Comput. Methods Appl. Mech. Engrg. 200 (2011) 896–904

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

Viscoelastic-stiffness tensor of anisotropic media from oscillatory numerical experiments

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A R T I C L E   I N F O

Article history:
Received 26 April 2010
Received in revised form 3 September 2010
Accepted 11 November 2010
Available online 19 November 2010

Keywords:
Finite elements
Viscoelasticity
Anisotropy
Stiffness tensor
Oscillatory numerical experiments

A B S T R A C T

A finely layered media behaves as an anisotropic medium when the dominat wavelengths are much larger than the layer thickness. If the constituent are anelastic, a generalization of Backus averaging predicts that the medium is effectively a transversely isotropic viscoelastic (TIV) medium. To test and validate the theory, we present a novel procedure to determine the complex and frequency-dependent stiffness components of a TIV medium. The methodology consists in performing numerical compressibility and shear harmonic tests on a representative sample of the material. These tests are described by a collection of non-coercive elliptic boundary-value problems formulated in the space-frequency domain, which are solved using a Galerkin finite-element procedure. Results on the existence and uniqueness of the continuous and discrete problems as well as optimal error estimates for the Galerkin finite-element method are derived. Numerical examples illustrates the implementation of the numerical oscillatory tests to determine the set of complex and frequency-dependent effective TIV coefficients and the associated phase velocities and quality factors for a periodic sequence of epoxy and glass thin layers. The results are compared to the analytical phase velocities and quality factors predicted by the Backus/Carcione theory.

1. Introduction

Many geological systems can be modeled as effective transversely isotropic and viscoelastic (TIV) media. Fine layering is a typical example which refers to the case when the dominant wavelength of the traveling waves are much larger than the average thicknesses of the single layers. When this occurs, the medium is effectively anisotropic with a TI symmetry. Backus [1] obtained the average elastic constants in the case when the single layers are transversely isotropic and elastic (lossless), with the symmetry axis perpendicular to the layering plane. Moreover, he assumed stationarity, i.e., in a given length of composite medium much smaller than the wavelength, the proportion of each material is constant (periodicity is not required). The equations were further generalized by Schoenberg and Muir [2] for anisotropic single constituents. Backus averaging was verified numerically by Carcione et al. [3], and generalized to the anelastic case by Carcione [4], in what constitutes the Backus/Carcione (BC) theory to describe anisotropic attenuation [5].

To test and validate the BC theory, this paper presents a novel approach to determine the complex coefficients defining a TIV medium. In particular, we consider the TIV equivalent medium to a finely layered material. The methodology consists in applying time-harmonic oscillatory tests to a numerical rock sample at a finite number of frequencies. Each test is defined using the viscoelastic wave equation of motion stated in the space-frequency domain, with appropriate boundary conditions, and solved with a finite-element method (FEM). These tests can be regarded as an upscaling method to obtain the effect of the fine layering scale on the macroscale. A similar approach was presented in [6] for isotropic fluid-saturated poroelastic media and it is generalized here for anisotropic media.

There exists an extensive literature on effective medium theories for media having two length scales in space, one small related to the microstructure and the other large and related to the shortest wavelength of the response to a given excitation. In [7], this problem is analyzed for the case of wave propagation in periodic composites using the Bloch expansion and homogenization techniques formulated in the space–time domain, obtaining an effective dispersive medium. The analysis yields and homogenized wave equation with and anisotropic effective bulk modulus that coincides in the limit with the Bloch expansion. These results are related to those presented in this paper in the sense that, by properly selecting a representative element of volume we have

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doi:10.1016/j.cma.2010.11.008

determined effective frequency dependent coefficients defining a TIV (dispersive) medium equivalent to the original finely layered material.

These upscaling techniques have also been successfully applied in problems related to flow in highly heterogeneous porous media. The case of steady flow in porous media with many spatial scales was studied in [8] using a multiscale finite element method; the relation between the multiscale method and the homogenized solution of the problem is also analyzed. The basic idea of multiscale finite element methods is to incorporate the small scale information into finite element basis functions and couple them through a global formulation of the problem. Flow in naturally fracture reservoirs also feature multiple scales, and upscaling is necessary for numerical simulations at the field scale. This problems is analyzed in [9] using an homogenized Stokes–Brinkman equation with an effective permeability tensor obtained solving a cell problem on a representative element of volume. This approach is based on two-scale asymptotics expansion as described in [10].

