must be of equal magnitude but opposite sign for all propagation directions, thus for these terms all coefficients of the product of direction cosines must change sign. The off-diagonal terms of the Kelvin–Christoffel matrix are given by Eqs. $(3)_4$, $(3)_5$ and $(3)_6$. The nine stiffnesses $c_{15}$, $c_{16}$, $c_{24}$, $c_{26}$, $c_{34}$, $c_{35}$, $c_{45}$, $c_{46}$ and $c_{56}$ are listed in Eq. (4), as being equal in both terms, thus they can change sign only if they vanish. The off-diagonal terms of the Kelvin–Christoffel matrix in a pair of companion matrices thus must have the form

$$\begin{aligned} \Gamma_{23} &= (c_{23} + c_{44})l_2l_3 + c_{36}l_1l_3 + c_{25}l_1l_2, \\ \Gamma_{13} &= c_{36}l_2l_3 + (c_{13} + c_{55})l_1l_3 + c_{14}l_1l_2, \\ \Gamma_{12} &= c_{25}l_2l_3 + c_{14}l_1l_3 + (c_{12} + c_{66})l_1l_2, \end{aligned} \tag{6}$$

and

$$\begin{aligned} \Gamma_{23}^* &= \left(c_{23}^* + c_{44}\right)l_2l_3 + c_{36}^*l_1l_3 + c_{25}^*l_1l_2, \\ \Gamma_{13}^* &= c_{36}^*l_2l_3 + \left(c_{13}^* + c_{55}\right)l_1l_3 + c_{14}^*l_1l_2, \\ \Gamma_{12}^* &= c_{25}^*l_2l_3 + c_{14}^*l_1l_3 + \left(c_{12}^* + c_{66}\right)l_1l_2, \end{aligned} \tag{7}$$

with

$$\Gamma_{23}^* = \pm\Gamma_{23}, \qquad \Gamma_{13}^* = \pm\Gamma_{13}, \qquad \Gamma_{12}^* = \pm\Gamma_{12}. \tag{8}$$

3. There are eight sign combinations of off-diagonal terms of the Kelvin–Christoffel matrix, each corresponding to a characteristic equation (1) with identical coefficients for the terms with $\lambda_m$, $m = 1, \ldots, 3$. The condition $\Gamma_{23}^*\Gamma_{13}^*\Gamma_{12}^* = \Gamma_{23}\Gamma_{13}\Gamma_{12}$ divides the corresponding eight slowness surfaces into two classes containing each four elements with the same product $\Gamma_{23}\Gamma_{13}\Gamma_{12}$. The members of the two classes are identical in the coordinate planes, but differ outside these planes. Slowness surfaces corresponding to an odd number of "+" signs in Table 1 are called "normal". This table shows the sign combinations for the two sets of four slownesses each. Any two anomalous companion media differ in the algebraic signs of precisely two off-diagonal terms of the Kelvin–Christoffel matrix. Inspection of Eqs. (6)–(8) shows that this is possible only if either all three or precisely two of the three stiffnesses $\{c_{14}, c_{25}, c_{36}\}$ vanish: if two of these stiffnesses would not vanish, all three off-diagonal terms would be affected and would have to change sign. The two slowness surfaces would share the

**Table 1**
Sign combinations for two sets of four slownesses.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\Gamma_{23}$ | + | + | − | − | − | − | + | + |
| $\Gamma_{13}$ | + | − | + | − | − | + | − | + |
| $\Gamma_{12}$ | + | − | − | + | − | + | + | − |
| $\Gamma_{23}\ \Gamma_{13}\ \Gamma_{12}$ | + | + | + | + | − | − | − | − |

intersections with the coordinate planes, but would not be identical outside these planes. It follows that anomalous polarization is possible for any stiffness matrix that can be brought – through rotation of the coordinate system and/or exchange of subscripts – into the following form (indicating the symmetry plane in each case):

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & c_{25} & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ c_{14} & 0 & 0 & c_{44} & 0 & 0 \\ 0 & c_{25} & 0 & 0 & c_{55} & 0 \\ 0 & 0 & c_{36} & 0 & 0 & c_{66} \end{pmatrix}, \tag{9}$$

(i) $(x, y)$-symmetry plane if $c_{36} \neq 0$, $c_{14} = c_{25} = 0$.
(ii) $(x, z)$-symmetry plane if $c_{25} \neq 0$, $c_{14} = c_{36} = 0$.
(iii) $(y, z)$-symmetry plane if $c_{14} \neq 0$, $c_{25} = c_{36} = 0$.

The media defined by these matrices have normal polarization in the symmetry plane and anomalous polarization in the other orthogonal planes. Anomalous companion media can have at most ten non-vanishing elements of the elastic matrix.

The formal conditions for the existence of anomalous companion pairs were derived without regard to the stability of the corresponding media. Only stable media can exist under the laws of physics. An elastic medium is stable if and only if the strain energy is positive for each deformation field. This means that all principal minors of the stiffness matrix must be positive (in this terminology, a "minor" is the determinant of the corresponding sub-matrix; the main diagonal of the sub-matrix corresponding to a "principal minor" is a non-empty subset of the main diagonal of the matrix) (Helbig, 1994). This is equivalent with the requirement that the stiffness matrix must be positive definite (see Carcione, 2007).

