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Existence and uniqueness of solutions of thermo-poroelasticity

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ABSTRACT

We present results on the existence and uniqueness of a new formulation of wave propagation in linear thermo-poroelastic isotropic media in bounded domains under appropriate boundary and initial conditions. The linear theory of thermo-poroelasticity describes wave propagation in fluid-saturated poroelastic media including the temperature. In the model analyzed here, the constitutive equations in Biot's theory are modified by introducing coupling temperature terms, while the heat equation is generalized by including the coupling with the elastodynamic field and relaxation terms, the latter to model finite velocities. This analysis shows the existence of a unique solution, given in terms of displacements of the solid and fluid phases and temperature, and proves its regularity in the space and time variables.

1. Introduction

The study of wave propagation in a fluid-saturated porous medium taking into account the effect of temperature has applications in many fields like hydrocarbon reservoirs and crustal rocks. In the case of elastic bodies, Zener [25–27] associated attenuation to stress inhomogeneities generating local heat currents. Treitel [24] analyzed seismic attenuation in the context of thermoelasticity. Biot [2] used differential equations based on the classical Fourier law of heat conduction to model waves in elastic media taking into account temperature, but the model yields infinite velocities due to the diffusive character of the heat equation. Lord and Shulman [14] analyzed linear thermoelasticity introducing time relaxation terms to avoid infinite speeds associated with the diffusive heat equation. The approach is based on Cattaneo's generalization of the Fourier equation [8].

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For fluid-saturated poroelastic media, the general theory of wave propagation was presented by Biot [3] but this theory does not take into account the temperature. Biot's theory predicts the existence of two compressional waves, one fast and one slow, and one shear wave. The slow P-wave is diffusive at low frequencies and becomes a truly propagation at high frequencies. The existence of solutions in the case of porous media are reported, for instance, in [19] and [5], but these works deal with a different theory and do not consider the relaxation terms, which constitute the main problem to prove the solution existence. A review of the existent literature on the subject can be found in [23]. In this work, we use a new form of the thermo-poroelastic equations as a generalization of the ones given in [22] and [6]. This formulation involves coupling Biot's equation of motion with a hyperbolic heat equation having a relaxation term to avoid infinite speed of heat conduction as in [14].

We formulate an initial boundary value problem (IBVP) for the thermo-poroelastic equations on an open bounded domain Ω with piecewise smooth boundary under boundary conditions specifying stress, fluid pressures and heat flux across $\Gamma = \partial\Omega$ and initial conditions on the solid and fluid displacements and temperature on Ω. To demonstrate the existence and uniqueness of the solution of the IBVP, following the ideas in [13], we first give a variational formulation of the IBVP and obtain a bounded sequence of smooth solutions. Then, we use a compactness argument to show the existence of the limit of the sequence in the weak$-^*$ topology. Assuming regularity of the solution of the IBVP for the thermoporoelasticity equations in the form used in [21], we demonstrate that the limit solution satisfies the given boundary and initial conditions. A similar approach was used in [20] to show existence and uniqueness of an IBVP for Biot's equations of motion.

2. The model equations

We consider a porous medium saturated by a single phase, compressible viscous fluid and assume that the whole aggregate is isotropic. Let $\mathbf{u}^s = (u_i^s)$ and $\mathbf{u}^f = (u_i^f)$ denote the averaged displacement vectors of the solid and relative fluid phases, respectively and set $\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f)$. Let $\boldsymbol{\varepsilon}(\mathbf{u}^s) = (\varepsilon_{ij}(\mathbf{u}^s))$ be the strain tensor of the solid. Also, let $\boldsymbol{\sigma}(\mathbf{u}, \theta) = (\sigma_{ij}(\mathbf{u}, \theta))$, and $p_f = p_f(\mathbf{u}, \theta)$ denote the stress tensor of the bulk material and the fluid pressure, respectively, with θ being the increment of temperature above a reference absolute temperature T_0 for the state of zero stress and strain. The stress-strain relations are [6]:

$$\sigma_{ij}(\mathbf{u}, \theta) = 2\mu\, \varepsilon_{ij}(u^s) + \delta_{ij}(\lambda_u \nabla \cdot \mathbf{u}^s + B\nabla \cdot \mathbf{u}^f - \beta\, \theta), \tag{1}$$

$$p_f(\mathbf{u}, \theta) = -B\, \nabla \cdot \mathbf{u}^s - M\nabla \cdot \mathbf{u}^f + \beta_f \theta, \tag{2}$$

where μ is the wet- or dry-rock shear modulus, ϕ is the porosity,

$$\lambda_u = \lambda + \alpha^2 M, \quad \alpha = 1 - \frac{K_m}{K_s}, \tag{3}$$

$$M = \left(\frac{\alpha - \phi}{K_s} + \frac{\phi}{K_f}\right)^{-1}, \quad B = \alpha\, M, \quad \beta = \beta_m + \beta_f, \tag{4}$$

with λ being the Lamé coefficients of the dry matrix and K_s, K_m and K_f denoting the bulk moduli of the solid grains composing the solid matrix, the dry matrix and the saturant fluid, respectively. The coefficient λ_u is the Lamé parameter of the saturated matrix. The positive coupling coefficients β_m and β_f are the coefficients of thermoelasticity of the frame (or matrix) and fluid, respectively.

2.1. Dynamical equations

Let

$$\rho_b = (1-\phi)\rho_s + \phi\rho_f$$

denote the mass density of the bulk material, with ρ_s and ρ_f being the mass densities of the solid grains and fluid, respectively. Let the positive definite matrix $\mathcal{P}$ and the nonnegative matrix $\mathcal{B}$ be defined by

$$\mathcal{P} = \begin{pmatrix} \rho_b I & \rho_f I \\ \rho_f I & gI \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0I & 0I \\ 0I & \dfrac{\eta}{\kappa} I \end{pmatrix}, \tag{5}$$

where I is the identity matrix in $R^{d\times d}$, with $d = 2, 3$, η is the fluid viscosity, κ is the permeability and $g = \dfrac{S\rho_f}{\phi}$, where S is the tortuosity.

Let $\mathcal{L}(\mathbf{u}, \theta)$ be the second-order differential operator defined by

$$\mathcal{L}(\mathbf{u}, \theta) = (\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}, \theta), -\nabla p_f(\mathbf{u}, \theta)).$$

Then Biot's dynamical equations taking into account temperature are [6]

$$\mathcal{P}\ddot{\mathbf{u}} + \mathcal{B}\dot{\mathbf{u}}^f - \mathcal{L}(\mathbf{u}, \theta)) = \mathbf{f}. \tag{6}$$

The heat equation is

$$\begin{aligned} &\tau\ c\ \ddot{\theta} + c\ \dot{\theta} - \nabla \cdot (\gamma \nabla \theta) + (1-\phi)\beta_m T_0 \nabla \cdot \dot{\mathbf{u}}^s + \phi\beta_f T_0 \nabla \cdot \dot{\mathbf{u}}^f \\ &+\tau(1-\phi)\beta_m T_o \nabla \cdot \ddot{\mathbf{u}}^s + \tau\phi\beta_f T_o \nabla \cdot \ddot{\mathbf{u}}^f = -q, \end{aligned} \tag{7}$$

where

$$\gamma = (1-\phi)\gamma_m + \phi\gamma_f, \tag{8}$$

is the bulk coefficient of heat conduction (or thermal conductivity), with γ_m and γ_f being the heat conduction of the solid frame and the fluid, respectively. Also,

$$c = (1-\phi)c_m + \phi c_f, \tag{9}$$

is the bulk specific heat of the unit volume in the absence of deformation, τ is a Maxwell-Vernotte-Cattaneo relaxation time and q is a heat source. These equations assume thermal equilibrium between the solid and the fluid, i.e., the temperature in both phases is the same. Thermal equilibrium is valid when the interstitial heat transfer coefficient between the solid and fluid is very large and the ratio of pore surface area to pore volume is sufficiently high.

Equation (7) is obtained as follows. The heat-balance equation for each phase is

$$\begin{aligned} &c_m(\tau_m\ddot{\theta} + \dot{\theta}) - \nabla \cdot (\gamma_m \nabla\theta) + T_0\beta_m \left(\nabla \cdot \dot{\mathbf{u}}^s + \tau_m \nabla \cdot \ddot{\mathbf{u}}_m\right) = 0, \\ &c_f(\tau_f\ddot{\theta} + \dot{\theta}) - \nabla \cdot (\gamma_f \nabla\theta) + T_0\beta_f \left(\nabla \cdot \dot{\mathbf{u}}^f + \tau_m \nabla \cdot \ddot{\mathbf{u}}_f\right) = 0. \end{aligned} \tag{10}$$

These equations have been taken from [16](eqs 2.1 and 2.2), by neglecting the advection term in the fluid equation and adding the inertial terms related to the deformation. Each of these equations is the same as that used in single-phase media (e.g. Carcione et al., 2018). Taking averages over an elemental volume of the medium for the solid and fluid phases, and adding the resulting equations, we obtain equation (7), where we have assumed $\tau_m = \tau_f$.

Next, we compare our equations with other formulations presented in the literature. Biot [2] and Deresiewicz [9] do not consider the relaxation term, leading to unphysical results (see [7]). McTigue [15] and Bonafede [4] treat the static problem, so that there are no inertial terms (accelerations) and no relaxation effects. Sharma [22] obtains similar equations, but with different coefficients, where $\beta = \beta_m + \alpha\beta_f$ [instead of equation (7)], and the heat equation is different.

The heuristic heat equation (7) reduces to that of linear thermoelasticity if $\phi = 0$ (no fluid) and to the heat equation of the fluid if $\phi = 1$, as expected. If one wishes to allow for heat transfer between the solid and the fluid, a starting point to do this is given by Nield and Bejan [16] (eqs. 2.11 and 2.12), where the inertial terms have to be included [those related to $\nabla \cdot \mathbf{u}_s$ and $\nabla \cdot \mathbf{u}_f$ in equation (7)]. Noda [18] (eq. 6) neglects the inertial terms in the temperature equation, but includes the non-linear advection term. This author relates these coefficients to the coefficients of thermal expansion, α_m and α_f, as $\beta_m = 3[(K_m + (\alpha - \phi)^2 M)\alpha_m + \phi(\alpha - \phi)M\alpha_f]$ and $\beta_f = 3\phi M[(\alpha - \phi)\alpha_m + \phi\alpha_f]$. The behavior of these quantities is such that for $\phi = 0$, $K_m = K_s$, $\alpha = 0$, $\beta_m = 3K_s\alpha_m$ and $\beta_f = 0$, and for $\phi = 1$, $K_m = 0$, $\alpha = 1$, $M = K_f$, $\beta_m = 0$ and $\beta_f = 3K_f\alpha_f$. Here, we consider β_m, β_f, γ and c as parameters, obtained from experiments or from a specific theoretical model.

2.2. The initial boundary value problem

The initial boundary value problem is formulated in the 2D case (with obvious extension to the 3D case) for the case of thermal equilibrium in an open bounded domain Ω with piecewise smooth boundary and a time interval $J = (0, T)$ as follows: Find $(\mathbf{u}, \theta)$ such that

$$\mathcal{P}\ddot{\mathbf{u}} + \mathcal{B}\dot{u}^f - \mathcal{L}(\mathbf{u}, \theta)) = \mathbf{f}, \quad (x,t) \in \Omega \times J, \tag{11}$$

$$\begin{aligned} &\tau\, c\, \ddot{\theta} + c\, \dot{\theta} - \nabla \cdot (\gamma \nabla \theta) + (1-\phi)\beta_m T_0 \nabla \cdot \dot{\mathbf{u}}^s + \phi \beta_f T_0 \nabla \cdot \dot{\mathbf{u}}^f \\ &+ \tau(1-\phi)\beta_m T_o \nabla \cdot \ddot{\mathbf{u}}^s + \tau \phi \beta_f T_o \nabla \cdot \ddot{\mathbf{u}}^f = -q \quad (x,t) \in \Omega \times J, \end{aligned} \tag{12}$$

with initial conditions

$$\mathbf{u}(x,0) = \mathbf{u}^0, \;\; x \in \Omega, \tag{13}$$

$$\dot{\mathbf{u}}(x,0) = \mathbf{u}^1, \;\; x \in \Omega, \tag{14}$$

$$\theta(x,0) = \theta^0, \;\; x \in \Omega, \tag{15}$$

$$\dot{\theta}(x,0) = \theta^1 \;\; x \in \Omega, \tag{16}$$

and boundary conditions

$$\boldsymbol{\sigma}(\mathbf{u}, \theta) \cdot \boldsymbol{\nu} = \mathbf{g}(x,t), \quad x \in \Gamma, \quad t \in J, \tag{17}$$

$$p_f(\mathbf{u}, \theta) = -\chi(x,t), \quad x \in \Gamma, \quad t \in J, \tag{18}$$

$$\gamma \nabla \theta \cdot \boldsymbol{\nu} = h(x,t), \quad x \in \Gamma, \quad t \in J. \tag{19}$$

In (11)-(12) $\mathbf{f} = (\mathbf{f}^s, \mathbf{f}^f)$ is an external force and q is a heat source.