The organization of the paper is as follows. Section 2 presents the finely-layered model and describes the Backus averaging technique [1]. In Section 3 we define the local boundary-value problem to determine the complex and frequency dependent coefficients defining the transversely isotropic medium. Section 4 presents a variational formulation of the boundary-value problem as well as the existence and uniqueness of the corresponding solutions. In Section 5, the FEM to solve the boundary value problems are formulated and optimal error estimates are derived. Section 6 presents numerical experiments applying the proposed methodology to compute the phase velocities and quality factors to a periodic sequence of epoxy and glass layers. The very good agreement between the numerical and analytical effective TIV coefficients and the corresponding phase velocities and quality factors provides a novel tool to validate the BC theory for the anelastic case.

2. The stress–strain relations

Let us consider wave propagation in a TIV medium. Let the Fourier transform in the time variable be defined as usual by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt, \tag{2.1}$$

where ω denotes the angular frequency. Let $\mathbf{x} = (x_1, x_2, x_3)$ and $u(\mathbf{x}) = (u_1, u_2, u_3)$ denote the time Fourier transform of the displacement vector of the viscoelastic medium. Here and in what follows we omit the $\hat{}$ symbol in the time-Fourier transformed variables to simplify the notation. Let σ_{ij} and $\epsilon_{ij}(u)$ denote the time Fourier transform of the stress and strain tensors of the viscoelastic material. The frequency-domain stress–strain relations of a general anisotropic medium, including attenuation, are:

$$\sigma_{jk}(u) = p_{jklm}(\omega)\epsilon_{lm}(u), \quad \epsilon_{lm}(u) = \frac{1}{2}\left(\frac{\partial u_l}{\partial x_m} + \frac{\partial u_m}{\partial u_l}\right), \tag{2.2}$$

where the coefficients $p_{jklm}(\omega)$ are complex and frequency dependent [5].

When the medium is composed of a sequence of isotropic linear viscoelastic horizontal layers Ω_n, $n = 1, \ldots, N$ the stress–strain relations on each Ω_n are:

$$\sigma_{jk}(u) = \lambda_n \delta_{jk} \nabla \cdot u + 2\mu_n \epsilon_{jk}(u), \tag{2.3}$$

where δ_{jk} is the Kroenecker delta and $\lambda_n(\omega)$ and $\mu_n(\omega)$ are the complex Lamé coefficients for the n-layer.

Let $\rho = \rho(\mathbf{x})$ be the mass density. The equation of motion is

$$-\omega^2 \rho u(\mathbf{x}, \omega) - \nabla \cdot \sigma(u(\mathbf{x}, \omega)) = 0, \tag{2.4}$$

where σ is given by (2.2) for a general medium and by (2.3) in the isotropic and viscoelastic case represented by the thin layers.

Let us consider x_1 and x_3 as the horizontal and vertical coordinates, respectively. As shown by Backus for the lossless case [1] and later generalized by Carcione for the anelastic case [4], the medium behaves as a homogeneous TIV medium with vertical x_3-axis of symmetry at long wavelengths. Denoting by τ_{ij} the stress tensor of the equivalent TIV medium, the corresponding stress–strain relations, stated in the space-frequency domain, are [4]:

$$\tau_{11}(u) = p_{11}\epsilon_{11}(u) + p_{12}\epsilon_{22}(u) + p_{13}\epsilon_{33}(u), \tag{2.5}$$
$$\tau_{22}(u) = p_{12}\epsilon_{11}(u) + p_{11}\epsilon_{22}(u) + p_{13}\epsilon_{33}(u), \tag{2.6}$$
$$\tau_{33}(u) = p_{13}\epsilon_{11}(u) + p_{13}\epsilon_{22}(u) + p_{33}\epsilon_{33}(u), \tag{2.7}$$
$$\tau_{23}(u) = 2p_{55}\epsilon_{23}(u), \tag{2.8}$$
$$\tau_{13}(u) = 2p_{55}\epsilon_{13}(u), \tag{2.9}$$
$$\tau_{12}(u) = 2p_{66}\epsilon_{12}(u), \tag{2.10}$$