## 3. Orthorhombic media

For orthorhombic media, we obtain the elasticity constants of the anomalous companions as:

$(x, y)$-plane

$$c^*_{13} = -(c_{13} + 2c_{55}),$$
$$c^*_{23} = -(c_{23} + 2c_{44}), \tag{10}$$

$(x, z)$-plane

$$c^*_{12} = -(c_{12} + 2c_{66}),$$
$$c^*_{23} = -(c_{23} + 2c_{44}), \tag{11}$$

$(y, z)$-plane

$$c^*_{12} = -(c_{12} + 2c_{66}),$$
$$c^*_{13} = -(c_{13} + 2c_{55}) \tag{12}$$

(Carcione, 2007), where we have indicated the symmetry plane where the polarization is normal.

Only companion pairs where $\{c_{23}, c_{13}, c_{12}\}$ and $\{c^*_{23}, c^*_{13}, c^*_{12}\}$ satisfy the stability conditions are meaningful. As mentioned before, stability imposes that all principal minors of the stiffness matrix must be positive The leading principal third-order minor

$$D_3 = c_{11}c_{22}c_{33} + 2c_{12}c_{23}c_{13} - c_{11}c^2_{23} - c_{22}c^2_{13} - c_{33}c^2_{12} \tag{13}$$

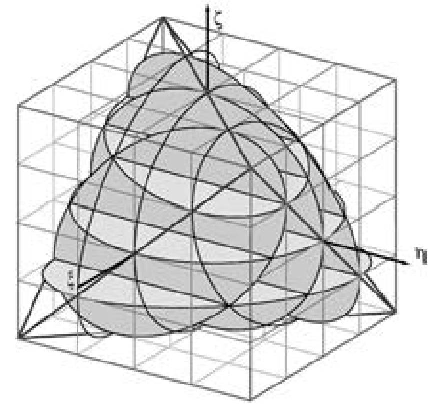

**Fig. 1.** An orthorhombic medium is stable if the "diagonal" stiffness components are positive and if the variables $\xi = c_{23}/\sqrt{c_{22}c_{33}}$, $\eta = c_{13}/\sqrt{c_{11}c_{33}}$ and $\zeta = c_{12}/\sqrt{c_{11}c_{22}}$ satisfy the four inequalities $\xi^2 < 1$, $\eta^2 < 1$, $\zeta^2 < 1$, and $2\xi\eta\zeta - \xi^2 - \eta^2 - \zeta^2 + 1 > 0$, i.e., if the point $(\xi, \eta, \zeta)$ lies inside the stability volume (the "tetrahedral cushion").

is positive if $c_{23}$, $c_{13}$ and $c_{12}$ satisfy

$$1 + 2\frac{c_{23}c_{13}c_{12}}{c_{11}c_{22}c_{33}} - \frac{c^2_{23}}{c_{22}c_{33}} - \frac{c^2_{13}}{c_{11}c_{33}} - \frac{c^2_{12}}{c_{11}c_{22}} > 0. \tag{14}$$

On the other hand, the second-order principal minors are positive if

$$-\sqrt{c_{22}c_{33}} < c_{23} < \sqrt{c_{22}c_{33}}, \qquad -\sqrt{c_{11}c_{33}} < c_{13} < \sqrt{c_{11}c_{33}},$$
$$-\sqrt{c_{11}c_{22}} < c_{12} < \sqrt{c_{11}c_{22}}, \qquad -\sqrt{c_{11}c_{44}} < c_{14} < \sqrt{c_{11}c_{44}} \tag{15}$$

(case (iii) in the discussion of Eq. (9)).

The equality corresponding to the inequality (14) can be regarded as a cubic equation in the three variables

$$\xi = \frac{c_{23}}{\sqrt{c_{22}c_{33}}}, \qquad \eta = \frac{c_{13}}{\sqrt{c_{11}c_{33}}}, \qquad \zeta = \frac{c_{12}}{\sqrt{c_{11}c_{22}}};$$
$$2\xi\eta\zeta - \xi^2 - \eta^2 - \zeta^2 + 1 = 0. \tag{16}$$

In a $\xi\eta\zeta$-system Eq. (16) describes a third-order surface with four singular points at, respectively, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$ and $(1, 1, 1)$. From each of the singular points extends a "flare" to infinity. The inner part of the cubic surface lies completely inside the cube defined by the singularities, which is also the range of validity prescribed by the inequalities (15). An orthorhombic medium is stable if the point $\{\xi, \eta, \zeta\}$ lies inside this "stability volume". The stability volume is shown in Fig. 1.

Since $c_{12}$ (and thus $\zeta$) remains unchanged, we can restrict the investigation to the intersection of the stability volume with the plane $\zeta = c_{12}/\sqrt{c_{11}c_{22}}$, e.g., for $\zeta_c = 0.7$ with the ellipse shown in Fig. 2. The shear-stiffnesses $c_{44}$, $c_{55}$ and $c_{66}$ are only constrained to be positive, but since – according to Eq. (10) – they interact with the off-diagonal stiffnesses, they are normalized in the same way:

$$\Xi = \frac{c_{44}}{\sqrt{c_{22}c_{33}}}, \qquad H = \frac{c_{55}}{\sqrt{c_{11}c_{33}}}, \qquad Z = \frac{c_{66}}{\sqrt{c_{11}c_{22}}}. \tag{17}$$