3. The existence and uniqueness results

In order to demonstrate the main result, we need to introduce some notation. For $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$, let $(\cdot,\cdot)_\Omega$ and $\langle\cdot,\cdot\rangle_\Gamma$ denote the $L^2(\Omega)$ and $L^2(\Gamma)$ inner products for scalar, vector, or matrix valued functions. Also, for $s \in \mathbb{R}$, $\|\cdot\|_{s,\Omega}$ and $\|\cdot\|_{s,\Gamma}$ will denote the usual norms for the Sobolev space $H^s(\Omega)$ and $H^s(\Gamma)$, respectively. If $X = \Omega$ or $X = \Gamma$, the subscript Ω may be omitted such that $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$ or $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_\Gamma$. Let

$$H(\mathrm{div};\Omega) = \{\mathbf{v} \in [L^2(\Omega)]^2 : \nabla\cdot\mathbf{v} \in L^2(\Omega)\},$$

provided with the norm

$$\|\mathbf{v}\|_{H(\mathrm{div};\Omega)} = \left[\|\mathbf{v}\|_0^2 + \|\nabla\cdot\mathbf{v}\|_0^2\right]^{1/2}.$$

We also will refer to the space

$$H^1(\mathrm{div};\Omega) = \{\mathbf{v} \in [H^1(\Omega)]^2 : \nabla\cdot\mathbf{v} \in H^1(\Omega)\}.$$

The following known results will be used [11]

$$\|\mathbf{v}\cdot\boldsymbol{\nu}\|_{-1/2,\Gamma} \le C\|\mathbf{v}\|_{H(\mathrm{div};\Omega)}, \tag{20}$$

$$\|\mathbf{v}\|_{0,\Gamma} \le C\|\mathbf{v}\|_{0,\Omega}^{1/2}\,\|\mathbf{v}\|_{1,\Omega}^{1/2} \le C\|\mathbf{v}\|_{1,\Omega}. \tag{21}$$

Here and in what follows C denotes a generic constant that may take different values at different places.

Also recall Korn's second inequality [10,17]

$$\int\limits_\Omega \left[\sum_{i,j}\left(\varepsilon_{ij}(\mathbf{v})\right)^2\right] d\Omega + \|\mathbf{v}\|_0^2 \ge C\|\mathbf{v}\|_1^2. \tag{22}$$

Next, we introduce the space

$$\mathcal{V} = \left[H^1(\Omega)\right]^2 \times H(\mathrm{div};\Omega),$$

provided with the natural norm

$$\|\mathbf{v}\|_{\mathcal{V}} = \left(\|\mathbf{v}^s\|_1^2 + \|\mathbf{v}^f\|_{H(\mathrm{div};\Omega)}^2\right)^{1/2}, \mathbf{v}^s \in [H^1(\Omega)]^2, \mathbf{v}^f \in H(\mathrm{div};\Omega).$$

Let $\mathcal{V}'$ be the dual space of $\mathcal{V}$, with the duality between $\mathcal{V}'$ and $\mathcal{V}$ denoted by $[\cdot,\cdot]$.

Note that $[H(\mathrm{div};\Omega)]'$ can be identified with a closed subspace of $[L^2(\Omega)]^3$, so that any element $\mathbf{u}$ in the dual space $\mathcal{V}'$ can be represented by $(u_1, u_2, u_3, u_4, u_5)$, where $u_1, u_2 \in H^{-1}(\Omega)$, $u_3, u_4, u_5 \in L^2(\Omega)$.

Next, let $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ denote the space of C^∞ functions having compact support in Ω, and by $\mathcal{D}'(\Omega)$ the space of distributions on Ω. Also, for any Banach space Y let

$$L^2(J,Y) = \{f : J \to Y : \|f\|_{J,Y}^2 = \int\limits_0^T \|f(t)\|_Y^2 dt < \infty\},$$

$$L^\infty(J,Y) = \{f : J \to Y : \|f\|_{J,Y}^\infty = ess.sup_{t\in J}\|f(t)\|_Y\}.$$

To obtain a variational formulation of the initial boundary value problem (11)-(19), multiply (11) by $\mathbf{v} = (\mathbf{v}^s, \mathbf{v}^f) \in \mathcal{V}$, (12) by $w \in H^1(\Omega)$, use integration by parts and the boundary conditions (17)-(19) to obtain

$$
\begin{aligned}
&(\mathcal{P}\ddot{\mathbf{u}}(x), \mathbf{v}) + \left(\frac{\eta}{\kappa}\dot{u}^f, \mathbf{v}^f\right) + \Lambda(\mathbf{u}, \mathbf{v}) - (\beta\theta, \nabla\cdot\mathbf{v}^s) - \left(\beta_f\theta, \nabla\cdot\mathbf{v}^f\right) \\
&+ \left(\tau\, c\, \ddot{\theta}, w\right) + \left(c\, \dot{\theta}, w\right) + (\gamma\nabla\theta, \nabla w) + ((1-\phi)\beta_m T_0 \nabla\cdot\dot{\mathbf{u}}^s, w) \\
&+ \left(\phi\beta_f T_0\nabla\cdot\dot{\mathbf{u}}^f, w\right) + (\tau(1-\phi)\beta_m T_0\nabla\cdot\ddot{\mathbf{u}}^s, w) + \left(\tau\phi\beta_f T_0\nabla\cdot\ddot{\mathbf{u}}^f, w\right) \\
&= (\mathbf{f}, \mathbf{v}) - (q, w) + \langle \mathbf{g}, \mathbf{v}^s\rangle + \left\langle \chi, \mathbf{v}^f\cdot\boldsymbol{\nu}\right\rangle + \langle h, w\rangle\,, \\
&\qquad \mathbf{v} = (\mathbf{v}^s, \mathbf{v}^f, w) \in \mathcal{V}\times H^1(\Omega),\ t\in J,
\end{aligned}
\tag{23}
$$

where $\Lambda(\mathbf{u}, \mathbf{v})$ is the bilinear form

$$
\Lambda(\mathbf{u}, \mathbf{v}) = \sum_{l,m} \left(\sigma_{lm}(\mathbf{u}), \varepsilon_{lm}(\mathbf{v}^s)\right) = \left(\mathcal{E}\ \widetilde{\epsilon}(\mathbf{u}), \widetilde{\epsilon}(\mathbf{v})\right). \tag{24}
$$

In (24), the matrix $\mathcal{E}$ and the column vector $\widetilde{\varepsilon}((\mathbf{u}))$ are defined by

$$
\mathcal{E} = \begin{pmatrix} \lambda_u + 2\mu & \lambda_u & B & 0 \\ \lambda_u & \lambda_u + 2\mu & B & 0 \\ B & B & M & 0 \\ 0 & 0 & 0 & 4\mu \end{pmatrix}, \quad \widetilde{\varepsilon}(\mathbf{u}) = \begin{pmatrix} \varepsilon_{11}(\mathbf{u}^s) \\ \varepsilon_{22}(\mathbf{u}^s) \\ \nabla\cdot\mathbf{u}^f \\ \varepsilon_{12}(\mathbf{u}^s) \end{pmatrix}. \tag{25}
$$

The term $(\mathcal{E}\ \widetilde{\epsilon}(\mathbf{u}), \widetilde{\epsilon}(\mathbf{u}))$ in (24) is associated with the strain energy of the system, so that the symmetric matrix $\mathcal{E}$ must be positive definite. Furthermore, $\Lambda(\mathbf{u}, \mathbf{v}) \le C\|\mathbf{u}\|_{\mathcal{V}}\|\mathbf{v}\|_{\mathcal{V}}$. Also, note that using (22), if $\lambda_*^{\mathcal{E}}$ is the minimum eigenvalue of $\mathcal{E}$,

$$
\Lambda(\mathbf{v}, \mathbf{v}) \ge C_2\|\mathbf{v}\|_{\mathcal{V}}^2 - \lambda_*^{\mathcal{E}}\|\mathbf{v}\|_0^2. \tag{26}
$$

Let

$$
\begin{aligned}
&M_0^2(\mathbf{f}, \mathbf{g}, q, h, \chi) = \|\mathbf{f}\|^2_{L^2(J,L^2(\Omega)]^2)} + \|\dot{\mathbf{f}}\|^2_{L^2(J,L^2(\Omega)]^2)} \\
&+\|\ddot{\mathbf{f}}\|^2_{L^2(J,L^2(\Omega)]^2)} + \|\mathbf{g}\|^2_{L^2(J,[H^1(\Omega)]^2)} + \|\dot{\mathbf{g}}\|^2_{L^\infty(J,[H^1(\Omega)]^2)} \\
&+\|\ddot{\mathbf{g}}\|^2_{L^\infty(J,[H^1(\Omega)]^2)} + \|\dddot{\mathbf{g}}\|^2_{L^2(J,[H^1(\Omega)]^2)} + \|\chi\|^2_{L^2(J,H^{1/2}(\Gamma))} \\
&+\|\dot{\chi}\|^2_{L^\infty(J,H^{1/2}(\Gamma))} + \|\ddot{\chi}\|^2_{L^\infty(J,H^{1/2}(\Gamma))} + \|\dddot{\chi}\|^2_{L^2(J,H^{1/2}(\Gamma))} \\
&+\|h\|^2_{L^\infty(J,H^1(\Omega))} + \|\dot{h}\|^2_{L^\infty(J,H^1(\Omega))} + \|\ddot{h}\|^2_{L^2(J,H^1(\Omega))},
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
N_0^2 &= \|\mathbf{u}^0\|_2^2 + \|\mathbf{u}^1\|_1^2 + \|\theta^0\|_2^2 + \|\theta^1\|_1^2 + \|\mathbf{f}(0)\|_0^2 + \|\dot{\mathbf{f}}(0)\|_0^2 \\
&+\|\ddot{\mathbf{f}}(0)\|_0^2 + \|\dot{\mathbf{g}}(0)\|^2_{1/2,\Gamma} + \|\ddot{\mathbf{g}}(0)\|^2_{1/2,\Gamma} + \|\dot{\boldsymbol{\chi}}(0)\|^2_{1/2,\Gamma} \\
&+\|\ddot{\boldsymbol{\chi}}(0)\|^2_{1/2,\Gamma} + \|\dot{h}(0)\|^2_{1/2,\Gamma} + \|\ddot{h}(0)\|^2_{1/2,\Gamma} + 1.
\end{aligned}
\tag{28}
$$

Now, we demonstrate the following theorem:

Theorem 1. *Let $\Omega \subset \mathbf{R}^2$ be an open bounded domain with piecewise smooth boundary. Assume that all coefficients in (1), (2) and (12) are in $C_B^0(\Omega)$ with gradients belonging to $[L^\infty(\Omega)]^2$, that $\mathcal{E}$ and $\mathcal{P}$ in (25) and (5) are positive definite and that $\mathcal{B}$ in (5) is nonnegative. Also assume that $M_0(\mathbf{f}, \mathbf{g}, q, h, \chi) < \infty$, $N_0 < \infty$. Then, there exists a unique solution $(\mathbf{u}, \theta)$ of problem (11)-(19) such that $\mathbf{u}, \dot{\mathbf{u}}, \ddot{\mathbf{u}} \in L^\infty(J, \mathcal{V})$, $\dddot{\mathbf{u}} \in L^\infty(J, [L^2(\Omega)]^4$, $\theta, \dot{\theta} \in L^\infty(J, H^1(\Omega)), \ddot{\theta} \in L^\infty(J, L^2(\Omega))$.*

Proof. The outline of the demonstration of the theorem is as follows:
i)- Construct a sequence of approximate solutions of problem (11)-(19) using the Galerkin method.
ii)- Obtain a priori bounds of the approximate solutions in terms of the initial and boundary conditions data.
iii)- Use a compactness argument to show the existence of the limit of a subsequence of the approximate solutions in the weak$-^*$ topology.
iv)- Show that such limit satisfy the given initial and boundary conditions.

To demonstrate the existence of a unique solution $(\mathbf{u}, \theta)$ of (11)-(19), following the ideas in [13] we take a sequence of functions $(\mathbf{v}_n = (\mathbf{v}^s_m, \mathbf{v}^f_m))_{n\geq 1}$ in $[H^2(\Omega)]^4$ and $(w_n)_{n\geq 1}$ in $H^2(\Omega)$ such that for all m $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ and $\{w_1, \cdots, w_m\}$ are linearly independent, the linear combinations of the $\mathbf{v}'_i s$ are dense in $[H^2(\Omega)]^4$ and the linear combinations of the $w'_i s$ are dense in $H^2(\Omega)$. Let

$$S^{\mathbf{u}}_m = \mathrm{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_m\},\ S^{\theta}_m = \mathrm{Span}\{w_1, \cdots, w_m\},\ \mathcal{V}_m = S^{\mathbf{u}}_m \times S^{\theta}_m.$$

Then, consider the solution of the set of initial boundary value problems: Find $(\mathbf{v}_m, \theta_m) \in \mathcal{V}_m$ such that

$$\begin{aligned}
&(\mathcal{P}\ddot{\mathbf{u}}_m(x), \mathbf{v}) + \left(\frac{\eta}{\kappa}\dot{\mathbf{u}}^f_m, \mathbf{v}^f\right) + \Lambda(\mathbf{u}_m, \mathbf{v}) - (\beta\theta_m, \nabla\cdot\mathbf{v}^s) \qquad (29)\\
&- \left(\beta_f\theta_m, \nabla\cdot\mathbf{v}^f\right) + \left(\tau\, c\, \ddot{\theta}_m, w\right) + \left(c\,\dot{\theta}_m, w\right)\\
&+ (\gamma\nabla\theta_m, \nabla w) + ((1-\phi)\beta_m T_0\nabla\cdot\dot{\mathbf{u}}^s_m, w) + \left(\phi\beta_f T_0\nabla\cdot\dot{\mathbf{u}}^f_m, w\right)\\
&+ (\tau(1-\phi)\beta_m T_0\nabla\cdot\ddot{\mathbf{u}}^s_m, w) + \left(\tau\phi\beta_f T_0\nabla\cdot\ddot{\mathbf{u}}^f_m, w\right)\\
&= (\mathbf{f}, \mathbf{v}) - (q, w) + \langle\mathbf{g}, \mathbf{v}^s\rangle + \left\langle\chi, \mathbf{v}^f\cdot\boldsymbol{\nu}\right\rangle\\
&\quad + \langle h, w\rangle, \quad \mathbf{v} = (\mathbf{v}^s, \mathbf{v}^f, w) \in \mathcal{V}_m, \quad t \in J,\\
&\mathbf{u}_m(x,0) \in S^{\mathbf{u}}_m, \quad \mathbf{u}_m(x,0) \xrightarrow[m\to\infty]{} \mathbf{u}^0 \quad \text{in} \quad [H^2(\Omega)]^4, \qquad (30)\\
&\dot{\mathbf{u}}_m(x,0) \in S^{\mathbf{u}}_m, \quad \dot{\mathbf{u}}_m(x,0) \xrightarrow[m\to\infty]{} \mathbf{u}^1 \quad \text{in} \quad [H^1(\Omega)]^4, \qquad (31)\\
&\theta_m(x,0) \in S^{\theta}_m, \quad \theta_m(x,0) \xrightarrow[m\to\infty]{} \theta^0 \quad \text{in} \quad H^2(\Omega), \qquad (32)\\
&\dot{\theta}_m(x,0) \in S^{\theta}_m, \quad \dot{\theta}_m(x,0) \xrightarrow[m\to\infty]{} \theta^1 \quad \text{in} \quad H^1(\Omega). \qquad (33)
\end{aligned}$$

