Bounds and averages of seismic quality factor Q

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ABSTRACT

An elastic two-phase composite, with no restriction on the shape of the two phases, has stiffness bounds given by the Reuss and Voigt equations, and a narrower range determined by the Hashin-Shtrikman bounds. Averages are given by the Voigt-Reuss-Hill, Hashin-Shtrikman, Gassmann, Backus and Wyllie equations. To obtain stiffness bounds and averages, we invoke the correspondence principle to compute the solution of the viscoelastic problem from the corresponding elastic solution. Then, seismic velocities and attenuation are established for the above - physical and heuristic - models which account for general geometrical shapes, unlike the Backus average. The approach is relevant to the seismic characterization of solid composites such as hydrocarbon source rocks.

Keywords: seismic attenuation, Voigt and Reuss bounds, Hashin-Shtrikman bounds, Reuss-Voigt-Hill average, Gassmann-Krief-Ciz-Shapiro average, Backus and Wyllie average, *Q* bounds

1. INTRODUCTION

The amplitude of seismic waves play an important role in determining the properties of rocks. Amplitude variations highly depend on frequency, since rocks are generally heterogeneous and anelastic. Depending on the wavelength, the behaviour of the signal differs. A composite has stiffness bounds given by the Reuss (or Wood) and Voigt equations, and by the Hashin-Shtrikmann (HS) equations. To obtain *Q* (quality factor) bounds, we invoke the correspondence principle to compute the solution of a viscoelastic problem from the corresponding elastic solution (e.g., *Carcione, 1992*; *Zhu et al., 2007*). The viscoelastic solution is obtained by substituting the elastic moduli by the corresponding complex moduli. The bounds can be used to test the validity of experimental measurements and theoretical models of attenuation and dispersion in rocks. It is important to point out possible limitations of the correspondence principle. Firstly, the boundary constraints should be independent of time for the principle to be valid (*Graham, 1968*). Then, the relaxation function should be separable in space and time, even if its time dependence not necessarily should be time translation invariant

(*Khazanovich, 2008*). On the other hand, *Hilton (2009)* states that relaxation moduli, compliances, and creep and relaxation functions properly allow the use of the elastic-viscoelastic correspondence principle. In any case, the principle is locally valid in homogeneous media, which is the case considered in this work, and in numerical modeling of wave propagation.

The stiffness of a composite medium can be obtained as the arithmetic averages of the Reuss and Voigt moduli (the Voigt-Reuss-Hill (VRH) average) and the lower and upper HS bounds. Similarly, Gassmann equation generalized to the case of a solid pore infill yields a physical average (*Ciz and Shapiro, 2007*). At high frequencies, a composite is heterogeneous and the wavefield can be represented by rays. In this case, the - heuristic - Wyllie time-average equation holds. At low frequencies, a finely-layered medium is effectively homogeneous and Backus averaging can be used to obtain the seismic properties (*Carcione et al., 1998*). *Qadrouh et al. (2017)* obtained averages for seismic Q, based on these two approaches. The Wyllie quality factor is higher than the Backus one, following the velocity trend, i.e., the higher the velocity (the stiffer the medium), the higher the attenuation.

Roscoe (1969) shows that for a mix of viscous media (moduli purely imaginary), the average viscosity lies at least between the Voigt and Reuss bounds, suggesting that attenuation behaves as the elastic moduli. *Chen and Lakes (1993)* performed a theoretical study of viscoelastic properties of composites to model high stiffness and high loss tangent (inverse quality factor). They obtained Voigt and Reuss averages as well as the HS bounds evaluated via the correspondence principle, as it is done here, showing that the lower and upper two-phase HS composites behave similarly to the Voigt and Reuss composites, respectively. Here, we find that the HS Q bounds lie within the Voigt and Reuss Q bounds.

On the other hand, *Gibiansky and Lakes (1997*) show that the quality factor associated with the bulk modulus of the composite can be no smaller than the minimal and no larger than the maximal quality factor of either phase (see their Eq. (5.3)). They also observe that adding small amounts of soft dissipative materials in a stiff matrix may dramatically increase the overall dissipation rate of the composite, and that the overall dissipation increases with the stiffness of the stiff phase.