A sub-region of the stability ellipse containing the points $\{\xi, \eta\}$ that allow the existence of anomalous companion pairs is easily established: For $\xi = \xi_c$ (with $-1 < \xi_c < 1$) the point $\{\xi, \eta\}$ corresponds to a stable medium if it satisfies

$$\xi^2 + \eta^2 - 2\zeta_c\xi\eta < 1 - \zeta^2_c, \tag{18}$$

i.e., if (for $\zeta_c = 0.7$) it lies inside the light grey ellipse in Fig. 2. An anomalous companion medium exists if also the point $\{\xi^*, \eta^*\}$ corresponds to a stable medium. For this, $\{\xi, \eta\}$ must satisfy in addition

$$(\xi - 2\Xi)^2 + (\eta - 2H)^2 - 2\zeta_c(\xi - 2\Xi)(\eta - 2H) < 1 - \zeta^2_c, \tag{19}$$

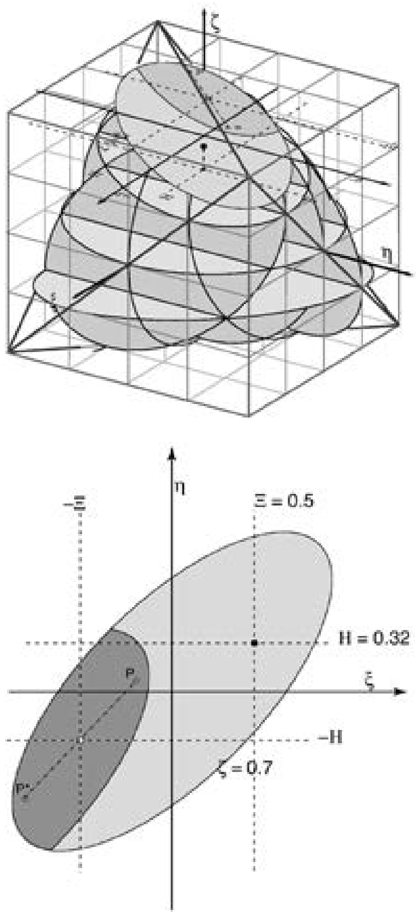


**Fig. 2.** Intersection of the stability volume of the original medium with that of the anomalous companion medium is shown for the plane $\zeta = 0.7$. The intersection is non-empty if – and only if – the point $(\Xi, H, \zeta)$ lies within the original stability volume.

i.e. it must lie in a shifted but similar ellipse. A point that satisfies both the inequalities (18) and (19) lies in the intersection of the two ellipses. In the example of Fig. 2, this "anomalous companion range" is indicated by a darker gray. Anomalous polarization is possible unless the intersection is empty.

It follows from Fig. 2 that a non-empty intersection of the two ellipses exists if the point $\boldsymbol{\Xi} = \{\Xi, H\}$ lies in the "open" ellipse: if the point would lie on the boundary, the "shifted" ellipse would move by a diameter in the opposite direction; the two ellipses would touch, and the intersection would be empty.

From this one can establish a simple rule over which range of off-diagonal stiffnesses of an orthorhombic medium with given diagonal stiffnesses anomalous polarization exists. As before we restrict the investigation to anomalous polarization in the $(x, z)$- and $(y, z)$-planes. The line $\{\xi, \eta\} = \{\Xi, H\}$ is parallel to the $z$-axis and lies completely in the quadrant enclosed by the positive $(x, z)$- and $(y, z)$-planes. If $\{0, 0\} < \{\Xi, H\} < \{1, 1\}$, the line intersects the stability volume. The segment of the line inside the stability volume corresponds to off-diagonal stiffnesses for which anomalous polarization is possible. By substituting $\Xi$ and $H$ for $\xi$ and $\eta$ in Eq. (16), one gets the quadratic equation

$$\zeta^2 - 2\Xi H\zeta - 1 + \Xi^2 + H^2 = 0, \quad \text{with the solutions}$$

$$\zeta_{1,2} = \Xi H \mp \sqrt{1 - \Xi^2 - H^2 + \Xi^2 H^2}. \tag{20}$$

For any $\xi_c$ satisfying $\xi_1 < \xi_c < \xi_2$ the anomalous companion range is non-empty, i.e., a range of companion pairs $\{\xi, \eta\}$, $\{\xi^*, \eta^*\}$ exists.

## 4. Monoclinic media

For monoclinic media, we obtain the elasticity constants of the anomalous companions as:

$(x, y)$-symmetry plane:

$$\begin{aligned} c^*_{36} &= -c_{36}, \\ c^*_{13} &= -(c_{13} + 2c_{55}), \\ c^*_{23} &= -(c_{23} + 2c_{44}), \end{aligned} \tag{21}$$

$(x, z)$-symmetry plane:

$$\begin{aligned} c^*_{25} &= -c_{25}, \\ c^*_{12} &= -(c_{12} + 2c_{66}), \\ c^*_{23} &= -(c_{23} + 2c_{44}), \end{aligned} \tag{22}$$

$(y, z)$-symmetry plane:

$$\begin{aligned} c^*_{14} &= -c_{14}, \\ c^*_{12} &= -(c_{12} + 2c_{66}), \\ c^*_{13} &= -(c_{13} + 2c_{55}) \end{aligned} \tag{23}$$

(Carcione, 2007).