Choose $\mathbf{v} = \dot{\mathbf{u}}_m, w = \dot{\theta}_m$ in (29) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[(\mathcal{P}\dot{\mathbf{u}}_m(x), \dot{\mathbf{u}}_m) + \Lambda(\mathbf{u}_m, \mathbf{u}_m) + (\gamma\nabla\theta_m, \nabla\theta_m) + \left(\tau\, c\,\dot{\theta}_m, \dot{\theta}_m\right)\right] \qquad (34)\\
&+ \left(\frac{\eta}{\kappa}\dot{\mathbf{u}}^f_m, \dot{\mathbf{u}}^f_m\right) - (\beta\theta_m, \nabla\cdot\dot{\mathbf{u}}^s_m) - \left(\beta_f\theta_m, \nabla\cdot\dot{\mathbf{u}}^f_m\right) + \left(c\,\dot{\theta}_m, \dot{\theta}_m\right)\\
&+ \left((1-\phi)\beta_m T_0\nabla\cdot\dot{\mathbf{u}}^s_m, \dot{\theta}_m\right) + \left(\phi\beta_f T_0\nabla\cdot\dot{\mathbf{u}}^f_m, \dot{\theta}_m\right)\\
&+ \left(\tau(1-\phi)\beta_m T_0\nabla\cdot\ddot{\mathbf{u}}^s_m, \dot{\theta}_m\right) + \left(\phi\beta_f T_0\nabla\cdot\ddot{\mathbf{u}}^f_m, \dot{\theta}_m\right)\\
&= (\mathbf{f}, \dot{\mathbf{u}}_m) - \left(q, \dot{\theta}_m\right) + \langle\mathbf{g}, \dot{\mathbf{u}}^s_m\rangle + \left\langle\chi, \dot{\mathbf{u}}^f_m\cdot\boldsymbol{\nu}\right\rangle + \left\langle h, \dot{\theta}_m\right\rangle, \quad t \in J.
\end{aligned}$$

Next, we obtain lower bounds satisfied by the terms $\left(\tau(1-\phi)\beta_m T_0\nabla\cdot\ddot{\mathbf{u}}^s_m, \dot{\theta}_m\right)$ and $\left(\phi\beta_f T_0\nabla\cdot\ddot{\mathbf{u}}^f_m, \dot{\theta}_m\right)$ in the right-hand side of (34). Take time derivative in (29) to obtain

$$\begin{aligned}
&(\mathcal{P}\dddot{\mathbf{u}}_m(x), \mathbf{v}) + \left(\frac{\eta}{\kappa}\ddot{u}^f_m, \mathbf{v}^f\right) + \Lambda(\dot{\mathbf{u}}_m, \mathbf{v}) - \left(\beta\dot{\theta}_m, \nabla\cdot\mathbf{v}^s\right) \qquad (35)\\
&- \left(\beta_f\dot{\theta}_m, \nabla\cdot\mathbf{v}^f\right) + \left(\tau\, c\,\dddot{\theta}_m, w\right) + \left(c\,\ddot{\theta}_m, w\right)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\gamma\nabla\dot{\theta}_m,\nabla w\right)+((1-\phi)\beta_m T_0\nabla\cdot\ddot{\mathbf{u}}^s_m,w)+\left(\phi\beta_f T_0\nabla\cdot\ddot{\mathbf{u}}^f_m,w\right)\\
&+(\tau(1-\phi)\beta_m T_0\nabla\cdot\dddot{\mathbf{u}}^s_m,w)+\left(\phi\beta_f T_0\nabla\cdot\dddot{\mathbf{u}}^f_m,w\right)\\
&\quad=\left(\dot{\mathbf{f}},\mathbf{v}\right)-(\dot{q},w)+\langle\dot{\mathbf{g}},\mathbf{v}^s\rangle+\left\langle\dot{\chi},\mathbf{v}^f\cdot\boldsymbol{\nu}\right\rangle\\
&\quad+\left\langle\dot{h},w\right\rangle,\quad \mathbf{v}=(\mathbf{v}^s,\mathbf{v}^f,w)\in\mathcal{V}_m,t\in J.
\end{aligned}
$$

Choosing $\mathbf{v}^s=\ddot{\mathbf{u}}^s_m,\mathbf{v}^f=0,w=0$ in (35) we get

$$
\begin{aligned}
&(\rho\dddot{\mathbf{u}}^s_m,\ddot{\mathbf{u}}^s_m)+\left(\rho_f\dddot{\mathbf{u}}^f_m,\ddot{\mathbf{u}}^s_m\right)+((\lambda_u+2\mu)\varepsilon_{11}(\dot{\mathbf{u}}^s_m),\varepsilon_{11}(\ddot{\mathbf{u}}^s_m))\\
&+((\lambda_u+2\mu)\varepsilon_{33}(\dot{\mathbf{u}})m^s),\varepsilon_{33}(\ddot{\mathbf{u}}^s_m))+(\lambda_u\varepsilon_{33}(\dot{\mathbf{u}}^s_m),\varepsilon_{11}(\ddot{\mathbf{u}}^s_m))\\
&+(\lambda_u\varepsilon_{11}(\dot{\mathbf{u}}^s_m),\varepsilon_{33}(\ddot{\mathbf{u}}^s_m))+(4\mu\varepsilon_{13}(\dot{\mathbf{u}}^s_m),\varepsilon_{13}(\ddot{\mathbf{u}}^s_m))+\left(B\nabla\dot{\mathbf{u}}^f_m,\nabla\ddot{\mathbf{u}}^s_m\right)\\
&-\left(\beta\dot{\theta}_m,\nabla\ddot{\mathbf{u}}^s_m\right)=\left(\dot{\mathbf{f}^s},\ddot{\mathbf{u}}^s_m\right)+\langle\dot{\mathbf{g}},\ddot{\mathbf{u}}^s_m\rangle\,.
\end{aligned}
\tag{36}
$$

Also, the choice $\mathbf{v}^s=0,\mathbf{v}^f=\ddot{\mathbf{u}}^f_m,w=0$ in (35) yields

$$
\begin{aligned}
&\left(\rho_f\dddot{\mathbf{u}}^s_m,\ddot{\mathbf{u}}^f_m\right)+\left(g\dddot{\mathbf{u}}^f_m,\ddot{\mathbf{u}}^f_m\right)+\left(\frac{\eta}{\kappa}\ddot{\mathbf{u}}^f_m,\ddot{\mathbf{u}}^f_m\right)+\left(B\nabla\dot{\mathbf{u}}^s_m,\nabla\ddot{\mathbf{u}}^f_m\right)\\
&+\left(M\nabla\dot{\mathbf{u}}^f_m,\nabla\ddot{\mathbf{u}}^f_m\right)-\left(\beta_f\dot{\theta}_m,\nabla\ddot{\mathbf{u}}^f_m\right)=\left(\dot{\mathbf{f}^f},\ddot{\mathbf{u}}^f_m\right)+\left\langle\dot{\chi},\ddot{\mathbf{u}}^f_m\cdot\boldsymbol{\nu}\right\rangle.
\end{aligned}
\tag{37}
$$

From (36)-(37) we get the relations

$$
\begin{aligned}
&\left(\tau(1-\phi)\beta_m T_0\nabla\cdot\ddot{\mathbf{u}}^s_m,\dot{\theta}_m\right)\ge C^m_\beta\left(\beta\nabla\ddot{\mathbf{u}}^s_m,\dot{\theta}\right)\\
&=C^m_\beta\left[(\rho\dddot{\mathbf{u}}^s_m,\ddot{\mathbf{u}}^s_m)+\left(\rho_f\dddot{\mathbf{u}}^f_m,\ddot{\mathbf{u}}^s_m\right)+((\lambda_u+2\mu)\varepsilon_{11}(\dot{\mathbf{u}}^s_m),\varepsilon_{11}(\ddot{\mathbf{u}}^s_m))\right.\\
&+((\lambda_u+2\mu)\varepsilon_{33}(\dot{\mathbf{u}})m^s),\varepsilon_{33}(\ddot{\mathbf{u}}^s_m))+(\lambda_u\varepsilon_{33}(\dot{\mathbf{u}}^s_m),\varepsilon_{11}(\ddot{\mathbf{u}}^s_m))\\
&+(\lambda_u\varepsilon_{11}(\dot{\mathbf{u}}^s_m),\varepsilon_{33}(\ddot{\mathbf{u}}^s_m))+(4\mu\varepsilon_{13}(\dot{\mathbf{u}}^s_m),\varepsilon_{13}(\ddot{\mathbf{u}}^s_m))+\left(B\nabla\dot{\mathbf{u}}^f_m,\nabla\ddot{\mathbf{u}}^s_m\right)\\
&\left.-\left(\dot{\mathbf{f}^s},\ddot{\mathbf{u}}^s_m\right)-\langle\dot{\mathbf{g}},\ddot{\mathbf{u}}^s_m\rangle\right],
\end{aligned}
\tag{38}
$$

and

$$
\begin{aligned}
&\left(\tau\phi\beta_f T_0\nabla\cdot\ddot{\mathbf{u}}^f_m,\dot{\theta}_m\right)\ge C^f_\beta\left(\beta_f\nabla\ddot{\mathbf{u}}^f_m,\dot{\theta}\right)\\
&=C^f_\beta\left[\left(\rho_f\dddot{\mathbf{u}}^s_m,\ddot{\mathbf{u}}^f_m\right)+\left(g\dddot{\mathbf{u}}^f_m,\ddot{\mathbf{u}}^f_m\right)+\left(\frac{\eta}{\kappa}\ddot{\mathbf{u}}^f_m,\ddot{\mathbf{u}}^f_m\right)+\left(B\nabla\dot{\mathbf{u}}^s_m,\nabla\ddot{\mathbf{u}}^f_m\right)\right.\\
&\left.+\left(M\nabla\dot{\mathbf{u}}^f_m,\nabla\ddot{\mathbf{u}}^f_m\right)-\left(\dot{\mathbf{f}^f},\ddot{\mathbf{u}}^f_m\right)-\left\langle\dot{\chi},\ddot{\mathbf{u}}^f_m\cdot\boldsymbol{\nu}\right\rangle\right],
\end{aligned}
\tag{39}
$$

where

$$
C^m_\beta=\inf_{x\in\Omega}\left(\frac{(1-\phi)\tau T_0\beta_m}{\beta}\right),\qquad C^f_\beta=\inf_{x\in\Omega}\left(\frac{\phi\tau T_0}{\beta_f}\right).
\tag{40}
$$

Using (38) and (39) in (34) we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[(\mathcal{P}\dot{\mathbf{u}}_m(x),\dot{\mathbf{u}}_m)+\left(\widehat{\mathcal{P}}\ddot{\mathbf{u}}_m(x),\ddot{\mathbf{u}}_m\right)+\Lambda(\mathbf{u}_m,\mathbf{u}_m)\right.\\
&\quad\left.+\widehat{\Lambda}(\dot{\mathbf{u}}_m,\dot{\mathbf{u}}_m)+(\gamma\nabla\theta_m,\nabla\theta_m)+\left(\tau\,c\,\dot{\theta}_m,\dot{\theta}_m\right)\right]\\
&+\left(\frac{\eta}{\kappa}\dot{\mathbf{u}}^f_m,\dot{\mathbf{u}}^f_m\right)+C^f_\beta\left(\frac{\eta}{\kappa}\ddot{\mathbf{u}}^f_m,\ddot{\mathbf{u}}^f_m\right)-(\beta\theta_m,\nabla\cdot\dot{\mathbf{u}}^s_m)-\left(\beta_f\theta_m,\nabla\cdot\dot{\mathbf{u}}^f_m\right)
\end{aligned}
\tag{41}
$$

$$+\left(c\,\dot\theta_m,\dot\theta_m\right)+\left((1-\phi)\beta_m T_0\nabla\cdot\dot{\mathbf{u}}^s_m,\dot\theta_m\right)+\left(\phi\beta_f T_0\nabla\cdot\dot{\mathbf{u}}^f_m,\dot\theta_m\right)$$
$$=(\mathbf{f},\dot{\mathbf{u}}_m)-\left(q,\dot\theta_m\right)+\langle\mathbf{g},\dot{\mathbf{u}}^s_m\rangle+\left\langle\chi,\dot{\mathbf{u}}^f_m\cdot\boldsymbol{\nu}\right\rangle+\left\langle h,\dot\theta_m\right\rangle$$
$$+C^m_\beta\left(\dot{\mathbf{f}}^s,\ddot{\mathbf{u}}^s_m\right)+C^f_\beta\left(\dot{\mathbf{f}}^f,\ddot{\mathbf{u}}^f_m\right)$$
$$+C^m_\beta\left\langle\dot{\mathbf{g}},\ddot{\mathbf{u}}^s_m\right\rangle+C^f_\beta\left\langle\dot\chi,\ddot{\mathbf{u}}^f_m\cdot\boldsymbol{\nu}\right\rangle,\quad t\in J,$$

where

$$\widehat\Lambda(\mathbf{u},\mathbf{v})=\left(\widehat{\mathcal{E}}\,\widetilde\epsilon(\mathbf{u}),\widetilde\epsilon(\mathbf{v})\right),\tag{42}$$

and

$$\widehat{\mathcal{E}}=\begin{pmatrix}C^m_\beta(\lambda_u+2\mu) & C^m_\beta\lambda_u & C^m_\beta B & 0\\ C^m_\beta\lambda_u & C^m_\beta(\lambda_u+2\mu) & C^m_\beta B & 0\\ C^f_\beta B & C^f_\beta B & C^f_\beta M & 0\\ 0 & 0 & 0 & 4\mu\end{pmatrix},\quad \widehat{\mathcal{P}}=\begin{pmatrix}C^m_\beta\rho_b I & C^m_\beta\rho_f I\\ C^f_\beta\rho_f I & C^f_\beta g I\end{pmatrix}.\tag{43}$$

Since the coefficients C^m_β and C^f_β are strictly positive, the positive definiteness of the matrices $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{E}}$ in (43) is inherited from that of the matrices $\mathcal{E}$ and $\mathcal{P}$ in (5) and (25). Furthermore,

$$\widehat\Lambda(\mathbf{v},\mathbf{v})\ge C_3\|\mathbf{v}\|^2_{\mathcal{V}}-\lambda^{\widehat{\mathcal{E}}}_*\|\mathbf{v}\|^2_0,\tag{44}$$

where $\lambda^{\widehat{\mathcal{E}}}_*$ denotes the minimum eigenvalue of $\widehat{\mathcal{E}}$.