Recently, *Gurevich and Makarynska (2012)* analyzed bounds for the stiffness moduli of a porous medium, that is, a mixture of an elastic solid and a linear Newtonian fluid. These bounds are extension of the HS bounds to the moduli of composite viscoelastic media. However, they do not deal with bounds for the phase velocity and quality factor, and their model is not based on solid-solid viscoelastic composites. Effects such as the squirt-flow loss or attenuation due to cracks, can be incorporated in the theory by modeling one of the phases with the bulk modulus associated with an specific porous-media theory, which is basically approximated by Zener kernels (e.g., *Carcione and Gurevich, 2011*), or one can use other approaches via the correspondence principle. In fact, these effects can be included in the complex and frequency-dependent bulk and shear moduli of each phase.

Mavko and Saxena (2016) use the the elastic-viscoelastic correspondence principle, showing that viscoelastic modeling can be done consistently. There are no Q bounds or averages in this work, which deals with creep functions and the correspondence principle, which is used in the present work. *Glubokovskikh and Gurevich (2017)* obtain bounds of the complex moduli, and their model is a particular case of porous media (fluid-solid mixture), where the fluid has viscosity and the grains are lossless. However, they do not obtain bounds of the wave-attenuation properties for fully anelastic solid-solid mixtures. Moreover, our approach is different.

In the present work, we establish bounds and averages on seismic *Q*, attenuation factor and wave velocities, based on different models. The bounds are general enough to be applied to any medium, where wave propagation undergoes dissipation and velocity dispersion, such as composite viscoelastic and poro-viscoelastic materials. In particular, the methodology is relevant to the study of hydrocarbon source rocks, where the medium is composed of two main solid phases, i.e., illite/smectite saturated with water and kerogen saturated with oil and gas (e.g., *Carcione, 2000*; *Carcione et al., 2011*). Comparison with experimental data validates the approach.

2. STIFFNESS BOUNDS

It is well known that an elastic two-phase composite, with no restriction on the shape of the two phases, has stiffness bounds given by the Reuss and Voigt averages (*Mavko et al., 2009*). To obtain the bounds in viscoelastic media, we use the the correspondence principle (e.g., *Carcione, 2014*). The viscoelastic solution is obtained by substituting the elastic moduli by the complex moduli (*Carcione, 2014*).

Let us consider a composite made of two materials, with - complex and frequency dependent - stiffness M_i, $i = 1, 2$, representing either the bulk and shear moduli K_i or μ_i, respectively, which correspond to the fundamental deformations of an isotropic solid. The Reuss average is the isostress average because it is equal to the ratio of the stress to the average strain when all constituents are assumed to have the same stress. On the other hand, in the Voigt average, the strain of the different phases is uniform. With these considerations it is easy to show that the complex Reuss and Voigt averages of a two-phase composite are

$$M^- = \left(\frac{\phi}{M_1} + \frac{1-\phi}{M_2} \right)^{-1} \quad \text{and} \quad M^+ = \phi M_1 + (1-\phi) M_2 , \tag{1}$$

respectively, where ϕ is the proportion of material 1, and superscripts + (–) denote the upper (Voigt) and lower (Reuss) bounds, respectively.

The Hashin-Shtrikman-Walpole bounds (*Hashin and Shtrikman, 1963*; *Mavko et al., 2009*) are narrower than the preceding bounds, and there is no assumption on the geometry of the single materials. The HS bounds for the bulk and shear moduli are

$$K_{HS}^{\pm} = K_1 + \frac{1-\phi}{(K_2 - K_1)^{-1} + \phi \left(K_1 + \frac{4}{3}\mu_\beta \right)^{-1}} \tag{2}$$

and

$$\mu_{HS}^{\pm} = \mu_1 + \frac{1-\phi}{\left(\mu_2-\mu_1\right)^{-1} + \phi\left(\mu_1 + \frac{\mu_\beta}{6}\frac{9K_\beta+8\mu_\beta}{K_\beta+2\mu_\beta}\right)^{-1}}, \tag{3}$$

where we obtain the upper bounds when K_β and μ_β are the maximum bulk and shear moduli of the single components, respectively, and the lower bounds when these quantities are the corresponding minimum moduli (*Mavko et al., 2009*). This identification is based on the lossless case. Generalization of the Reuss, Voigt and HS bounds to the case of *n* layers is straightforward (*Mavko et al., 2009*).