It is shown in Carcione (2007) that, in the case of an $(x, y)$-symmetry plane, the relation between the anomalous and normal polarizations is simply

$$\mathbf{u}^* = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{u}. \tag{24}$$

This means that the polarization direction of two companion media are everywhere symmetric to a line parallel to – in this case – the $z$-axis.

## 5. Examples

In this section, the theoretical results are further verified by means of full wave numerical modeling. Firstly, we consider the examples given by Helbig and Schoenberg (1987) in a plane containing the symmetry axis of a transversely isotropic medium. Since this is an axis of rotational invariance, a 2-D simulation is enough to analyze the polarization behavior. On the other hand, for monoclinic media we use a 3-D modeling technique in order to analyze the polarization features out of the symmetry planes.

The forward modeling codes are based on the Fourier pseudospectral method for computing the spatial derivatives and Chebyshev expansions of the evolution operator as the time-integration technique. The details about the 2-D and 3-D algorithms can be found in Carcione et al. (1992, 1988), respectively. These algorithms possess spectral accuracy for band-limited signals and are not affected by temporal or spatial numerical dispersion.

### 5.1. Transversely isotropic media

In order to simulate a case illustrated by Helbig and Schoenberg (1987) in their Figs. 2 and 3 (case 1, transversely isotropic media with vertical symmetry axis), we adopt their convention for defining the material properties:

$$A = \frac{c_{13}}{c_{55}} + 1, \qquad B = \frac{c_{11}}{c_{55}} - 1, \quad \text{and} \quad C = \frac{c_{33}}{c_{55}} - 1,$$

with $A = 1.2$, $B = 2.24$ and $C = 2.61$. The parameters of the medium with anomalous polarization are the same except for a

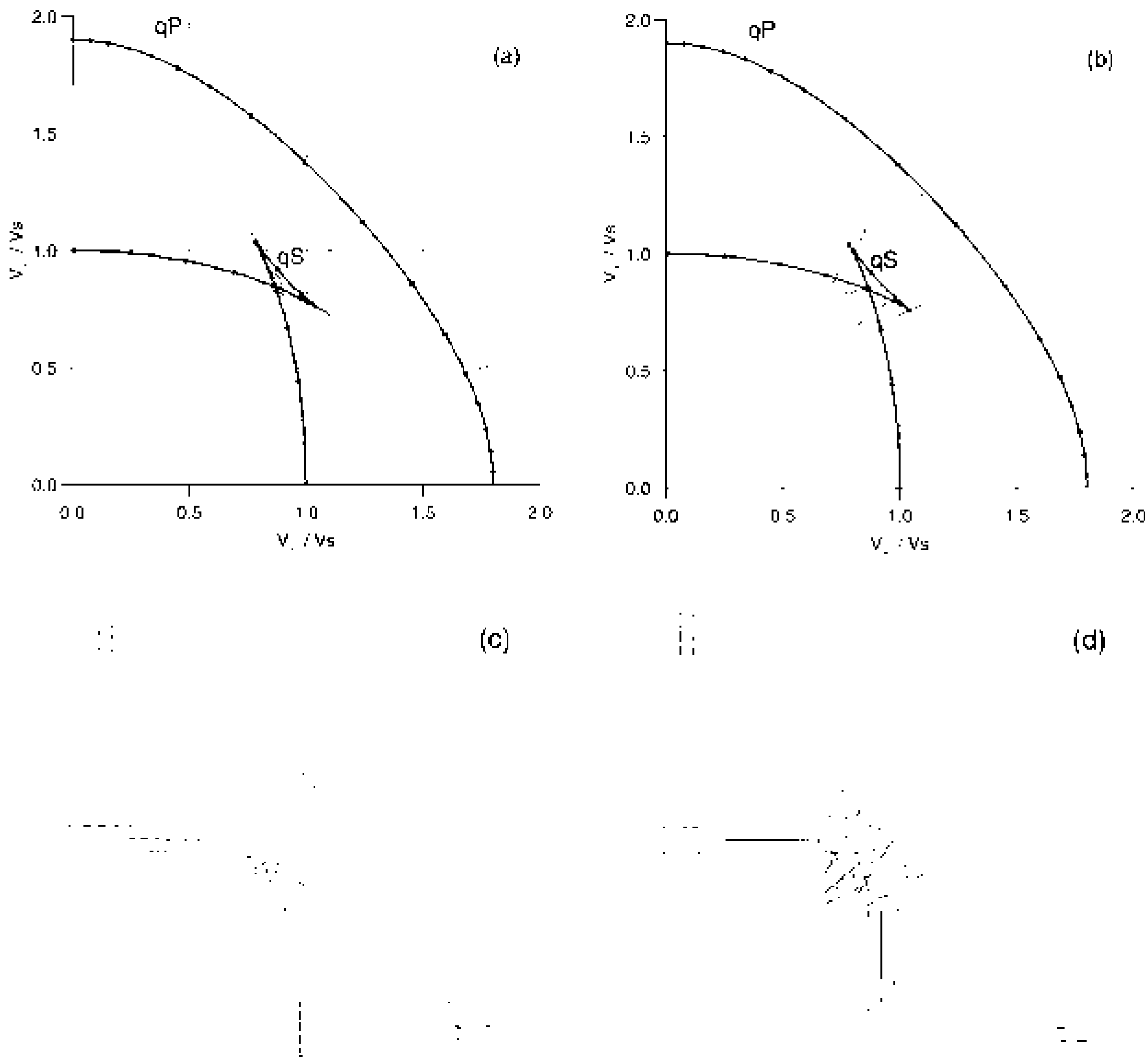


**Fig. 3.** Normalized ray velocity sections and snapshots for a transversely isotropic medium. Figures (b) and (d) correspond to the anomalous medium.

change of sign for $A$. The size of the numerical mesh is $225 \times 225$ grid points, and the motion is initiated by a directional force making an angle $\pi/4$ with the symmetry axis.