Next, take ζ_1,ζ_2 such that

$$\Lambda_{\zeta 1}(\mathbf{v},\mathbf{v})=\Lambda(\mathbf{v},\mathbf{v})+\zeta_1\|\mathbf{v}\|^2_0\ge C_3\|\mathbf{v}\|^2_{\mathcal{V}},\tag{45}$$
$$\widehat\Lambda_{\zeta_2}(\mathbf{v},\mathbf{v})=\widehat\Lambda(\mathbf{v},\mathbf{v})+\zeta_2\|\mathbf{v}\|^2_0\ge C_4\|\mathbf{v}\|^2_{\mathcal{V}},\tag{46}$$

and add to (41) the inequalities

$$\zeta\frac{d}{dt}\|\mathbf{u}_m\|^2_0\le\zeta\left(\|\mathbf{u}_m\|^2_0+\|\dot{\mathbf{u}}_m\|^2_0\right),\quad \zeta=\zeta_1,\zeta_2$$
$$\frac{d}{dt}\|\gamma^{1/2}\theta_m\|^2_0\le\left(\|\gamma^{1/2}\theta_m\|^2_0+\|\dot\gamma^{1/2}\dot\theta_m\|^2_0\right),$$

to obtain

$$\frac{1}{2}\frac{d}{dt}\Big[(\mathcal{P}\dot{\mathbf{u}}_m,\dot{\mathbf{u}}_m)+\left(\widehat{\mathcal{P}}\ddot{\mathbf{u}}_m,\ddot{\mathbf{u}}_m\right)+\Lambda_{\zeta_1}(\mathbf{u}_m,\mathbf{u}_m)\tag{47}$$
$$+\widehat\Lambda_{\zeta_2}(\dot{\mathbf{u}}_m,\dot{\mathbf{u}}_m)+(\gamma\nabla\theta_m,\nabla\theta_m)+\left(\tau\,c\,\dot\theta_m,\dot\theta_m\right)\Big]$$
$$+\left(\frac{\eta}{\kappa}\dot{\mathbf{u}}^f_m,\dot{\mathbf{u}}^f_m\right)+C^f_\beta\left(\frac{\eta}{\kappa}\ddot{\mathbf{u}}^f_m,\ddot{\mathbf{u}}^f_m\right)$$
$$\le C\left(\|\mathbf{u}_m\|^2_0+\|\dot{\mathbf{u}}_m\|^2_0+\|\ddot{\mathbf{u}}_m\|^2_0+\|\theta_m\|^2_0+\|\dot\theta_m\|^2_0+\|\mathbf{f}\|^2_0+\|\dot{\mathbf{f}}\|^2_0+\|q\|^2_0\right)$$
$$+\left(\beta\theta_m,\nabla\cdot\dot{\mathbf{u}}^s_m\right)+\left(\beta_f\theta_m,\nabla\cdot\dot{\mathbf{u}}^f_m\right)$$
$$-\left((1-\phi)\beta_m T_0\nabla\cdot\dot{\mathbf{u}}^s_m,\dot\theta_m\right)-\left(\phi\beta_f T_0\nabla\cdot\dot{\mathbf{u}}^f_m,\dot\theta_m\right)$$
$$+\langle\mathbf{g},\dot{\mathbf{u}}^s_m\rangle+\left\langle\chi,\dot{\mathbf{u}}^f_m\cdot\boldsymbol{\nu}\right\rangle+\left\langle h,\dot\theta_m\right\rangle$$
$$+C^m_\beta\left\langle\dot{\mathbf{g}},\ddot{\mathbf{u}}^s_m\right\rangle+C^f_\beta\left\langle\dot\chi,\ddot{\mathbf{u}}^f_m\cdot\boldsymbol{\nu}\right\rangle,\quad t\in J.$$

Next, we obtain bounds for the integrals in time of the last seven terms in the right-hand side of (47). First,

$$\left| \int_0^t \left[(\beta\theta_m(s), \nabla \cdot \dot{\mathbf{u}}_m^s(s)) + \left(\beta_f \theta_m(s), \nabla \cdot \dot{\mathbf{u}}_m^f(s)\right)\right] ds \right| \tag{48}$$

$$\leq C \int_0^t \left(\|\theta_m(s)\|_0 \left(\|\nabla \cdot \dot{\mathbf{u}}_m^s(s)\|_0 + \|\nabla \cdot \dot{\mathbf{u}}_m^f(s)\|_0 \right)\right) ds$$

$$\leq C \int_0^t \left(\|\theta_m(s)\|_0^2 + \|\dot{\mathbf{u}}_m(s)\|_{\mathcal{V}}^2 \right) ds.$$

Similarly,

$$\left| \int_0^t \left((1-\phi)\beta_m T_0 \nabla \cdot \dot{\mathbf{u}}_m^s(s), \dot{\theta}_m(s)\right) ds \right| \tag{49}$$

$$+ \left| \int_0^t \left(\phi \beta_f T_0 \nabla \cdot \dot{\mathbf{u}}_m^f(s), \dot{\theta}_m(s)\right) ds \right| \leq C \int_0^t \left(\|\dot{\theta}_m(s)\|_0^2 + \|\dot{\mathbf{u}}_m(s)\|_{\mathcal{V}}^2 \right) ds.$$

Next, using (21),

$$\left| \int_0^t \langle \mathbf{g}(s), \dot{\mathbf{u}}_m^s(s) \rangle \right| \leq C \int_0^t \|\mathbf{g}(s)\|_1 \|\dot{\mathbf{u}}_m^s(s)\|_1 ds \tag{50}$$

$$\leq C \left(\|\mathbf{g}\|_{L^2(J,[H^1(\Omega)]^2)}^2 + \int_0^t \|\dot{\mathbf{u}}_m(s)\|_{\mathcal{V}}^2 \right) ds.$$

Also, using (20),

$$\left| \int_0^t \left\langle \chi(s), \dot{\mathbf{u}}_m^f(s) \cdot \boldsymbol{\nu} \right\rangle ds \right| \leq C \int_0^t \|\dot{\mathbf{u}}_m^f(s) \cdot \boldsymbol{\nu}\|_{-1/2,\Gamma} \|\chi(s)\|_{1/2,\Gamma} ds \tag{51}$$

$$\leq C \int_0^t \left(\|\chi(s)\|_{1/2,\Gamma}^2 + \|\dot{\mathbf{u}}_m^f(s)\|_{H(\mathrm{div};\Omega)}^2 \right) ds$$

$$\leq C \left(\|\chi\|_{L^2(J,H^{1/2}(\Gamma))}^2 + \int_0^t \|\dot{\mathbf{u}}_m(s)\|_{\mathcal{V}}^2 ds \right).$$

A bound for the last three terms in the right-hand side can be obtained using integration by parts in time and (20)- (21) as follows.

$$\left| \int_0^t \left\langle h(s), \dot{\theta}_m(s) \right\rangle ds \right| \tag{52}$$

$$\left|\langle h(t),\theta_m(t)\rangle - \langle h(0),\theta_m(0)\rangle - \int_0^t \left\langle \dot{h}(s),\theta_m(s)\right\rangle ds\right|$$
$$\le \epsilon\|\theta_m(t)\|_1^2 + C\left(\|\theta_m(0)\|_1^2 + \|h\|^2_{L^\infty(J,H^1(\Omega))}\right.$$
$$\left. +(\|\dot{h}\|^2_{L^2(J,H^1(\Omega))} + \int_0^t \|\theta_m(s)\|_1^2 ds\right),$$
$$\left|\int_0^t \langle \dot{\mathbf{g}}(s), \ddot{\mathbf{u}}_m^s(s))\rangle\, ds\right| \tag{53}$$
$$\left|\langle \dot{\mathbf{g}}(t), \dot{\mathbf{u}}_m^s(t)\rangle - \langle \dot{\mathbf{g}}(0), \dot{\mathbf{u}}_m^s(0)\rangle - \int_0^t \langle \ddot{\mathbf{g}}(s), \dot{\mathbf{u}}_m(s)\rangle\, ds\right|$$
$$\le \epsilon\|\dot{\mathbf{u}}_m(t)\|_{\mathcal{V}}^2 + \le C\left(\|\dot{\mathbf{u}}_m(0)\|_{\mathcal{V}}^2 + \|\dot{\mathbf{g}}\|^2_{L^\infty(J,[H^1(\Omega)]^2)}\right.$$
$$\left. +\|\ddot{\mathbf{g}}\|^2_{L^2(J,[H^1(\Omega)]^2)} + \int_0^t \|\dot{\mathbf{u}}_m(s)\|_{\mathcal{V}}^2 ds\right),$$

and

$$\left|\int_0^t \left\langle \dot{\chi}(s), \ddot{\mathbf{u}}_m^f(s)\cdot\boldsymbol{\nu}\right\rangle ds\right| \tag{54}$$
$$\left|\left\langle \dot{\chi}(t), \dot{\mathbf{u}}_m^f(t)\cdot\boldsymbol{\nu}\right\rangle - \left\langle \dot{\chi}(0), \dot{\mathbf{u}}_m^f(0)\cdot\boldsymbol{\nu}\right\rangle - \int_0^t \left\langle \ddot{\chi}(s), \dot{\mathbf{u}}_m^f(s)\cdot\boldsymbol{\nu}\right\rangle ds\right|$$
$$\le \epsilon\|\dot{\mathbf{u}}_m(t)\|_{\mathcal{V}}^2 + C\left(\|\dot{\mathbf{u}}_m(0)\|_{\mathcal{V}}^2 + (\|\dot{\chi}\|^2_{L^\infty(J,H^{1/2}(\Gamma))}\right.$$
$$\left. +\|\ddot{\chi}\|^2_{L^2(J,H^{1/2}(\Gamma))} + \int_0^t \|\dot{\mathbf{u}}_m(s)\|_{\mathcal{V}}^2 ds\right).$$

Then, integrate in time in (47) and use the estimates (48)-(54) to obtain

$$\begin{aligned}
&\|\mathcal{P}^{1/2}\dot{\mathbf{u}}_m(t)\|_0^2 + \|\widehat{\mathcal{P}}^{1/2}\ddot{\mathbf{u}}_m(t)\|_0^2 + C_2\|\mathbf{u}_m(t)\|_{\mathcal{V}}^2 + C_3\|\dot{\mathbf{u}}_m(t)\|_{\mathcal{V}}^2 \\
&+\|\gamma^{1/2}\theta_m(t)\|_1^2 + \|(\tau\, c)^{1/2}\dot{\theta}_m(t)\|_0^2 \\
&\le \epsilon\left(\|\dot{\mathbf{u}}_m(t)\|_{\mathcal{V}}^2 + \|\theta_m(t)\|_1^2\right) + C\left(H_0^2 + M_0^2\left(\mathbf{f},\mathbf{g},q,h,\chi\right)\right) \\
&+ \int_0^t \left[\|\mathbf{u}_m(s)\|_0^2 + \|\dot{\mathbf{u}}_m(s)\|_{\mathcal{V}}^2 + \|\ddot{\mathbf{u}}_m(s)\|_0^2 + \|\theta_m(s)\|_1^2 + \|\dot{\theta}_m(s)\|_0^2\right] ds,
\end{aligned} \tag{55}$$

where

$$H_0^2 = \|\mathbf{u}_m(0)\|_{\mathcal{V}}^2 + \|\dot{\mathbf{u}}_m(0)\|_{\mathcal{V}}^2 + \|\ddot{\mathbf{u}}_m(0)\|_0^2 + \|\theta_m(0)\|_1^2 + \|\dot{\theta}_m(0)\|_0^2. \tag{56}$$

The term $\|\ddot{\mathbf{u}}_m(0)\|_0^2$ in (56) can be estimated as follows. Choose $\mathbf{v} = \ddot{\mathbf{u}}_m, w = 0$ in (29), use integration by parts in the $\Lambda(\mathbf{u}_m, \mathbf{u}_m)$-term and set $t = 0$ to obtain

$$(\mathcal{P}\ddot{\mathbf{u}}_m(0),\ddot{\mathbf{u}}_m(0))+\left(\frac{\eta}{\kappa}\dot{\mathbf{u}}^f_m(0),\ddot{\mathbf{u}}^f_m(0)\right)-(\mathcal{L}(\mathbf{u}_m(0),\theta_m(0)),\ddot{\mathbf{u}}_m(0)) \\ =(\mathbf{f}(0),\ddot{\mathbf{u}}_m(0))\,. \tag{57}$$