The arithmetic average of the upper and lower bounds are frequently used to obtain an approximation of the effective stiffness of composites. It can be shown that if the two media have the same shear modulus ($\mu_1 = \mu_2$), the lower and upper HS bounds of the bulk modulus coincide and satisfy (*Hill, 1963*)

$$\frac{1}{K_{HS}+\frac{4}{3}\mu} = \frac{\phi}{K_1+\frac{4}{3}\mu} + \frac{1-\phi}{K_2+\frac{4}{3}\mu}, \tag{4}$$

where K_{HS} is the Hill average. This equation is the Reuss average of the P-wave modulus.

3. STIFFNESS AVERAGES

Stiffness averages can be obtained as the VRH equation

$$M = \frac{1}{2}\left(M^- + M^+\right), \tag{5}$$

and the HS averages

$$K = \frac{1}{2}\left(K_{HS}^- + K_{HS}^+\right) \quad \text{and} \quad \mu = \frac{1}{2}\left(\mu_{HS}^- + \mu_{HS}^+\right). \tag{6}$$

On the other hand, poroelasticity provides other expressions, based on the Gassmann-Krief-Ciz-Shapiro equations

$$K = \frac{K_2 - \bar{K} + \phi\bar{K}\left(\frac{K_2}{K_1}-1\right)}{1-\phi-\frac{\bar{K}}{K_2}+\frac{\phi K_2}{K_1}}, \quad \bar{K} = K_2\left(1-\phi\right)^{3/(1-\phi)}, \tag{7}$$

and

$$\mu = \frac{\mu_2 - \bar{\mu} + \phi\bar{\mu}\left(\frac{\mu_2}{\mu_1}-1\right)}{1-\phi-\frac{\bar{\mu}}{\mu_2}+\frac{\phi \mu_2}{\mu_1}}, \quad \bar{\mu} = \mu_2\left(1-\phi\right)^{3/(1-\phi)}, \tag{8}$$

where material 1 is the (solid) pore infill, material 2 is the mineral, and $\bar{K}$ and $\bar{\mu}$ denote the dry-rock moduli obtained with Krief equation (*Ciz and Shapiro, 2007*; *Carcione et al., 2011*). This model is not restricted to the Krief equation, since other approaches, such that of *Eshelby (1956)*, can be used to obtain the dry-rock moduli.

Similarly, the Backus and Wyllie equations provide averages, with Backus average valid for finely-layered media (effectively transversely isotropic at low frequencies). Let p_i be the material proportion, such that $\sum_i p_i = 1$. For two layers, we have $p_1 = \phi$ and $p_2 = 1 - \phi$. At the long wavelength limit, the complex stiffnesses, normal to the stratification (33 and 55 components), are given by the Backus averages:

$$K = \left(\sum_i \frac{p_i}{K_i + \frac{4\mu_i}{3}} \right)^{-1} \quad \text{and} \quad \mu = \left(\sum_i \frac{p_i}{\mu_i} \right)^{-1}, \tag{9}$$

respectively (e.g., *Qadrouh et al., 2017*).

In the case that the signal wavelength is very short (infinite frequency), the stiffnesses are

$$K = \rho \left(\sum_i p_i \sqrt{\frac{\rho_i}{K_i + \frac{4\mu_i}{3}}} \right)^{-2} \quad \text{and} \quad \mu = \rho \left(\sum_i p_i \sqrt{\frac{\rho_i}{\mu_i}} \right)^{-2}, \tag{10}$$

based on the Wyllie or time average equation (*Qadrouh et al., 2017*), where $\rho = \rho_1 \phi + \rho_2 (1 - \phi)$ is the arithmetic average of the density.

4. BOUNDS ON SEISMIC PROPERTIES

The phase velocity c_p, attenuation factor α and quality factor Q are, respectively,

$$c_p = \left[\mathrm{Re}\left(\frac{1}{c} \right) \right]^{-1}, \tag{11}$$

$$\alpha = -\omega \, \mathrm{Im}\left(\frac{1}{c} \right), \tag{12}$$

and

$$Q = \frac{\mathrm{Re}\left(c^2 \right)}{\mathrm{Im}\left(c^2 \right)}, \tag{13}$$

where $\omega = 2\pi f$ is the angular frequency, c denotes c_P or c_S, being the complex and frequency-dependent P-wave and S-wave velocities

$$c_P = \sqrt{\frac{K + \frac{4\mu}{3}}{\rho}} \quad \text{and} \quad c_S = \sqrt{\frac{\mu}{\rho}} , \tag{14}$$

respectively (*Mainardi, 2010*; *Carcione, 2014*).