The snapshots are shown in Fig. 3, where only one quadrant of the 2-D model is represented. Figures (a) and (b) represent the ray (group) velocity section for normal and anomalous polarization, respectively, with the "tadpoles" indicating the polarization directions. Figures (c) and (d) are the respective snaphots with the segments representing the displacement vector every third grid point of the mesh. The ray velocities are normalized with respect to the shear velocity $\sqrt{c_{55}/\rho}$. As predicted by the theory, the $z$-axis (the vertical axis in the figure) bisects the angle between the normal and the anomalous polarizations. In general, the anomaly is more pronounced around $\pi/4$, where the faster wave is transversely polarized and the slower wave is longitudinally polarized. Moreover, note that the fastest branch of the triplication event has longitudinal polarization.

For media of at least orthorhombic symmetry and $\{c_{11}, c_{22}, c_{33}\} > \{c_{44}, c_{55}, c_{66}\}$ in their natural coordinate system, the polarization of the qP wave is parallel or antiparallel to the propagation direction for propagation along the coordinate axes. The polarization in a coordinate plane is called *normal* if the angle $\delta$ between propagation direction and the polarization direction is acute everywhere, i.e., $-\pi/2 < \delta < \pi/2$, else it is called anomalous. If under normal polarization, $\delta = 0$ is chosen at one of the two axes of the coordinate plane, then one has $\delta = 0$ at the other axis. Under anomalous polarization, one has $\delta = \pm\pi$ at the other axis.

In normal polarization media, the scalar product of the propagation vector and the longitudinal polarization vector is always positive, but for anomalous polarization may range from $-1$ to 1 for every anomalous quadrant.

### 5.2. *Monoclinic media*

We first consider a monoclinic medium having the $(y, z)$-plane as symmetry plane. Two of the four polarization distributions corresponding to the "normal" slowness surface – with sign combinations on the left-hand side of Table 1 – are shown in Fig. 4. This figure shows the intersections of the slowness surface with the three planes of symmetry, and the polarization vector for the fastest (innermost) sheet. The "zones" of anomalous polarization are clearly visible in the figure.

The stiffness matrix of the monoclinic medium with normal polarization is

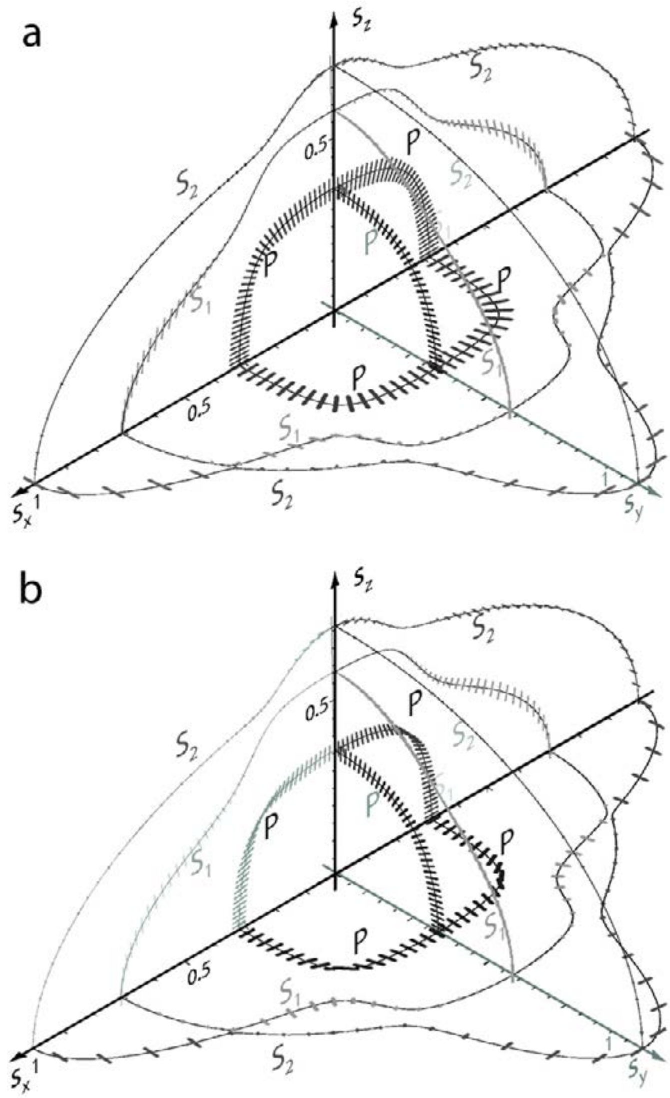


**Fig. 4.** Intersection of the slowness surfaces of the normally polarized medium (a) and anomalously polarized medium (b) with the coordinates planes. The normally polarized medium corresponds to the stiffness matrix (25). The polarization vectors are indicated. Anomalous polarization occurs in the $(x, y)$- and $(x, z)$-planes. The angle between the qP polarization and the propagation direction goes beyond $\pi/2$ in the midrange of the two planes (close to the letter "P").