Hence,

$$\|\ddot{\mathbf{u}}_m(0)\|_0\le C\left(\|\dot{\mathbf{u}}_m(0)\|_0+\|\mathbf{u}_m(0)\|_2+\|\theta_m(0)\|_1+\|\mathbf{f}(0)\|_0\right)\le N_0. \tag{58}$$

Thanks to (30)-(33), all other terms in (56) are bounded in terms of $\|\mathbf{u}^0\|_2, \|\mathbf{u}^1\|_1, \|\theta^0\|_2$ and $\|\theta^1\|_1$, so that

$$H_0^2\le N_0^2. \tag{59}$$

Thus, using in (55) that $\mathcal{P}$ and $\widehat{\mathcal{P}}$ are positive definite, absorbing the $\epsilon-$ terms in the left-hand side of (55) and applying Gronwall's Lemma [12,1], we get the estimate

$$\|\mathbf{u}_m(t)\|^2_{L^\infty(J,\mathcal{V})}+\|\dot{\mathbf{u}}_m\|^2_{L^\infty(J,\mathcal{V})}+\|\ddot{\mathbf{u}}_m\|^2_{L^\infty(J,[L^2(\Omega)]^4)} \\ +\|\theta_m\|^2_{L^\infty(J,H^1(\Omega))}+\|\dot{\theta}_m\|^2_{L^\infty(J,L^2(\Omega))}\le C\left(N_0^2+M_0^2\left(\mathbf{f},\mathbf{g},q,h,\chi\right)\right). \tag{60}$$

Next, we obtain additional estimates for time derivatives of $\mathbf{u}$ and θ. Take time derivative in (35) and choose $\mathbf{v}^s=\ddot{\mathbf{u}}^s_m, \mathbf{v}^f=0$ and $\mathbf{v}^s=0, \mathbf{v}^f=\ddot{\mathbf{u}}^f_m$ to obtain the estimates

$$\left(\tau(1-\phi)\beta_m T_0\nabla\cdot\dddot{\mathbf{u}}^s_m,\ddot{\theta}_m\right)\ge C^m_\beta\left(\beta\nabla\dddot{\mathbf{u}}^s_m,\ddot{\theta}\right) \\ =C^m_\beta\left[\left(\rho\,\dddot{\mathbf{u}}^s_m,\ddot{\mathbf{u}}^s_m\right)+\left(\rho_f\,\dddot{\mathbf{u}}^f_m,\ddot{\mathbf{u}}^s_m\right)+\left((\lambda_u+2\mu)\varepsilon_{11}(\ddot{\mathbf{u}}^s_m),\varepsilon_{11}(\dddot{\mathbf{u}}^s_m)\right)\right. \\ +\left((\lambda_u+2\mu)\varepsilon_{33}(\ddot{\mathbf{u}})m^s),\varepsilon_{33}(\dddot{\mathbf{u}}^s_m)\right)+\left(\lambda_u\varepsilon_{33}(\ddot{\mathbf{u}}^s_m),\varepsilon_{11}(\dddot{\mathbf{u}}^s_m)\right) \\ +\left(\lambda_u\varepsilon_{11}(\ddot{\mathbf{u}}^s_m),\varepsilon_{33}(\dddot{\mathbf{u}}^s_m)\right)+\left(4\mu\varepsilon_{13}(\ddot{\mathbf{u}}^s_m),\varepsilon_{13}(\ddot{\mathbf{u}}^s_m)\right)+\left(B\nabla\ddot{\mathbf{u}}^f_m,\nabla\dddot{\mathbf{u}}^s_m\right) \\ \left.-\left(\ddot{\mathbf{f}}^s,\dddot{\mathbf{u}}^s_m\right)-\langle\ddot{\mathbf{g}},\dddot{\mathbf{u}}^s_m\rangle\right], \tag{61}$$

and

$$\left(\tau\phi\beta_f T_0\nabla\cdot\dddot{\mathbf{u}}^f_m,\ddot{\theta}_m\right)\ge C^f_\beta\left(\beta_f\nabla\dddot{\mathbf{u}}^f_m,\ddot{\theta}\right) \\ =C^f_\beta\left[\left(\rho_f\,\dddot{\mathbf{u}}^s_m,\dddot{\mathbf{u}}^f_m\right)+\left(g\,\dddot{\mathbf{u}}^f_m,\dddot{\mathbf{u}}^f_m\right)+\left(\frac{\eta}{\kappa}\ddot{\mathbf{u}}^f_m,\dddot{\mathbf{u}}^f_m\right)+\left(B\nabla\ddot{\mathbf{u}}^s_m,\nabla\dddot{\mathbf{u}}^f_m\right)\right. \\ \left.+\left(M\nabla\ddot{\mathbf{u}}^f_m,\nabla\dddot{\mathbf{u}}^f_m\right)-\left(\ddot{\mathbf{f}}^f,\dddot{\mathbf{u}}^f_m\right)-\left\langle\ddot{\chi},\dddot{\mathbf{u}}^f_m\cdot\boldsymbol{\nu}\right\rangle\right]. \tag{62}$$

Thus, using (61) and (62) in the time derivative of (35) for the choice $\mathbf{v}=\ddot{\mathbf{u}}, w=\ddot{\theta}$, using the argument leading to (41) we obtain

$$\frac{1}{2}\frac{d}{dt}\left[(\mathcal{P}\ddot{\mathbf{u}}_m(x),\ddot{\mathbf{u}}_m)+\left(\widehat{\mathcal{P}}\dddot{\mathbf{u}}_m(x),\dddot{\mathbf{u}}_m\right)+\Lambda(\dot{\mathbf{u}}_m,\dot{\mathbf{u}}_m)\right. \\ \left.+\widehat{\Lambda}(\ddot{\mathbf{u}}_m,\ddot{\mathbf{u}}_m)+\left(\gamma\nabla\dot{\theta}_m,\nabla\dot{\theta}_m\right)+\left(\tau\,c\,\ddot{\theta}_m,\ddot{\theta}_m\right)\right] \\ +\left(\frac{\eta}{\kappa}\ddot{\mathbf{u}}^f_m,\ddot{\mathbf{u}}^f_m\right)+C^f_\beta\left(\frac{\eta}{\kappa}\dddot{\mathbf{u}}^f_m,\dddot{\mathbf{u}}^f_m\right)-\left(\beta\dot{\theta}_m,\nabla\cdot\ddot{\mathbf{u}}^s_m\right)-\left(\beta_f\dot{\theta}_m,\nabla\cdot\ddot{\mathbf{u}}^f_m\right) \\ +\left(c\,\ddot{\theta}_m,\ddot{\theta}_m\right)+\left((1-\phi)\beta_m T_0\nabla\cdot\ddot{\mathbf{u}}^s_m,\ddot{\theta}_m\right)+\left(\phi\beta_f T_0\nabla\cdot\ddot{\mathbf{u}}^f_m,\ddot{\theta}_m\right) \\ =\left(\dot{\mathbf{f}},\ddot{\mathbf{u}}_m\right)-\left(\dot{q},\ddot{\theta}_m\right)+\langle\dot{\mathbf{g}},\ddot{\mathbf{u}}^s_m\rangle+\left\langle\dot{\chi},\ddot{\mathbf{u}}^f_m\cdot\boldsymbol{\nu}\right\rangle+\left\langle\dot{h},\ddot{\theta}_m\right\rangle \\ +C^m_\beta\left(\ddot{\mathbf{f}}^s,\dddot{\mathbf{u}}^s_m\right)+C^f_\beta\left(\ddot{\mathbf{f}}^f,\dddot{\mathbf{u}}^f_m\right) \\ +C^m_\beta\left\langle\ddot{\mathbf{g}},\dddot{\mathbf{u}}^s_m\right\rangle+C^f_\beta\left\langle\ddot{\chi},\dddot{\mathbf{u}}^f_m\cdot\boldsymbol{\nu}\right\rangle,\quad t\in J. \tag{63}$$

Next, add to (63) the inequalities

$$\zeta \frac{d}{dt}\|\dot{\mathbf{u}}_m\|_0^2 \le \zeta\left(\|\dot{\mathbf{u}}_m\|_0^2 + \|\ddot{\mathbf{u}}_m\|_0^2\right), \quad \zeta = \zeta_1, \zeta_2$$
$$\frac{d}{dt}\|\gamma^{1/2}\dot{\theta}_m\|_0^2 \le \left(\|\gamma^{1/2}\dot{\theta}_m\|_0^2 + \|\dot{\gamma}^{1/2}\ddot{\theta}_m\|_0^2\right),$$

to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[(\mathcal{P}\ddot{\mathbf{u}}_m, \ddot{\mathbf{u}}_m) + \left(\widehat{\mathcal{P}}\dddot{\mathbf{u}}_m, \dddot{\mathbf{u}}_m\right) + \Lambda_{\zeta_1}(\dot{\mathbf{u}}_m, \dot{\mathbf{u}}_m) \\
&\quad + \widehat{\Lambda}_{\zeta_2}(\ddot{\mathbf{u}}_m, \ddot{\mathbf{u}}_m) + \left(\gamma\nabla\dot{\theta}_m, \nabla\dot{\theta}_m\right) + \left(\tau\, c\, \ddot{\theta}_m, \ddot{\theta}_m\right)\Big] \\
&\le C\left(\|\dot{\mathbf{u}}_m\|_0^2 + \|\ddot{\mathbf{u}}_m\|_0^2 + \|\dddot{\mathbf{u}}_m\|_0^2 + \|\dot{\theta}_m\|_0^2 + \|\ddot{\theta}_m\|_0^2 + \|\dot{\mathbf{f}}\|_0^2 + \|\ddot{\mathbf{f}}\|_0^2 + \|\dot{q}\|_0^2\right) \\
&+ \left(\beta\dot{\theta}_m, \nabla\cdot\ddot{\mathbf{u}}_m^s\right) + \left(\beta_f\dot{\theta}_m, \nabla\cdot\ddot{\mathbf{u}}_m^f\right) \\
&- \left((1-\phi)\beta_m T_0\nabla\cdot\ddot{\mathbf{u}}_m^s, \ddot{\theta}_m\right) - \left(\phi\beta_f T_0\nabla\cdot\ddot{\mathbf{u}}_m^f, \ddot{\theta}_m\right) \\
&+ \langle\dot{\mathbf{g}}, \ddot{\mathbf{u}}_m^s\rangle + \left\langle\dot{\chi}, \ddot{\mathbf{u}}_m^f\cdot\boldsymbol{\nu}\right\rangle + \left\langle\dot{h}, \ddot{\theta}_m\right\rangle \\
&+ C_\beta^m\langle\ddot{\mathbf{g}}, \ddot{\mathbf{u}}_m^s\rangle + C_\beta^f\left\langle\ddot{\chi}, \ddot{\mathbf{u}}_m^f\cdot\boldsymbol{\nu}\right\rangle, \quad t \in J.
\end{aligned} \tag{64}$$

Next we derive bounds for the integrals in time of the last nine terms in the right-hand side of (64). First,

$$\begin{aligned}
&\left|\int_0^t \left[\left(\beta\dot{\theta}_m(s), \nabla\cdot\ddot{\mathbf{u}}_m^s(s)\right) + \left(\beta_f\dot{\theta}_m(s), \nabla\cdot\ddot{\mathbf{u}}_m^f(s)\right)\right] ds\right| \\
&\le C\int_0^t \left(\|\dot{\theta}(s)\|_0^2 + \|\ddot{\mathbf{u}}_m(s)\|_{\mathcal{V}}^2\right) ds,
\end{aligned} \tag{65}$$

and

$$\begin{aligned}
&\left|\int_0^t \left((1-\phi)\beta_m T_0\nabla\cdot\ddot{\mathbf{u}}_m^s(s), \ddot{\theta}_m(s)\right) ds\right| \\
&+ \left|\left(\phi\beta_f T_0\nabla\cdot\ddot{\mathbf{u}}_m^f(s), \ddot{\theta}_m(s)\right) ds\right| \le C\int_0^t \left(\|\ddot{\theta}_m(s)\|_0^2 + \|\ddot{\mathbf{u}}_m(s)\|_{\mathcal{V}}^2\right) ds.
\end{aligned} \tag{66}$$

Also, using (20) and (21)

$$\begin{aligned}
&\left|\int_0^t \langle\dot{\mathbf{g}}(s), \ddot{\mathbf{u}}_m^s(s)\rangle\right| + \left|\int_0^t \left\langle\dot{\chi}(s), \ddot{\mathbf{u}}_m^f(s)\cdot\boldsymbol{\nu}\right\rangle ds\right| \\
&\le C\left(\|\dot{\mathbf{g}}\|_{L^2(J,[H^1(\Omega)]^2)}^2 + \|\dot{\chi}\|_{L^2(J,H^{1/2}(\Gamma))}^2 + \int_0^t \|\ddot{\mathbf{u}}_m(s)\|_{\mathcal{V}}^2 ds\right).
\end{aligned} \tag{67}$$