5. EXAMPLES

Let us consider a composite medium. For the first example, we assume Poisson media for each of the components (the Lamé constants are equal and $K = 5\mu/3$), but the theory is general and not restricted to this type of media (see the next examples). Moreover, anelasticity is described by Zener elements, such that the complex modulus is

$$M = M_0 \frac{1 + \mathrm{i}\omega\tau a}{1 + \frac{\mathrm{i}\omega\tau}{a}} , \quad a = Q_0^{-1} + \sqrt{1 + Q_0^{-2}} , \tag{15}$$

where M represents either K or μ, Q_0 is the minimum quality factor at the frequency $f_0 = 1/2\pi\tau$, which is the centre frequency of the relaxation peak, and $\mathrm{i} = \sqrt{-1}$. The quantities f_0, c_0 and Q define the media (e.g., *Carcione, 2014*). When $Q = \infty$, $a = 1$ and $M = M_0$, the medium is lossless. Moreover $M(\omega = 0) = M_0$ and $M(\omega = \infty) = a^2 M_0$. For each single medium of the two-phase composite, we assume:

$$c_{0S} = \frac{c_{0P}}{\sqrt{3}} , \quad \rho\,[\mathrm{g\,cm^{-3}}] = 1.74 c_{0P}^{1/4}\,[\mathrm{km\,s^{-1}}] , \quad E_0 = \rho c_{0P}^2 , \quad \mu_0 = \rho c_{0S}^2 ,$$

$$K_0 = E_0 - \frac{4}{3}\mu_0 , \quad Q_{0K} = 10 K_0 \ (K_0 \text{ in GPa}), \quad \text{and} \quad Q_{0\mu} = Q_{0K}\frac{\mu_0}{K_0} ,$$

where the subscript "0" means zero frequency or relaxed.

Gibiansky and Lakes (1997) show that the quality factor associated with the bulk modulus of the composite can be no smaller than the minimal and no larger than the maximal quality factor of either phase (their Eq. (5.3)). Figure 1 shows that the bounds are within the values of the minimal and maximal quality factors, indicated in the plot. Also, the Reuss and Voigt Qs provide lower and upper bounds, as well as the corresponding HS bounds, indicating that they follow the trend of the moduli (real part) and therefore the velocities, as reported by *Roscoe (1969)*. Moreover, as indicated by *Chen and Lakes (1993)*: "*In the stiffness-loss map, the lower and upper two-phase Hashin composites behave similarly to the Voigt and Reuss composites, respectively. This is in contrast to the usual plots of elastic stiffness vs volume fraction, in which the Hashin bounds can differ greatly from the Voigt/Reuss ones*".

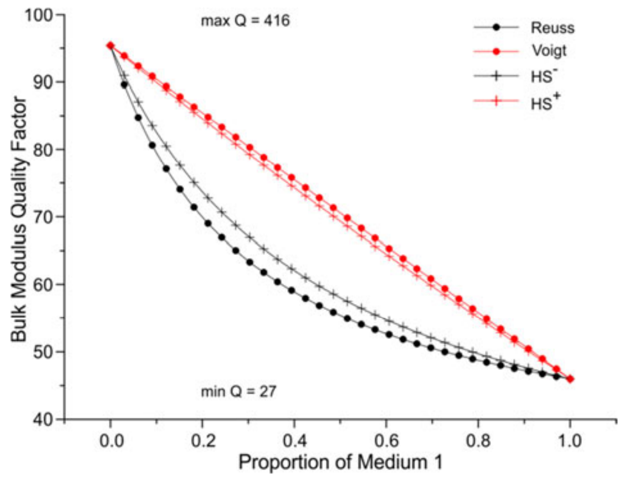

Fig. 1. Bulk modulus Q as a function of the proportion of Medium 1 for different bounds and frequency $f = 25$ Hz.