with

$$p_{11} = \left\langle E - \lambda^2 E^{-1} \right\rangle + \left\langle E^{-1} \right\rangle^{-1} \left\langle E^{-1}\lambda \right\rangle^2, \tag{2.11a}$$
$$p_{12} = \left\langle 2\lambda\mu E^{-1} \right\rangle + \left\langle E^{-1} \right\rangle^{-1} \left\langle E^{-1}\lambda \right\rangle^2, \tag{2.11b}$$
$$p_{13} = \left\langle E^{-1} \right\rangle^{-1} \left\langle E^{-1}\lambda \right\rangle, \tag{2.11c}$$
$$p_{33} = \left\langle E^{-1} \right\rangle^{-1}, \tag{2.11d}$$
$$p_{55} = \langle \mu^{-1} \rangle^{-1}, \tag{2.11e}$$
$$p_{66} = \langle \mu \rangle, \tag{2.11f}$$

where λ and μ represent λ_n and μ_n, $E = \lambda + 2\mu$, and $\langle\cdot\rangle$denotes the thickness weighted average. The p_{IJ} are the complex and frequency-dependent Voigt stiffnesses to be determined with the harmonic experiments. The conversion between the Voigt stiffnesses and the stiffnesses of the 4th-rank tensors is

$$\begin{aligned} p_{IJ} &= p_{ijkl}, \\ I &= i\delta_{ij} + (1 - \delta_{ij})(9 - i - j), \\ J &= k\delta_{kl} + (1 - \delta_{kl})(9 - k - l). \end{aligned} \tag{2.12}$$

The phase velocities and quality factors for the quasi-compressional (qP), vertically-polarized quasi-shear (qSV) and horizontally-polarized shear (SH) waves can be computed using the complex velocities, that are given in terms of the coefficients in (2.11a) by the relations [11,5]:

$$\begin{aligned} v_{\mathrm{qP}} &= (2\rho)^{-1/2}\sqrt{p_{11}l_1^2 + p_{33}l_3^2 + p_{55} + A}, \\ v_{\mathrm{qSV}} &= (2\rho)^{-1/2}\sqrt{p_{11}l_1^2 + p_{33}l_3^2 + p_{55} - A}, \\ v_{\mathrm{SH}} &= \rho^{-1/2}\sqrt{p_{66}l_1^2 + p_{55}l_3^2}, \\ A &= \sqrt{\left[(p_{11} - p_{55})l_1^2 + (p_{55} - p_{33})l_3^2\right]^2 + 4\left[\left(p_{(13)} + p_{55}\right)l_1 l_3\right]^2}, \end{aligned} \tag{2.13}$$

where $l_1 = \sin\theta$ and $l_3 = \cos\theta$ are the directions cosines, θ is the propagation angle between the wavenumber vector and the x_3-symmetry axis. The corresponding phase velocity and quality factors for homogeneous waves are given by [5]:

$$v_{p,\zeta} = \left[\mathrm{Re}\left(\frac{1}{v_\zeta}\right)\right]^{-1}, \tag{2.14}$$

$$Q_\zeta = \frac{\mathrm{Re}(v_\zeta^2)}{\mathrm{Im}(v_\zeta^2)}, \quad \zeta = \mathrm{qP}, \mathrm{qSV}, \mathrm{SH}. \tag{2.15}$$

These equations hold for any TIV medium, in particular for the equivalent medium described by the BC theory. To test and validate this theory, we will present in the next section a novel numerical procedure to determine the coefficients in (2.11a) and the corresponding phase velocities and quality factors. We will show that for this purpose it is sufficient to perform a collection of oscillatory tests on representative 2D samples of the viscoelastic material.

3. Determination of the stiffneses

We show how that stiffnesses p_{11}, p_{33}, p_{13} and p_{55} can be determined by taking a 2D representative square sample $\Omega = (0,L)^2$ of the TIV material in the (x_1,x_3)-plane.