Using integration by parts in time and (20)- (21),

$$\left|\int_0^t \left\langle \dot{h}(s), \ddot{\theta}_m(s) \right\rangle ds\right| \tag{68}$$
$$= \left|\left\langle \dot{h}(t), \dot{\theta}_m(t) \right\rangle - \left\langle \dot{h}(0), \dot{\theta}_m(0) \right\rangle - \int_0^t \left\langle \ddot{h}(s), \dot{\theta}_m(s) \right\rangle ds\right|$$
$$\leq \epsilon\|\dot{\theta}_m(t)\|_1^2 + C\left(\|\dot{\theta}_m(0)\|_1^2 + \|\dot{h}\|^2_{L^\infty(J,H^1(\Omega))}\right.$$
$$\left. +(\|\ddot{h}\|^2_{L^2(J,H^1(\Omega))} + \int_0^t \|\dot{\theta}_m(s)\|_1^2 ds\right),$$
$$\left|\int_0^t \langle \dot{\mathbf{g}}(s), \ddot{\mathbf{u}}^s_m(s)) \rangle\, ds\right| \tag{69}$$
$$= \left|\langle \dot{\mathbf{g}}(t), \dot{\mathbf{u}}^s_m(t) \rangle - \langle \dot{\mathbf{g}}(0), \dot{\mathbf{u}}^s_m(0) \rangle - \int_0^t \langle \ddot{\mathbf{g}}(s), \dot{\mathbf{u}}_m(s) \rangle\, ds\right|$$
$$\leq \epsilon\|\dot{\mathbf{u}}_m(t)\|^2_{\mathcal{V}} + C\left(\|\dot{\mathbf{u}}_m(0)\|^2_{\mathcal{V}}\right.$$
$$\left. +\|\dot{\mathbf{g}}\|^2_{L^\infty(J,[H^1(\Omega)]^2)} + \|\ddot{\mathbf{g}}\|^2_{L^2(J,[H^1(\Omega)]^2)} + \int_0^t \|\dot{\mathbf{u}}_m(s)\|^2_{\mathcal{V}} ds\right),$$

and

$$\left|\int_0^t \left\langle \ddot{\chi}(s), \ddot{\mathbf{u}}^f_m(s)\cdot\boldsymbol{\nu} \right\rangle ds\right| \tag{70}$$
$$= \left|\left\langle \ddot{\chi}, \dot{\mathbf{u}}^f_m\cdot\boldsymbol{\nu} \right\rangle(t) - \left\langle \ddot{\chi}, \dot{\mathbf{u}}^f_m\cdot\boldsymbol{\nu} \right\rangle(0) - \int_0^t \left\langle \dddot{\chi}, \dot{\mathbf{u}}^f_m\cdot\boldsymbol{\nu} \right\rangle(s) ds\right|$$
$$\leq \epsilon\|\dot{\mathbf{u}}_m(t)\|^2_{\mathcal{V}} + C\left(\|\dot{\mathbf{u}}_m(0)\|^2_{\mathcal{V}}\right.$$
$$\left. +\|\ddot{\chi}\|^2_{L^\infty(J,H^{1/2}(\Gamma))} + \|\dddot{\chi}\|^2_{L^2(J,H^{1/2}(\Gamma))} + \int_0^t \|\dot{\mathbf{u}}_m(s)\|^2_{\mathcal{V}} ds\right).$$

Then, integrate in time in (64), use that $\mathcal{P}$ and $\widehat{\mathcal{P}}$ are positive definite, apply the estimates (65)-(70), absorb the ϵ-terms in the left-hand side of the resulting inequality and apply Gronwall's Lemma [12,1] to get

$$\begin{aligned} &\|\dot{\mathbf{u}}_m(t)\|^2_{L^\infty(J,\mathcal{V})} + \|\ddot{\mathbf{u}}_m\|^2_{L^\infty(J,\mathcal{V})} + \|\dddot{\mathbf{u}}_m\|^2_{L^\infty(J,[L^2(\Omega)]^4)} \\ &+\|\dot{\theta}_m\|^2_{L^\infty(J,H^1(\Omega)} + \|\ddot{\theta}_m\|^2_{L^\infty(J,L^2(\Omega)} \\ &\leq C\left(H_1^2 + M_0^2\left(\mathbf{f},\mathbf{g},q,h,\chi\right)\right), \end{aligned} \tag{71}$$

where

$$H_1^2 = \|\dot{\mathbf{u}}_m(0)\|^2_{\mathcal{V}} + \|\ddot{\mathbf{u}}_m(0)\|^2_{\mathcal{V}} + \|\dddot{\mathbf{u}}_m(0)\|_0^2 + \|\dot{\theta}_m(0)\|_1^2 + \|\ddot{\theta}_m(0)\|_0^2. \tag{72}$$

Let us bound the term $\|\dddot{\mathbf{u}}_m(0)\|_0^2$ in (72). Choose $\mathbf{v} = \dddot{\mathbf{u}}_m, w = 0$ in (35), use integration by parts in the $\Lambda(\dot{\mathbf{u}}_m, \dddot{\mathbf{u}}_m)$-term and set $t = 0$ to obtain

$$(\mathcal{P}\ddot{\mathbf{u}}_m(0), \ddot{\mathbf{u}}_m(0)) + \left(\frac{\eta}{\kappa}\ddot{\mathbf{u}}^f_m(0), \ddot{\mathbf{u}}^f_m(0)\right) - \left(\mathcal{L}(\dot{\mathbf{u}}_m(0), \dot{\theta}_m(0)), \ddot{\mathbf{u}}_m(0)\right)$$
$$= \left(\dot{\mathbf{f}}(0), \ddot{\mathbf{u}}_m(0)\right). \tag{73}$$

Hence,

$$\|\dddot{\mathbf{u}}_m(0)\|_0 \le C\left(\|\ddot{\mathbf{u}}_m(0)\|_0 + \|\dot{\mathbf{u}}^s_m(0)\|_2 + \|\dot{\mathbf{u}}^f_m(0)\|_{H^1(\mathrm{div};\Omega)}\right.$$
$$\left. + \|\dot{\theta}_m(0)\|_1 + |\dot{\mathbf{f}}(0)\|_0\right). \tag{74}$$

To bound $\|\dot{\mathbf{u}}^s_m(0)\|_2$ and $\|\dot{\mathbf{u}}^f_m(0)\|_{H^1(\mathrm{div};\Omega)}$ in (74), we assume that problem (11)-(19) satisfy the regularity assumption [21]

$$\|\mathbf{u}^s\|_2 + \|\mathbf{u}^f\|_{H^1(\mathrm{div};\Omega)} + \|\theta\|_2 \tag{75}$$
$$\le C\left(\|\mathbf{f}\|_0 + \|\mathbf{g}\|_{1/2,\Gamma} + \|\boldsymbol{\chi}\|_{1/2,\Gamma} + \|h\|_{1/2,\Gamma}\right), \quad t \in J,$$

and that (75) also holds for time derivatives of $\mathbf{u}^s, \mathbf{u}^f$ and θ. Hence,

$$\|\dot{\mathbf{u}}^s(0)\|_2 + \|\dot{\mathbf{u}}^f(0)\|_{H^1(\mathrm{div};\Omega)} \tag{76}$$
$$\le C\left(\|\dot{\mathbf{f}}(0)\|_0 + \|\dot{\mathbf{g}}(0)\|_{1/2,\Gamma} + \|\dot{\boldsymbol{\chi}}(0)\|_{1/2,\Gamma} + \|\dot{h}(0)\|_{1/2,\Gamma}\right).$$

Then, assuming in (35) that

$$\dot{\mathbf{u}}_m(0) \xrightarrow[m\to\infty]{} \dot{\mathbf{u}}(0) \quad \text{in} \quad [H^2(\Omega)]^4, \tag{77}$$

we get

$$\|\dot{\mathbf{u}}^s_m(0)\|_2 + \|\dot{\mathbf{u}}^f_m(0)\|_{H^1(\mathrm{div};\Omega)} \tag{78}$$
$$\le \|\dot{\mathbf{u}}^s_m(0) - \dot{\mathbf{u}}^s(0)\|_2 + \|\dot{\mathbf{u}}^s(0)\|_2 + \|\dot{\mathbf{u}}^f_m(0) - \dot{\mathbf{u}}^f(0)\|_{H^1(\mathrm{div};\Omega)}$$
$$+ \|\dot{\mathbf{u}}^f(0)\|_{H^1(\mathrm{div};\Omega)}$$
$$\le \epsilon + C\left(\|\dot{\mathbf{f}}(0)\|_0 + \|\dot{\mathbf{g}}(0)\|_{1/2,\Gamma} + \|\dot{\boldsymbol{\chi}}(0)\|_{1/2,\Gamma} + \|\dot{h}(0)\|_{1/2,\Gamma}\right) \le N_0.$$

To bound $\|\ddot{\mathbf{u}}_m(0)\|^2_{\mathcal{V}}$ and $\|\ddot{\theta}_m(0)\|^2_0$ in the expression for H_1 in (72), using that second derivatives at $t = 0$ satisfy (75) we see that

$$\|\ddot{\mathbf{u}}(0)\|_{\mathcal{V}} + \|\ddot{\theta}(0)\|_0 \le \|\ddot{\mathbf{u}}^s(0)\|_2 + \|\ddot{\mathbf{u}}^f(0)\|_{H^1(\mathrm{div};\Omega)} + \|\ddot{\theta}(0)\|_2 \tag{79}$$
$$\le C\left(\|\ddot{\mathbf{f}}(0)\|_0 + \|\ddot{\mathbf{g}}(0)\|_{1/2,\Gamma} + \|\ddot{\boldsymbol{\chi}}(0)\|_{1/2,\Gamma} + \|\ddot{h}(0)\|_{1/2,\Gamma}\right).$$

Thus, assuming in (35) that

$$\ddot{\mathbf{u}}_m(0) \xrightarrow[m\to\infty]{} \ddot{\mathbf{u}}(0) \quad \text{in} \quad [H^1(\Omega)]^4, \tag{80}$$
$$\ddot{\theta}_m(0) \xrightarrow[m\to\infty]{} \ddot{\theta}(0) \quad \text{in} \quad H^1(\Omega), \tag{81}$$

we get the estimates

$$\|\ddot{\mathbf{u}}_m(0)\|_{\mathcal{V}} \le \|\ddot{\mathbf{u}}_m(0) - \ddot{\mathbf{u}}(0)\|_{\mathcal{V}} + \|\ddot{\mathbf{u}}(0)\|_{\mathcal{V}} \tag{82}$$
$$\le \epsilon + C\left(\|\ddot{\mathbf{f}}(0)\|_0 + \|\ddot{\mathbf{g}}(0)\|_{1/2,\Gamma} + \|\ddot{\boldsymbol{\chi}}(0)\|_{1/2,\Gamma} + \|\ddot{h}(0)\|_{1/2,\Gamma}\right) \le N_0,$$
$$\|\ddot{\theta}_m(0)\|_0 \le \|\ddot{\theta}_m(0) - \ddot{\theta}(0)\|_1 + \|\ddot{\theta}(0)\|_1 \tag{83}$$
$$\le \epsilon + C\left(\|\ddot{\mathbf{f}}(0)\|_0 + \|\ddot{\mathbf{g}}(0)\|_{1/2,\Gamma} + \|\ddot{\boldsymbol{\chi}}(0)\|_{1/2,\Gamma} + \|\ddot{h}(0)\|_{1/2,\Gamma}\right) \le N_0.$$

Next, using (30)-(33), (74), (78), (82) and (83), we have

$$H_1^2 \leq \|\mathbf{u}^1\|_2^2 + \|\theta^1\|_0^2 + \|\ddot{\mathbf{f}}(0)\|_0^2 + \|\ddot{\mathbf{g}}(0)\|_{1/2,\Gamma}^2 + \|\ddot{\boldsymbol{\chi}}(0)\|_{1/2,\Gamma}^2 + \|\ddot{h}(0)\|_{1/2,\Gamma}^2 + \|\dot{\mathbf{f}}(0)\|_0^2 \leq N_0^2. \tag{84}$$

Combining (60), (71) and the bounds for H_0 and H_1 in (59) and (84), we conclude that

$$\|\mathbf{u}_m(t)\|_{L^\infty(J,\mathcal{V})}^2 + \|\dot{\mathbf{u}}_m\|_{L^\infty(J,\mathcal{V})}^2 + \|\ddot{\mathbf{u}}_m\|_{L^\infty(J,\mathcal{V})}^2 + \|\dddot{\mathbf{u}}_m\|_{L^\infty(J,[H^1(\Omega)]^4)}^2 + \|\theta_m\|_{L^\infty(J,H^1(\Omega))}^2 + \|\dot{\theta}_m\|_{L^\infty(J,H^1(\Omega))}^2 + \|\ddot{\theta}_m\|_{L^\infty(J,L^2(\Omega))}^2 \leq C\left(N_0^2 + M_0^2\left(\mathbf{f}, \mathbf{g}, q, h, \chi\right)\right). \tag{85}$$

It follows from (85) that there exist subsequences of $(\mathbf{u}_m)_{m\geq 1}$ and $(\theta_m)_{m\geq 1}$, that to avoid notation we denote again by $(\mathbf{u}_m)_{m\geq 1}$ and $(\theta_m)_{m\geq 1}$, such that

$$\mathbf{u}_m \xrightarrow[m\to\infty]{} \mathbf{u} \quad in \quad L^\infty(J,\mathcal{V}) \quad weak -^* \tag{86}$$

$$\dot{\mathbf{u}}_m \xrightarrow[m\to\infty]{} \dot{\mathbf{u}} \quad in \quad L^\infty(J,\mathcal{V}) \quad weak -^* \tag{87}$$

$$\ddot{\mathbf{u}}_m \xrightarrow[m\to\infty]{} \ddot{\mathbf{u}} \quad in \quad L^\infty(J,\mathcal{V}) \quad weak -^* \tag{88}$$

$$\theta_m \xrightarrow[m\to\infty]{} \theta \quad in \quad L^\infty(J,H^1(\Omega)) \quad weak -^* \tag{89}$$

$$\dot{\theta}_m \xrightarrow[m\to\infty]{} \dot{\theta} \quad in \quad L^\infty(J,H^1(\Omega)) \quad weak-^*, \tag{90}$$

$$\ddot{\theta}_m \xrightarrow[m\to\infty]{} \ddot{\theta} \quad in \quad L^\infty(J,L^2(\Omega)) \quad weak -^*. \tag{91}$$

Thus,

$$\int_0^T [\mathbf{v}, \mathbf{u}_m](t)dt \xrightarrow[m\to\infty]{} \int_0^T [\mathbf{v}, \mathbf{u}](t)dt \quad \forall \mathbf{v} \in L^1(J,\mathcal{V}'), \tag{92}$$

and similarly for (87)-(91).