Gibiansky and Lakes (1997) have noticed that adding small amounts of soft dissipative materials in a stiff matrix dramatically increases the overall dissipation of the composite, and that the overall dissipation increases with the stiffness of the stiff phase. The model described above has the following properties of the two single phases: $K_1 = 4.6$ GPa, $\mu_1 = 2.7$ GPa, $Q_{K1} = 46$, $Q_{\mu 1} = 27$, $K_2 = 9.5$ GPa, $\mu_2 = 41$ GPa, $Q_{K2} = 95$, and $Q_{\mu 2} = 416$, where these are relaxed bulk and shear moduli and minimum quality factors at the relaxation peak frequency of the Zener model. Medium 1 is soft compared to Medium 2, in particular the shear modulus is more than one order of magnitude smaller. Figure 2 shows the HS lower and upper bound, where we can see a pronounced decrease of the quality factor (higher loss).

Figure 3 compares the HS bounds of the bulk modulus to those of a medium with $K_2 = 30$ GPa, i.e., we have doubled the bulk modulus of Medium 2, keeping the other properties constant. The lower HS bound shows lower quality factors (higher dissipation). The HS of the shear modulus are not affected (not shown). The upper bound has the opposite behavior, i.e., the attenuation decreases. Thus, increasing the bulk modulus of one of the components, the lower-bound quality factor of the composite medium decreases (attenuation increases), and the possibility of making composite microstructures providing high stiffness and high loss exist, as already reported by *Chen and Lakes (1993)*.

Figures 4 and 5 show the P- and S-wave quality-factor and velocity bounds and averages, respectively, as a function of the proportion of Medium 1. For P waves, the Backus and Reuss averages are equal, when the medium is of Poisson type. For S waves, both averages are the same in general. The HS bounds are tighter than the Reuss-Voigt bounds, as expected. The lower HS bound is close to the Reuss bound, at least for this example. All the averages lie within the HS bounds, unlike the Backus average, and the

Gassmann moduli at high proportions, when exceeding a critical porosity value of 0.4 approximately. This violation is more remarkable in the velocities. This happens because the Gassmann moduli approach the Reuss bound. The isostress assumption behind this bound does not apply for isotropic composites made of solids, i.e., when each component has a finite shear modulus. Isostress is easily attained by a finely-layered medium but this is not isotropic. In fact, the Backus modulus (the 33-component) holds for an equivalent anisotropic medium, while the HS bounds hold for isotropic media. If Medium 1 is replaced by a fluid, the lower HS bound equals the Reuss bound, and Gassmann bulk moduli lie within the HS bounds. It can be shown that the 11-component, corresponding

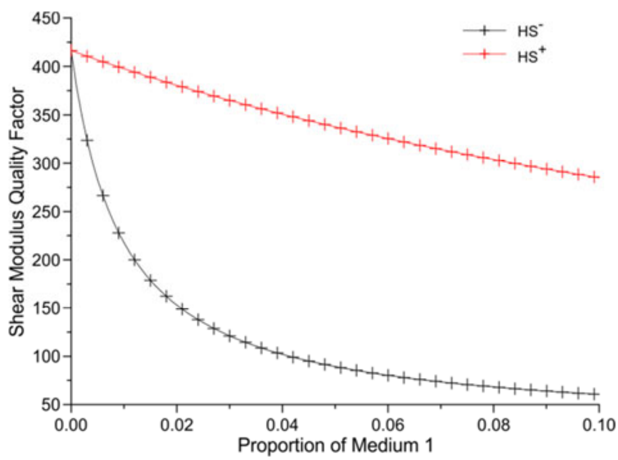

Fig. 2. Shear modulus Q for HS bounds as a function of the proportion of Medium 1 (softer medium).

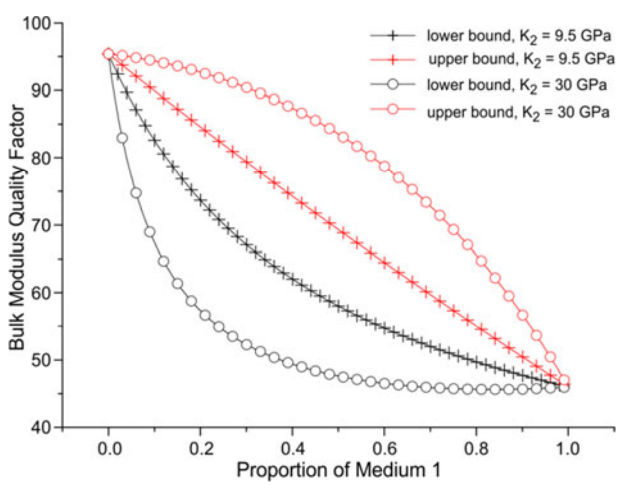

Fig. 3. Bulk modulus Q for HS bounds as a function of the proportion of Medium 1.