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ c_{14} & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix} = \begin{pmatrix} 10 & 2 & 1.5 & 0.8 & 0 & 0 \\ 2 & 9 & 1 & 0 & 0 & 0 \\ 1.5 & 1 & 8 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{25}$$

(normalized by $\rho\times$ MPa, where $\rho$ is the density in kg/m$^3$). Then, according to Eq. (23), the companion medium has $c^*_{12} = -4$ GPa, $c^*_{13} = -5.5$ GPa and $c^*_{14} = -0.8$ GPa. The size of the numerical mesh is $165 \times 165 \times 165$ grid points, and the motion is initiated by a directional force making an angle $\pi/4$ with the $z$-axis (the vertical axis in the plots).

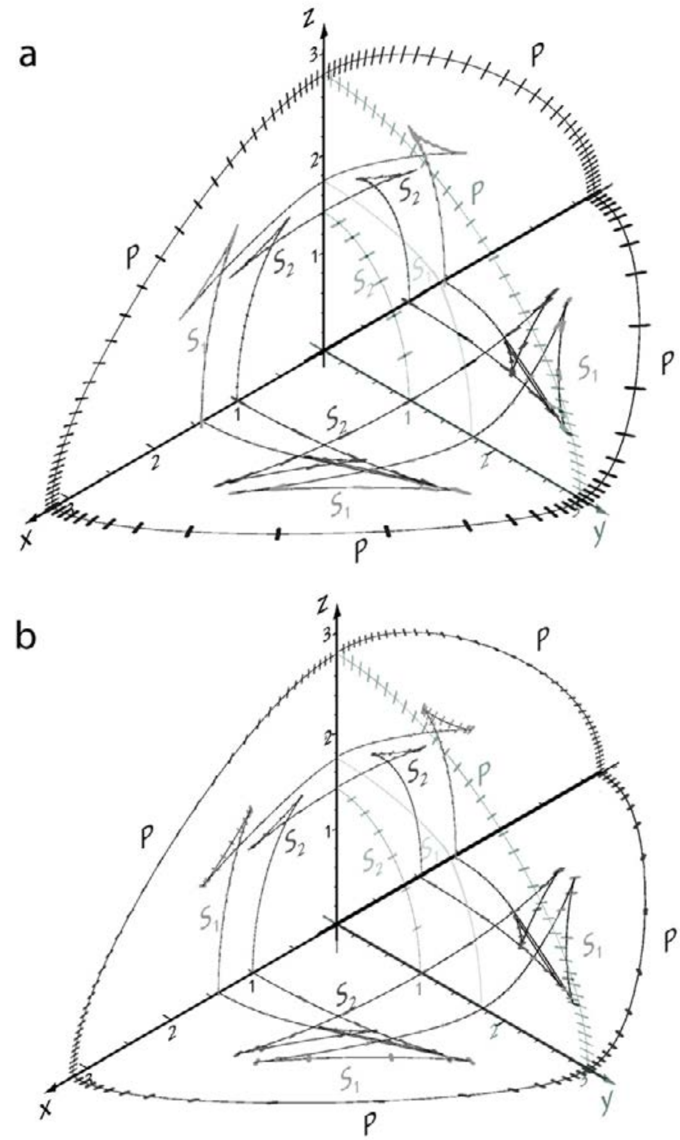


**Fig. 5.** Intersection of the wave (ray) surfaces of the normally polarized medium (a) and anomalously polarized medium (b) with the coordinates planes. The normally polarized medium corresponds to the stiffness matrix (25). The polarization vectors are indicated. Anomalous polarization occurs in the $(x, y)$- and $(x, z)$-planes. The angle between the qP polarization and the propagation direction goes beyond $\pi/2$ in the midrange of the two planes (close to the letter "P").

Fig. 5 shows the group velocities in three perpendicular Cartesian planes, where one of them (i.e., the $(y, z)$-plane) is the symmetry plane. The polarization is indicated on the curves; when it is not plotted, the particle motion is perpendicular to the respective plane (cross-plane shear waves). The snapshots corresponding the numerical simulation are shown in Fig. 6. As can be seen, the companion medium has anomalous polarization in the $(x, y)$- and $(x, z)$-planes, while the polarizations in the $(y, z)$-plane are unaltered. The anomaly is more pronounced about 45° where the polarization of the fastest wave is quasi-transverse and the cusp lid is essentially longitudinal. Moreover, the cross-plane shear wave with polarization perpendicular to the symmetry plane can clearly be seen.