Next, we observe that for each $\mathbf{v} \in S_m^{\mathbf{u}}$ and $g(t) \in L^1(J)$, $\mathcal{P}\mathbf{v}g(t) \in L^1(J,[L^2(\Omega)]^4)$ and consequently from (88)

$$\int_0^T \left(\mathcal{P}\ddot{\mathbf{u}}_m, \mathbf{v}\right) g(t)dt \xrightarrow[m\to\infty]{} \int_0^T \left(\mathcal{P}\ddot{\mathbf{u}}, \mathbf{v}\right) g(t)dt. \tag{93}$$

Hence,

$$\left(\mathcal{P}\ddot{\mathbf{u}}_m, \mathbf{v}\right) \xrightarrow[m\to\infty]{} \left(\mathcal{P}\ddot{\mathbf{u}}, \mathbf{v}\right) \quad in \quad L^\infty(J) \quad weak -^*. \tag{94}$$

Similarly, from (86)-(91), it follows that for each $\mathbf{v} \in S_m^{\mathbf{u}}, w \in S_m^\theta$, all other inner product terms in the left-hand side of (29) converge in $L^\infty(J) \quad weak-^*$ when $m \to \infty$.

Thus taking limit in m in (29), we see that

$$\left(\mathcal{P}\ddot{\mathbf{u}}(x), \mathbf{v}\right) + \left(\frac{\eta}{\kappa}\dot{\mathbf{u}}^f, \mathbf{v}^f\right) + \Lambda(\mathbf{u}, \mathbf{v}) - (\beta\theta, \nabla\cdot\mathbf{v}^s) - \left(\beta_f\theta, \nabla\cdot\mathbf{v}^f\right) + \left(\tau\, c\, \ddot{\theta}, w\right) + \left(c\, \dot{\theta}, w\right) + \left(\gamma\nabla\theta, \nabla w\right) \tag{95}$$

$$
\begin{aligned}
&+\left((1-\phi)\beta_m T_0 \nabla\cdot\dot{\mathbf{u}}^s, w\right)+\left(\phi\beta_f T_0\nabla\cdot\dot{\mathbf{u}}^f, w\right)\\
&+\left(\tau(1-\phi)\beta_m T_0 \nabla\cdot\ddot{\mathbf{u}}^s, w\right)+\left(\tau\phi\beta_f T_0\nabla\cdot\ddot{\mathbf{u}}^f, w\right)\\
&=(\mathbf{f},\mathbf{v})-(q,w)+\langle\mathbf{g},\mathbf{v}^s\rangle\\
&\quad+\left\langle\chi,\mathbf{v}^f\cdot\boldsymbol{\nu}\right\rangle+\langle h,w\rangle,\quad \mathbf{v}=(\mathbf{v}^s,\mathbf{v}^f,w)\in\mathcal{V}_m,\quad t\in J.
\end{aligned}
$$

The density of $\mathcal{V}_m$ in $[H^2(\Omega)]^5$ implies that (95) also holds for any $\mathbf{v}\in[H^2(\Omega)]^4, w\in H^2(\Omega)$. Then the density of $[C^\infty(\Omega)]^5$ in $[H^2(\Omega)]^5$ and in $\mathcal{V}\times H^1(\Omega)$ implies the validity of (95) in $\mathcal{V}\times H^1(\Omega)$.

Next, taking in (95) $\mathbf{v}=(\mathbf{v}^s,\mathbf{v}^f)\in[\mathcal{D}(\Omega)]^4, w\in\mathcal{D}(\Omega)$,

$$
\begin{aligned}
&\Lambda(\mathbf{u},\mathbf{v})-(\beta\theta,\nabla\cdot\mathbf{u}^s)-\left(\beta_f\theta,\nabla\cdot\mathbf{u}^f\right)=-\left(\mathcal{L}(\mathbf{u},\theta),\mathbf{v}\right), \qquad (96)\\
&(\gamma\nabla\theta,\nabla w)=-\left(\nabla\cdot(\gamma\nabla\theta),w\right).
\end{aligned}
$$

Thus, equation (95) reduces to

$$
\begin{aligned}
&(\mathcal{P}\ddot{\mathbf{u}}(x),\mathbf{v})+\left(\frac{\eta}{\kappa}\dot{\mathbf{u}}^f,\mathbf{v}^f\right)-(\mathcal{L}(\mathbf{u},\theta),\mathbf{v}) \qquad (97)\\
&+\left(\tau\, c\,\ddot{\theta},w\right)+\left(c\,\dot{\theta},w\right)-\left(\nabla\cdot(\gamma\nabla\theta),w\right)+\left((1-\phi)\beta_m T_0\nabla\cdot\dot{\mathbf{u}}^s,w\right)\\
&+\left(\phi\beta_f T_0\nabla\cdot\dot{\mathbf{u}}^f,w\right)+\left(\tau(1-\phi)\beta_m T_0\nabla\cdot\ddot{\mathbf{u}}^s,w\right)+\left(\tau\phi\beta_f T_0\nabla\cdot\ddot{\mathbf{u}}^f,w\right)\\
&\quad=(\mathbf{f},\mathbf{v})-(q,w),\quad \mathbf{v}=(\mathbf{v}^s,\mathbf{v}^f,w)\in[\mathcal{D}(\Omega)]^5,\quad t\in J.
\end{aligned}
$$

Hence

$$
\mathcal{P}\ddot{\mathbf{u}}+\frac{\eta}{\kappa}\dot{\mathbf{u}}^f-\mathcal{L}(\mathbf{u},\theta)=\mathbf{f},\quad \text{in}\quad [\mathcal{D}'(\Omega)]^4,\quad t\in J, \qquad (98)
$$

$$
\begin{aligned}
&\tau\, c\,\ddot{\theta}+c\,\dot{\theta}-\nabla\cdot(\gamma\nabla\theta)+(1-\phi)\beta_m T_0\nabla\cdot\dot{\mathbf{u}}^s+\phi\beta_f T_0\nabla\cdot\dot{\mathbf{u}}^f \qquad (99)\\
&+\tau(1-\phi)\beta_m T_0\nabla\cdot\ddot{\mathbf{u}}^s+\tau\phi\beta_f T_0\nabla\cdot\ddot{\mathbf{u}}^f=-q\quad\text{in}\quad\mathcal{D}'(\Omega),\quad t\in J.
\end{aligned}
$$

Since $\mathbf{f}\in[L^2(\Omega)]^4$ and by (88) $\ddot{\mathbf{u}}\in\mathcal{V},\dot{\mathbf{u}}^f\in H(\mathrm{div};\Omega)$, then $\mathcal{L}(\mathbf{u},\theta)\in[L^2(\Omega)]^4$ so that (98) also hold as functions in $[L^2(\Omega)]^4$.

Also $q\in L^2(J,L^2(\Omega))$ and by (87), (88), (90) and (91) $\dot{\theta}\in L^\infty(J,(H^1(\Omega)),\ \ddot{\theta}\in L^\infty(J,L^2(\Omega))$ and $\nabla\cdot\dot{\mathbf{u}}^s,\nabla\cdot\ddot{\mathbf{u}}^s,\nabla\cdot\dot{\mathbf{u}}^f,\nabla\cdot\ddot{\mathbf{u}}^f\in L^\infty(J,L^2(\Omega))$, respectively. Thus, $\nabla\cdot(\gamma\nabla\theta)\in L^2(J,L^2(\Omega))$ so that (99) also hold in $L^2(\Omega)$.

To verify that $(\mathbf{u},\theta)$ satisfy the boundary conditions, we observe that since $\mathcal{L}(\mathbf{u},\theta)\in[L^2(\Omega)]^4$, $\nabla\cdot(\gamma\nabla\theta)\in L^2(\Omega))$, we can integrate by parts in (95) to get

$$
\begin{aligned}
&(\mathcal{P}\ddot{\mathbf{u}},\mathbf{v})+\left(\frac{\eta}{\kappa}\dot{\mathbf{u}}^f,\mathbf{v}^f\right)-\mathcal{L}(\mathbf{u},\theta)+\langle\boldsymbol{\sigma}\cdot\boldsymbol{\nu},\mathbf{v}^s\rangle-\left\langle p_f,\mathbf{v}^f\cdot\boldsymbol{\nu}\right\rangle+\langle\gamma\nabla\theta\cdot\boldsymbol{\nu},w\rangle\\
&+\left((1-\phi)\beta_m T_0\nabla\cdot\dot{\mathbf{u}}^s,w\right)+\left(\phi\beta_f T_0\nabla\cdot\dot{\mathbf{u}}^f,w\right)\\
&+\left(\tau(1-\phi)\beta_m T_0\nabla\cdot\ddot{\mathbf{u}}^s,w\right)+\left(\tau\phi\beta_f T_0\nabla\cdot\ddot{\mathbf{u}}^f,w\right) \qquad (100)\\
&=(\mathbf{f},\mathbf{v})-(q,w)+\langle\mathbf{g},\mathbf{v}^s\rangle+\left\langle\chi,\mathbf{v}^f\cdot\boldsymbol{\nu}\right\rangle+\langle h,w\rangle,\\
&\quad\mathbf{v}=(\mathbf{v}^s,\mathbf{v}^f,w)\in\mathcal{V}\times H^1(\Omega),t\in J.
\end{aligned}
$$

Using (98) and (99) in (100) we obtain

$$
\begin{aligned}
&\langle\boldsymbol{\sigma}\cdot\boldsymbol{\nu},\mathbf{v}^s\rangle=\langle\mathbf{g},\mathbf{v}^s\rangle\quad \mathbf{v}^s\in[H^1(\Omega)]^2,t\in J, \qquad (101)\\
&-\left\langle p_f,\mathbf{v}^f\cdot\boldsymbol{\nu}\right\rangle=\left\langle\chi,\mathbf{v}^f\cdot\boldsymbol{\nu}\right\rangle\quad \mathbf{v}^f\in H(\mathrm{div};\Omega),t\in J, \qquad (102)\\
&\langle\gamma\nabla\theta\cdot\boldsymbol{\nu},w\rangle=\langle h,w\rangle,\quad w\in H^1(\Omega). \qquad (103)
\end{aligned}
$$

From (101) and (103), we see that the boundary conditions (17) and (19) are satisfied. Next, recall that for any $z \in H^{-1/2}(\Gamma)$ there exists $\mathbf{v}^f \in H(\mathrm{div};\Omega)$ such that $\mathbf{v}^f \cdot \boldsymbol{\nu} = z$ [11], and consequently (102) implies that the boundary condition (18) is satisfied.

To verify that the initial conditions are satisfied, we take a function $\varphi(t) \in C^\infty[0,T]$ such that $\varphi(0) = 1, \varphi(T) = 0$. From (87) we know that

$$\int_0^T (\dot{\mathbf{u}}_m, \mathbf{v})\,\varphi(t)dt \xrightarrow[m\to\infty]{} \int_0^T (\dot{\mathbf{u}}, \mathbf{v})\,\varphi(t)dt. \tag{104}$$

Also, since $(\dot{\mathbf{u}}_m, \mathbf{v})$ and $(\dot{\mathbf{u}}, \mathbf{v})$ are continuous in $[0,T]$, using integration by parts,

$$\begin{aligned}\int_0^T (\dot{\mathbf{u}}_m, \mathbf{v})\,\varphi(t)dt &= (\mathbf{u}_m, \mathbf{v})\,|_0^T - \int_0^T (\mathbf{u}_m, \mathbf{v})\,\varphi'(t)dt \\ &= -\,(\mathbf{u}_m(0), \mathbf{v}) - \int_0^T (\mathbf{u}_m, \mathbf{v})\,\varphi'(t)dt\end{aligned} \tag{105}$$

and

$$\int_0^T (\dot{\mathbf{u}}, \mathbf{v})\,\varphi(t)dt = -\left(\mathbf{u}^0, \mathbf{v}\right) - \int_0^T (\mathbf{u}, \mathbf{v})\,\varphi'(t)dt. \tag{106}$$

Taking limit in m in (105) we get

$$\int_0^T (\dot{\mathbf{u}}, \mathbf{v})\,\varphi(t)dt = -\mathrm{limit}_{m\to\infty}\,(\mathbf{u}_m(0), \mathbf{v}) - \int_0^T (\mathbf{u}, \mathbf{v})\,\varphi'(t)dt. \tag{107}$$

Since from (30) $\mathbf{u}_m(0) \xrightarrow[m\to\infty]{} \mathbf{u}^0$ strongly in $[L^2(\Omega)]^4$, from (105)-(107) and the uniqueness of the weak limit in $[L^2(\Omega)]^4$ we conclude that

$$\mathrm{limit}_{m\to\infty}\mathbf{u}_m(0) = \mathbf{u}^0. \tag{108}$$

A similar argument shows that the other initial conditions (14)-(16) are satisfied.