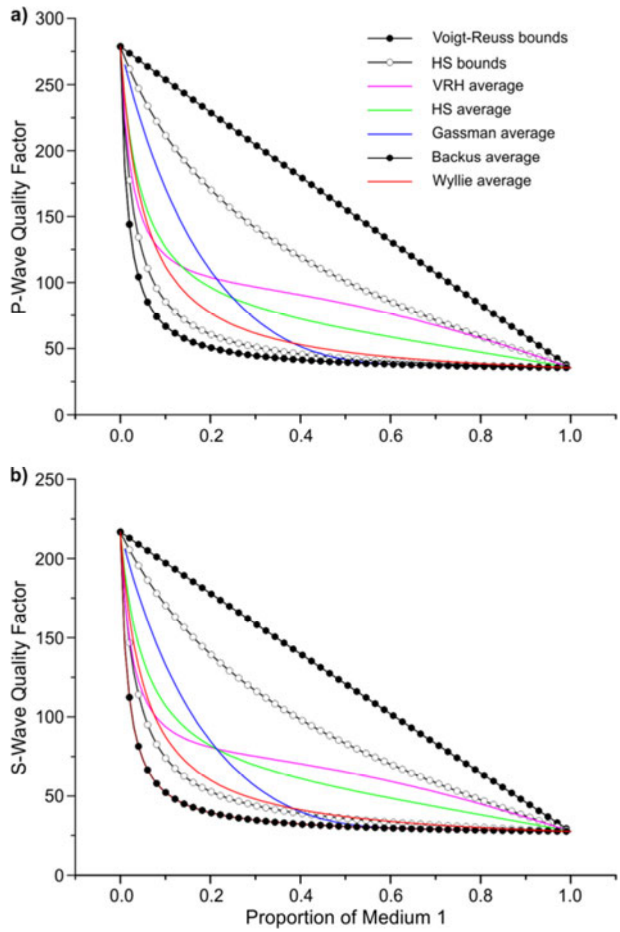

Fig. 4. Quality factor *Q* for different bounds and averages as a function of the proportion of Medium 1 for **a)** P wave, and **b)** S wave. The Backus average (33-component) coincides with the Reuss bound. The properties are $c_{0P1} = 2$ km s^{-1}, $c_{0P2} = 5$ km s^{-1} and frequencies $f_{01} = f_0 = 25$ Hz. The frequency is $f = 25$ Hz.

to propagation along the stratification, is higher than the upper HS bound but lower than the Voigt bound (P wave), whereas the 66-component is equal to the S-wave Voigt bound.

Let us now compare our theoretical results with experimental data. Unfortunately, there are no experiments at the seismic frequency band, but *Biwa et al. (2003)* have performed measurements at the ultrasonic frequency band for a composite made of layers of epoxy and carbon fiber. Epoxy has the properties: $K = (1.053 - \mathrm{i}0.112)$ GPa, $\mu = (1.58 - \mathrm{i}0.128)$ GPa, and $\rho = 1230$ kg m^{-3}, and carbon fiber is lossless with $K = 14.99$ GPa, $\mu = 24$ GPa, and $\rho = 1670$ kg m^{-3} along the propagation direction of the