The last example considers a monoclinic medium having the $(x, y)$-plane as symmetry plane. The stiffness matrix of the monoclinic medium is

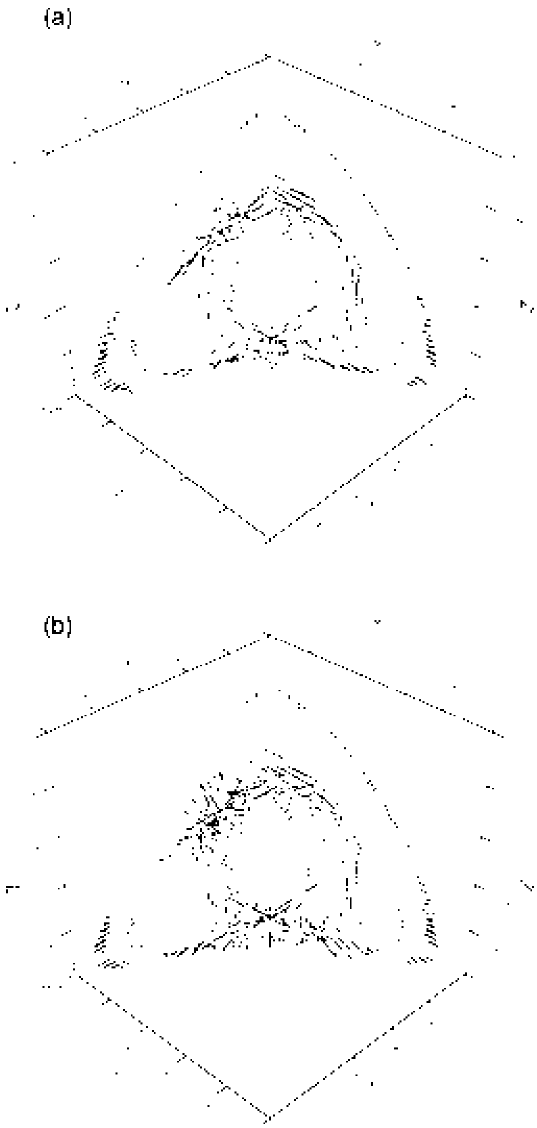


**Fig. 6.** Snapshots of the displacement vector corresponding to the case shown in Fig. 5.

$$\mathbf{C}=\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0\\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0\\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36}\\ 0 & 0 & 0 & c_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{55} & 0\\ 0 & 0 & c_{36} & 0 & 0 & c_{66} \end{pmatrix} =\begin{pmatrix} 10 & 2.68 & 3.1 & 0 & 0 & 0\\ 2.68 & 8 & -3.46 & 0 & 0 & 0\\ 3.1 & -3.46 & 6 & 0 & 0 & 1\\ 0 & 0 & 0 & 2.08 & 0 & 0\\ 0 & 0 & 0 & 0 & 1.55 & 0\\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

Then, according to Eq. (21), the companion medium has $c^*_{13}=-6.2$ GPa, $c^*_{23}=-0.7$ GPa and $c^*_{36}=-1$ GPa. The respective group velocities and snapshots are shown in Figs. 7 and 8. The companion medium should have anomalous polarization in the $(y,z)$- and $(x,z)$-planes, while the polarizations in the $(x,y)$-plane should not change. However, note that in (a) the polarization is anomalous in the $(y,z)$-plane, and has changed to normal in (b). Conversely, the polarization in the $(x,z)$-plane has changed from normal to anomalous.

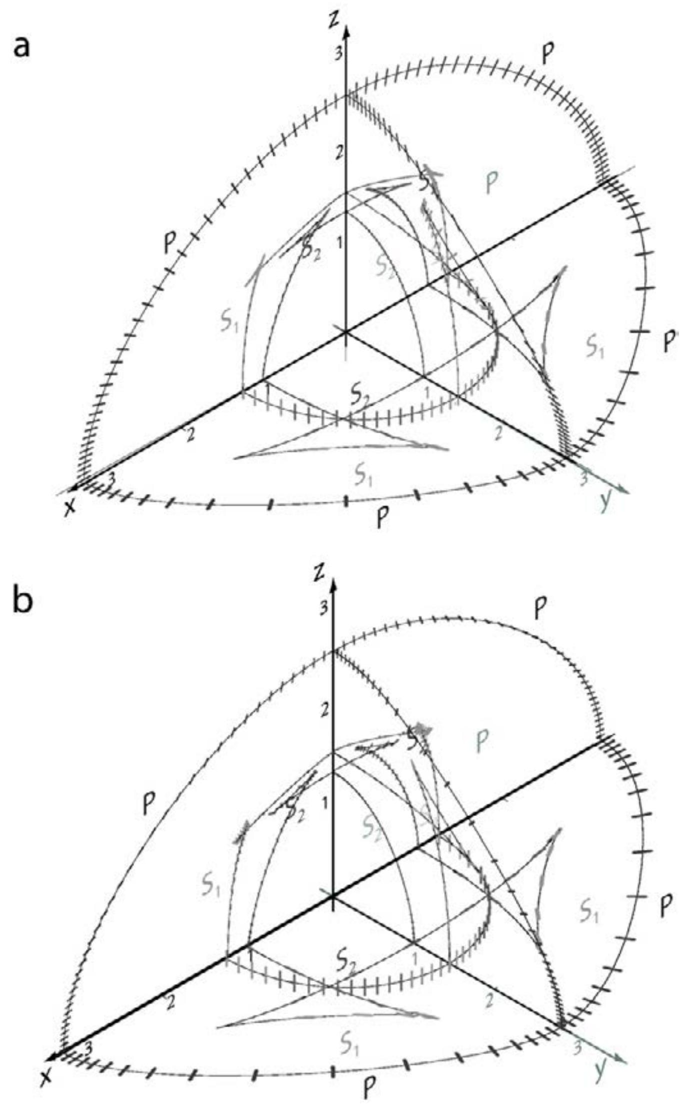


**Fig. 7.** Intersection of the wave (ray) surfaces of the medium defined by the stiffness matrix (26) (a) and that of the companion medium (b) with the coordinates planes. Polarization vectors are indicated. Note that in (a) the polarization is anomalous in the $(y,z)$-plane. Anomalous polarizations should occur in the $(x,z)$- and $(y,z)$-planes. However, note that in the $(y,z)$-plane the polarization has changed to normal, while in the $(x,z)$-plane it is now anomalous.