To demonstrate uniqueness, let $(\mathbf{u}_1, \theta_1), (\mathbf{u}_2, \theta_2)$ be two solutions of problem (11)-(19). Then $(\mathbf{u}, \theta) = (\mathbf{u}_1 - \mathbf{u}_2, \theta_1 - \theta_2)$ satisfy the equations

$$\mathcal{P}\ddot{\mathbf{u}} + \mathcal{B}\dot{u}^f - \mathcal{L}(\mathbf{u}, \theta)) = 0, \quad [L^2(\Omega)]^4 \times J, \tag{109}$$

$$\begin{aligned}&\tau\, c\,\ddot{\theta} + c\,\dot{\theta} - \nabla\cdot(\gamma\nabla\theta) + (1-\phi)\beta_m T_0 \nabla\cdot\dot{\mathbf{u}}^s + \phi\beta_f T_0\nabla\cdot\dot{\mathbf{u}}^f \\ &+\tau(1-\phi)\beta_m T_o\nabla\cdot\ddot{\mathbf{u}}^s + \tau\phi\beta_f T_o\nabla\cdot\ddot{\mathbf{u}}^f = 0, \quad L^2(\Omega)\times J,\end{aligned} \tag{110}$$

with vanishing initial and boundary conditions. Then a repetition of the argument leading to (95) shows that

$$\begin{aligned}&(\mathcal{P}\ddot{\mathbf{u}}(x), \mathbf{v}) + \left(\frac{\eta}{\kappa}\dot{\mathbf{u}}^f, \mathbf{v}^f\right) + \Lambda(\mathbf{u}, \mathbf{v}) - (\beta\theta, \nabla\cdot\mathbf{v}^s) - \left(\beta_f\theta, \nabla\cdot\mathbf{v}^f\right) \\ &+ \left(\tau\, c\,\ddot{\theta}, w\right) + \left(c\,\dot{\theta}, w\right) + (\gamma\nabla\theta, \nabla w)\end{aligned} \tag{111}$$

$$
\begin{aligned}
&+((1-\phi)\beta_m T_0 \nabla \cdot \dot{\mathbf{u}}^s, w) + \left(\phi\beta_f T_0 \nabla \cdot \dot{\mathbf{u}}^f, w\right) \\
&+(\tau(1-\phi)\beta_m T_0 \nabla \cdot \ddot{\mathbf{u}}^s, w) + \left(\tau\phi\beta_f T_0 \nabla \cdot \ddot{\mathbf{u}}^f, w\right) = 0 \\
&\quad \mathbf{v} = (\mathbf{v}^s, \mathbf{v}^f, w) \in \mathcal{V} \times H^1(\Omega), \quad t \in J.
\end{aligned}
$$

Next, choose $\mathbf{v} = \dot{\mathbf{u}}, w = \dot{\theta}$ in (111) and follow the ideas leading to (60) to obtain the inequality

$$
\begin{aligned}
&\|\mathbf{u}(t)\|^2_{L^\infty(J,\mathcal{V})} + \|\dot{\mathbf{u}}\|^2_{L^\infty(J,\mathcal{V})} + \|\ddot{\mathbf{u}}\|^2_{L^\infty(J,[L^2(\Omega)]^4)} \\
&+\|\theta\|^2_{L^\infty(J,H^1(\Omega))} + \|\dot{\theta}\|^2_{L^\infty(J,L^2(\Omega))} \le 0,
\end{aligned} \tag{112}
$$

so that $\mathbf{u}(t) \equiv 0, \theta(t) \equiv 0$ and uniqueness is demonstrated.

This completes the proof.

Next we analyze the case of the model equations in [6], where the constitutive and dynamic equations are

$$
\sigma_{ij}(\mathbf{u},\theta) = 2\mu\,\varepsilon_{ij}(u^s) + \delta_{ij}(\lambda_u\,\nabla\cdot\mathbf{u}^s + B\nabla\cdot\mathbf{u}^f - \beta\,\theta), \tag{113}
$$

$$
p_f(\mathbf{u},\theta) = -B\,\nabla\cdot\mathbf{u}^s - M\nabla\cdot\mathbf{u}^f + \beta_f\theta, \tag{114}
$$

$$
\mathcal{P}\ddot{\mathbf{u}} + \mathcal{B}\dot{\mathbf{u}}^f - \mathcal{L}(\mathbf{u},\theta)) = \mathbf{f}, \quad (x,t) \in \Omega \times J, \tag{115}
$$

$$
\begin{aligned}
&\tau\ c\ \ddot{\theta} + c\ \dot{\theta} - \nabla\cdot(\gamma\nabla\theta) + \beta T_0\left(\nabla\cdot\dot{\mathbf{u}}^s + \nabla\cdot\dot{\mathbf{u}}^f\right) \\
&+\tau\beta T_o\left(\nabla\cdot\ddot{\mathbf{u}}^s + \nabla\cdot\ddot{\mathbf{u}}^f\right) = -q \quad (x,t) \in \Omega \times J.
\end{aligned} \tag{116}
$$

The proof of the existence and uniqueness of the solution $(\mathbf{u},\theta)$ of an IBVP for (115)-(116) with the initial conditions (13)-(15) and the boundary conditions (17)-(19) follows with minor changes in the arguments given in the proof Theorem 1. In fact: Note that (116) can be obtained by changing $(1-\phi)\beta_m$ and $\phi\beta_f$ by β in (7). Then an inspection of the argument given in the proof of Theorem 1 shows that for this problem $C^m_\beta = 1, C^f_\beta = \frac{\beta}{\beta_f}$, so that and existence and uniqueness for an IBVP for (115)-(116) can be demonstrated with identical argument of that of Theorem 1 using these new definitions of the coefficients C^m_β and C^f_β.

4. Thermoelasticity

We consider the equations in [14] and [7] in an open bounded domain Ω with piecewise smooth boundary. With $\widehat{\boldsymbol{\sigma}} = (\widehat{\sigma}_{ij})$ denoting the stress tensor in a linear isotropic thermoelastic medium, the constitutive equations are

$$
\widehat{\sigma}_{ij}(\mathbf{u}^s,\theta) = 2\mu\,\varepsilon_{ij}(\mathbf{u}^s) + \delta_{ij}\left(\lambda\nabla\cdot\mathbf{u}^s - \beta\,\theta\right). \tag{117}
$$

The IBVP can be formulated as follows. Find $(\mathbf{u}^s,\theta)$ such that

$$
\rho\ddot{\mathbf{u}}^s - \nabla\cdot\widehat{\boldsymbol{\sigma}}(\mathbf{u}^s) = \mathbf{f}^s, \quad (x,t) \in \Omega \times J, \tag{118}
$$

$$
\begin{aligned}
&\tau\,c\,\ddot{\theta} + c\,\dot{\theta} - \nabla\cdot(\gamma\nabla\theta) + \beta T_0\nabla\cdot\dot{\mathbf{u}}^s \\
&\quad +\tau\beta T_o\nabla\cdot\ddot{\mathbf{u}}^s = -q \quad (x,t) \in \Omega \times J,
\end{aligned} \tag{119}
$$

$$
\mathbf{u}^s(x,0) = \mathbf{u}^{s,0},\ x \in \Omega, \tag{120}
$$

$$
\dot{\mathbf{u}}^s(x,0) = \mathbf{u}^{s,1},\ x \in \Omega, \tag{121}
$$

$$
\theta(x,0) = \theta^0,\ x \in \Omega, \tag{122}
$$

$$\dot{\theta}(x,0) = \theta^1 \;\; x \in \Omega, \tag{123}$$

$$\widehat{\boldsymbol{\sigma}}(\mathbf{u}^s,\theta)\cdot\boldsymbol{\nu} = \mathbf{g}(x,t), \quad x \in \Gamma, t \in J, \tag{124}$$

$$\gamma\nabla\theta\cdot\boldsymbol{\nu} = h(x,t), \quad x \in \Gamma, t \in J. \tag{125}$$

The IBVP for thermoelasticity formulated in (118)-(125) can be obtained by omitting the fluid terms and setting $\phi = 0, \beta_m = \beta_f \equiv \beta$ in equations (1) and (7). Thus the argument leading to (23) yields the following variational formulation for (118)-(125): Find $(\mathbf{u}^s,\theta)$ such that

$$\begin{aligned}
&(\mathcal{P}\ddot{\mathbf{u}}^s,\mathbf{v}) + \Lambda(\mathbf{u}^s,\mathbf{v}) - (\beta\theta,\nabla\cdot\mathbf{u}^s) + \left(\tau\, c\, \ddot{\theta}, w\right) \\
&+ \left(c\,\dot{\theta}, w\right) + (\gamma\nabla\theta,\nabla w) + (\beta T_0\nabla\cdot\dot{\mathbf{u}}^s, w) + (\tau\beta T_0\nabla\cdot\ddot{\mathbf{u}}^s, w) \\
&= (\mathbf{f}^s,\mathbf{v}) - (q,w) + \langle\mathbf{g},\mathbf{v}\rangle + \langle h,w\rangle\,, \quad (\mathbf{v},w) \in \times[H^1(\Omega)]^3,\; t \in J,
\end{aligned} \tag{126}$$

where $\Lambda(\mathbf{u}^s,\mathbf{v})$ is the bilinear form

$$\Lambda(\mathbf{u}^s,\mathbf{v}) = \sum_{l,m}(\widehat{\sigma}_{lm}(\mathbf{u}^s),\varepsilon_{lm}(\mathbf{v}^s)) = \left(\widehat{\mathcal{E}}\;\widetilde{\epsilon}(\mathbf{u}^s),\widetilde{\epsilon}(\mathbf{v})\right). \tag{127}$$

In (127), the matrix $\widehat{\mathcal{E}}$ and the column vector $\widetilde{\varepsilon}((\mathbf{u}^s))$ are defined by

$$\widehat{\mathcal{E}} = \begin{pmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & 4\mu \end{pmatrix}, \;\; \widetilde{\varepsilon}(\mathbf{u}^s) = \begin{pmatrix} \varepsilon_{11}(\mathbf{u}^s) \\ \varepsilon_{22}(\mathbf{u}^s) \\ \varepsilon_{12}(\mathbf{u}^s) \end{pmatrix}. \tag{128}$$

The term $\left(\widehat{\mathcal{E}}\;\widetilde{\epsilon}(\mathbf{u}),\widetilde{\epsilon}(\mathbf{u})\right)$ in (127) represents the strain energy of the system, so that $\widehat{\mathcal{E}}$ must be positive definite. Furthermore,

$$\Lambda(\mathbf{u}^s,\mathbf{v}) \le C\|\mathbf{u}\|_1\|\mathbf{v}\|_1. \tag{129}$$

Also, using Korn's second inequality (22), if $\lambda_*^{\widehat{\mathcal{E}}}$ is the minimum eigenvalue of $\widehat{\mathcal{E}}$,

$$\Lambda(\mathbf{v},\mathbf{v}) \ge C_2\|\mathbf{v}\|_1^2 - \lambda_*^{\widehat{\mathcal{E}}}\|\mathbf{v}\|_0^2. \tag{130}$$

Next, using (129)-(130) and ignoring all fluid terms, the argument used in the proof of Theorem 1 can be applied to the thermoelasticity case to demonstrate the following existence and uniqueness Theorem for (117)-(125): Let

$$\begin{aligned}
&M_1^2\left(\mathbf{f}^s,\mathbf{g},q,h,\right) = \|\mathbf{f}^s\|^2_{L^2(J,L^2(\Omega)]^2)} + \|\dot{\mathbf{f}^s}\|^2_{L^2(J,L^2(\Omega)]^2)} + \|\ddot{\mathbf{f}^s}\|^2_{L^2(J,L^2(\Omega)]^2)} \\
&\|\mathbf{g}\|^2_{L^2(J,[H^1(\Omega)]^2)} + +\|\dot{\mathbf{g}}\|^2_{L^\infty(J,[H^1(\Omega)]^2)} + \|\ddot{\mathbf{g}}\|^2_{L^\infty(J,[H^1(\Omega)]^2)} + \|\dddot{\mathbf{g}}\|^2_{L^2(J,[H^1(\Omega)]^2)} \\
&+\|h\|^2_{L^\infty(J,H^1(\Omega))} + \|\dot{h}\|^2_{L^\infty(J,H^1(\Omega))} + \|\ddot{h}\|^2_{L^2(J,H^1(\Omega))}, \\
&N_1^2 = \|\mathbf{u}^{s,0}\|_2^2 + \|\mathbf{u}^{s,1}\|_1^2 + \|\theta^0\|_2^2 + \|\theta^1\|_1^2 + \|\mathbf{f}(0)\|_0^2 + \|\dot{\mathbf{f}^s}(0)\|_0^2 + \|\ddot{\mathbf{f}^s}(0)\|_0^2 \\
&\qquad +\|\dot{\mathbf{g}}(0)\|^2_{1/2,\Gamma} + \|\dot{\boldsymbol{\chi}}(0)\|^2_{1/2,\Gamma} + \|\dot{h}(0)\|^2_{1/2,\Gamma} \\
&\qquad +\|\ddot{\mathbf{g}}(0)\|^2_{1/2,\Gamma} + \|\ddot{\boldsymbol{\chi}}(0)\|^2_{1/2,\Gamma} + \|\ddot{h}(0)\|^2_{1/2,\Gamma} + 1.
\end{aligned}$$

Theorem 2. *Let $\Omega \subset \mathbf{R}^2$ be an open bounded domain with piecewise smooth boundary. Assume that all coefficients in (117) and (119) are in $C^0_B(\Omega)$ with gradients belonging to $[L^\infty(\Omega)]^2$ and that the matrix $\widehat{\mathcal{E}}$ in (128) is positive definite. Also assume that $M_1(\mathbf{f},\mathbf{g},q,h,\chi) < \infty$, $N_1 < \infty$. Then, there exists a unique solution $(\mathbf{u}^s,\theta)$ of problem (118)-(125) such that $\mathbf{u}^s,\dot{\mathbf{u}}^s,\ddot{\mathbf{u}}^s \in L^\infty(J,[H^1(\Omega)]^2)$, $\dddot{\mathbf{u}}^s \in L^\infty(J,[L^2(\Omega)]^2$, $\theta,\dot{\theta} \in L^\infty(J,H^1(\Omega)),\ddot{\theta} \in L^\infty(J,L^2(\Omega))$.*

5. Conclusions

The differential equations of thermo-poroelasticity combine Biot's equations of motion in fluid-saturated porous media with a new heat equation with relaxation times to avoid infinite signal speeds. In this approach, the heat equation contains coupling terms consisting of first and second order time derivatives of the dilatations of the solid and fluid phases weighted by the corresponding thermal expansions of the phases. Under appropriate assumptions on the coefficients, the initial conditions and the open bounded domain where the solution is searched, an argument is given to ensure that these additional terms still allow the existence and uniqueness of the solution of the initial boundary value problem. The existence and uniqueness of the solutions of the corresponding problem for a previous thermo-poroelasticity formulation and single-phase thermoelasticity are shown to follow immediately from the novel formulation.

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