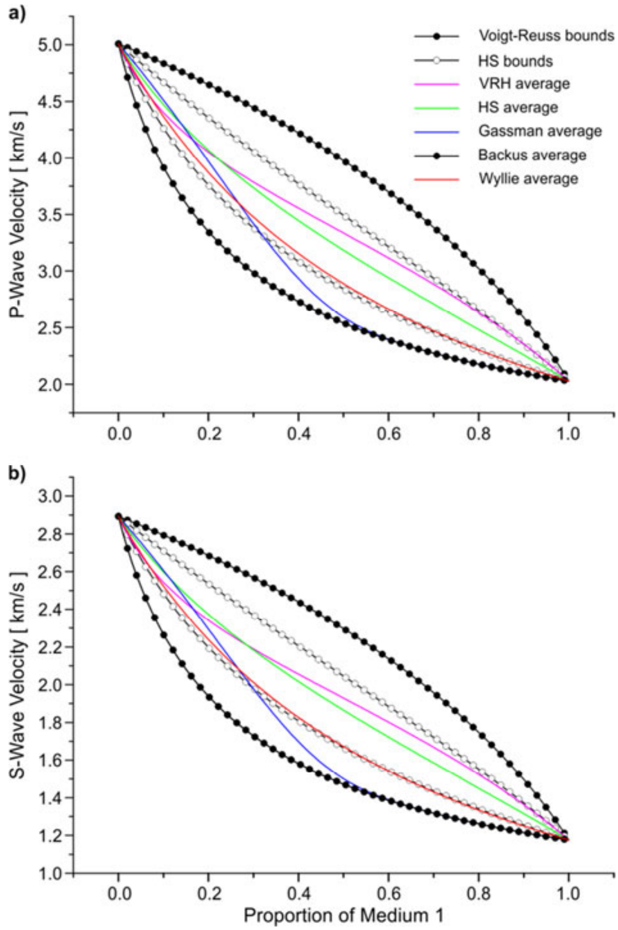

Fig. 5. The same as in Fig. 4, but for P- and S-wave velocities (see Fig. 1 for more details).

measurements. The shear modulus is the axial one, parallel to the fibers (axially polarized, according to *Biwa et al., 2003*). Figures 6 and 7 show the P-wave attenuation factor (Eq. (12)) as a function of the proportion of epoxy, compared to the experimental data, for frequencies of 3 and 7 MHz, respectively. In spite of some scatter of the measured values due not only to instrumentation errors but also to inherent sample variations (*Biwa et al., 2003*), the agreement is very good and the experimental values lie within the bounds, between the Backus and Wyllie averages.

On the other hand, Figures 8 and 9 show the S-wave attenuation factors as a function of the proportion of epoxy, compared to the experimental data, for frequencies of 2 and 3 MHz, respectively. The attenuation factors correspond to the shear wave travelling along the layers. The Backus average coincides with the Reuss bound. The experimental values are within the HS bounds and close to the Wyllie average. In summary, despite its simplicity and the fact that it is heuristic, the model shows a very good agreement with the data.

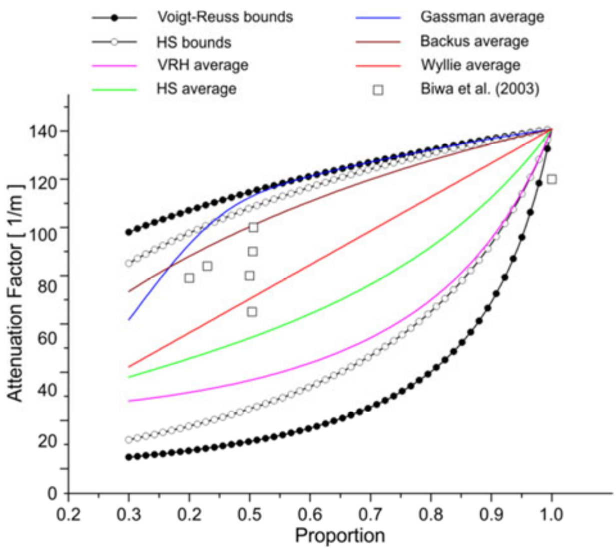

Fig. 6. P-wave attenuation factor for different bounds and averages as a function of the proportion of epoxy in a composite material made of epoxy and carbon fiber at frequency of 3 MHz, compared with experimental data of *Biwa et al. (2003)*.

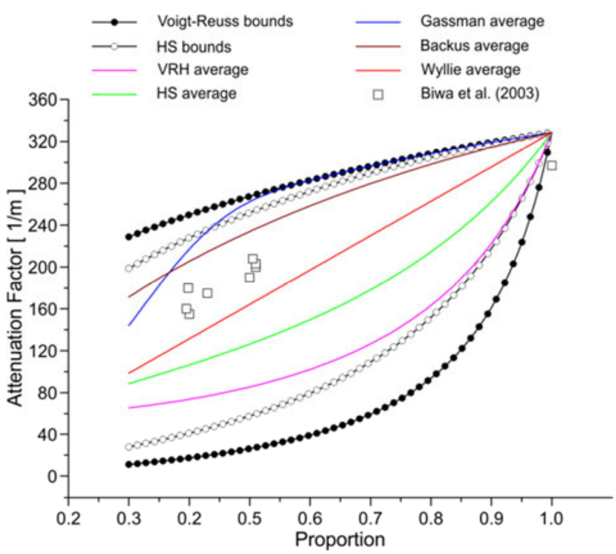

Fig. 7. The same as in Fig. 6, but at frequency of 7 MHz.