## 6. Conclusions

A pair of "anomalous companion media" shares the characteristic surfaces slowness- and wave surface, but the nature of the polarization differs in precisely two coordinate planes. It follows that at most four media with identical slowness surfaces form a set (provided all four are stable), from which at most six anomalous companion pairs can be formed. If the number of coordinate planes with anomalous polarizations is even (i.e., if the medium with all polarizations normal is a member of the set), the set is called the "even". The corresponding "odd" set consists of (at most) three media with anomalous polarization in a single coordinate plane and one medium with anomalous polarization in all three coordinate planes. The slowness- and wave surfaces of the media of the odd and even sets have the identical intersections with the coordinate planes, but are different outside the coordinate planes. Note that the condition $\{c_{11},c_{22},c_{33}\}>\{c_{44},c_{55},c_{66}\}$ is not necessary for a medium to be stable. For media with $\{c_{44},c_{55},c_{66}\}>\{c_{11},c_{22},c_{33}\}$ the intersection of the two stability volumes (Fig. 2)

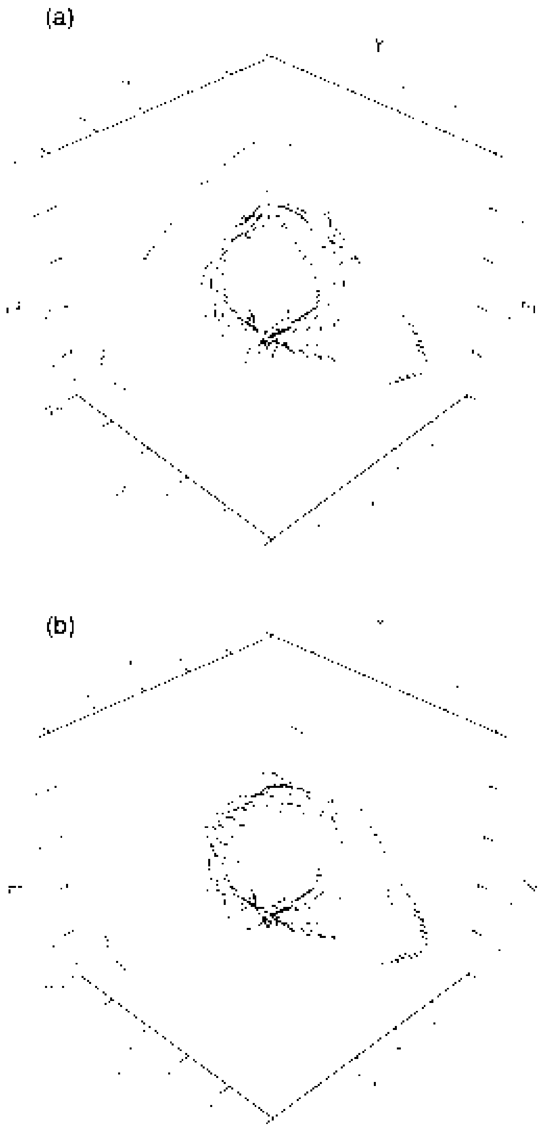

**Fig. 8.** Snapshots of the displacement vector corresponding to the case shown in Fig. 7.

would be empty. If the sets of compressional and shear stiffness are not disjoint (as is, e.g., possible for wood), *single* anomalous companion pairs can exist.

The axis common to the two planes in which the polarization differs in the two media of an anomalous companion pair bisects the polarization directions for any propagation direction (not only in the planes of symmetry).

It is known that anomalous companion media exist in transversely isotropic (Helbig and Schoenberg, 1987) and in orthorhombic media (Carcione and Helbig, 2000) for certain ranges of the components of the stiffness tensor. We show in this paper that anomalous companion pairs exist for monoclinic media. The natural coordinate system for a monoclinic medium is that in which the stiffness component $c_{45}(c_{46}, c_{56})$ vanishes. This is always possible through a simple rotation: if the stiffness matrix of a monoclinic medium has $c_{45} \neq 0$, the rotation about the $z$-axis by

$$\alpha = \frac{1}{2}\arctan\left(\frac{2c_{45}}{c_{55} - c_{44}}\right)$$

makes $c_{45}$ vanish. In this coordinate system the monoclinic stiffness matrix of a medium that admits anomalous companions differs from that of an orthorhombic medium that admits anomalous companions by one added matrix element from the set $\{c_{14}, c_{25}, c_{36}\}$. The added stiffness element must come from a range proscribed by stability. However, for orthorhombic "root" media that admit anomalous polarization, this set is always non-empty. Two examples of such pairs are shown, one from an even set, one from odd set.

This analysis does not exclude that media with a symmetry less than monoclinic may have anomalous companion pairs. The conditions for the existence of these media will be investigated in a future work.

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