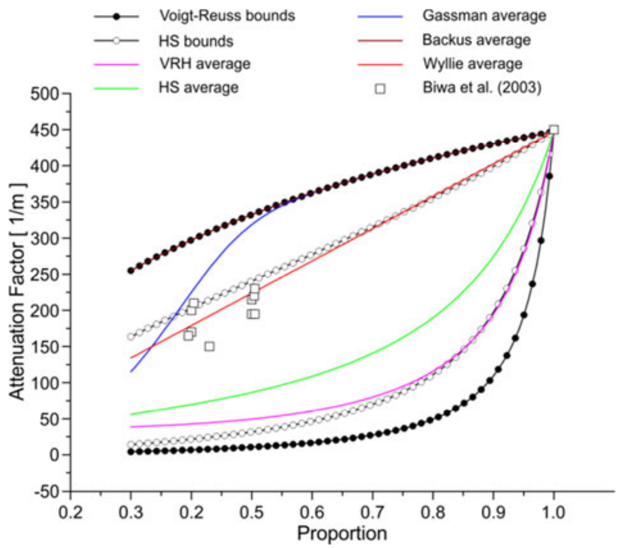

Fig. 8. The same as in Fig. 6, but for S-wave attenuation factor at frequency of 2 MHz, compared with experimental data of *Biwa et al. (2003)*. The Backus average coincides with the upper Voigt-Reuss bound.

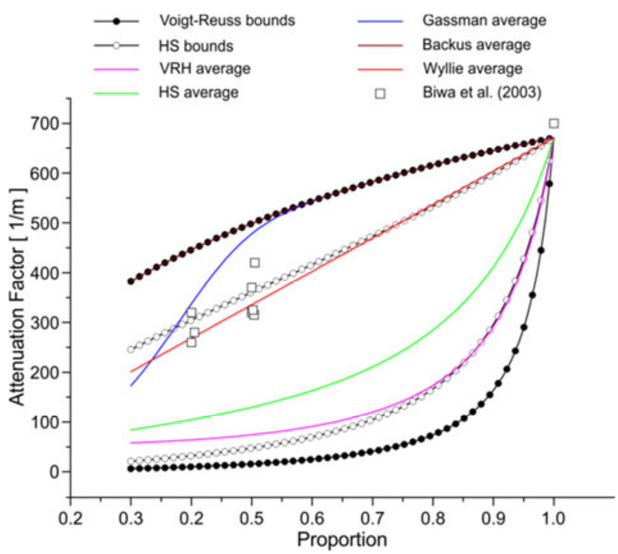

Fig. 9. The same as in Fig. 8, but at frequency of 3 MHz.

6. CONCLUSIONS

We have analized bounds and averages of wave attenuation of a two-phase medium, where there is no assumption on the geometry of the components, unlike the Backus average, which assumes a finely-stratified medium, corresponding to the low-frequency limit. The approach is based on the correspondence principle and the Voigt-Reuss and Hashin-Shtrikman lower and upper bounds. Although it is an heuristic extension, satisfies basic physical properties and is in agreement with experimental data.

For P waves and a Poisson medium, the Backus and Reuss averages are equal, as well as for S waves in general. As expected, the HS bounds are tighter than the Reuss-Voigt bounds. All the averages lie within the HS bounds, unlike the Backus average, and the Gassmann moduli at high proportions, when exceeding a critical porosity value of 0.4 approximately. This violation happens because the Gassmann moduli approach the Reuss bounds, and the isostress assumption does not apply for isotropic (solid) composites, whereas it is valid for Backus averaging.

Comparison of the theoretical attenuation factors with experimental data, shows a very good agreement, with the experimental values lying within the HS bounds, and between the Backus and Wyllie averages for P waves, and between the HS lower-bound curve and the Wyllie average for S waves.

Since the quality factor can be related to porosity, permeability and fluid viscosity and saturation, these averages can be useful for evaluating reservoir properties. Indeed, theories such as the mesoscopic or wave-induced fluid-flow loss, which consider these properties, are now believed to model the correct level of attenuation and velocity dispersion at seismic frequencies.

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