Set $\Gamma = \partial\Omega = \Gamma^L \cup \Gamma^B \cup \Gamma^R \cup \Gamma^T$, where

$$\Gamma^L = \{(x_1,x_3) \in \Gamma : x_1 = 0\}, \quad \Gamma^R = \{(x_1,x_3) \in \Gamma : x_1 = L\},$$
$$\Gamma^B = \{(x_1,x_3) \in \Gamma : x_3 = 0\}, \quad \Gamma^T = \{(x_1,x_3) \in \Gamma : x_3 = L\}.$$

Denote by ν the unit outer normal on Γ and let χ be a unit tangent on Γ so that $\{\nu,\chi\}$ is an orthonormal system on Γ.

To obtain the complex stiffness $p_{33}(\omega)$, let us consider the solution of the equation:

$$-\omega^2 \rho u(x_1,x_3,\omega) - \nabla\cdot\sigma(u(x_1,x_3,\omega)) = 0, \tag{3.1}$$

with boundary conditions:

$$\sigma(u)\nu\cdot\nu = -\Delta P, \quad (x_1,x_3) \in \Gamma^T, \tag{3.2}$$
$$\sigma(u)\nu\cdot\chi = 0, \quad (x_1,x_3) \in \Gamma^L \cup \Gamma^R \cup \Gamma^T, \tag{3.3}$$
$$u\cdot\nu = 0, \quad (x_1,x_3) \in \Gamma^L \cup \Gamma^R, \tag{3.4}$$
$$u = 0, \quad (x_1,x_3) \in \Gamma^B. \tag{3.5}$$

For this set of boundary conditions the solid is not allowed to move on the bottom boundary Γ^B, a uniform compression is applied on the boundary Γ^T (i.e., a uniform compression parallel to the symmetry axis) and no tangential external forces are applied on the boundaries $\Gamma^L \cup \Gamma^R \cup \Gamma^T$. Consequently, $\epsilon_{11} = \epsilon_{22} = 0$ and this experiment will yield the value of $p_{33}(\omega)$ in (2.7) as follows.

Denoting by V the original volume of the sample, its (complex) oscillatory volume change, $\Delta V(\omega)$, allows us to define the effective P-wave complex stiffness $p_{33}(\omega)$, by using the relation:

$$\frac{\Delta V(\omega)}{V} = -\frac{\Delta P}{p_{33}(\omega)}, \tag{3.6}$$

valid for a viscoelastic homogeneous medium in the quasi-static case, i.e., for wavelengths much larger than the size of the sample. After solving (3.1) with the boundary conditions (3.2)–(3.5), the vertical displacements $u_3(x_1,L,\omega)$ on Γ^T allow us to obtain an average vertical displacement $u_3^T(\omega)$ suffered by the boundary Γ^T. Then, for each frequency ω, the volume change produced by this compressibility test can be approximated by $\Delta V(\omega) \approx L u_3^T(\omega)$, which enable us to compute $p_{33}(\omega)$ by using the relation (3.6).

To determine the complex coefficient $p_{11}(\omega)$, we solve (3.1) with the following boundary conditions:

$$\sigma(u)\nu\cdot\nu = -\Delta P, \quad (x_1,x_3) \in \Gamma^R, \tag{3.7}$$
$$\sigma(u)\nu\cdot\chi = 0, \quad (x_1,x_3) \in \Gamma^R \cup \Gamma^B \cup \Gamma^T, \tag{3.8}$$
$$u\cdot\nu = 0, \quad (x_1,x_3) \in \Gamma^B \cup \Gamma^T, \tag{3.9}$$
$$u = 0, \quad (x_1,x_3) \in \Gamma^L. \tag{3.10}$$

In this oscillatory test, the solid is not allowed to move on the left boundary Γ^L, a uniform compression is applied on the boundary Γ^R (i.e., a uniform compression perpendicular to the symmetry axis) and no tangential external forces are applied on the boundaries $\Gamma^B \cup \Gamma^R \cup \Gamma^T$. Consequently, $\epsilon_{22} = \epsilon_{33} = 0$ and this experiment will yield the value of $p_{11}(\omega)$ in (2.5) by measuring the volume change of the sample as explained above for $p_{33}(\omega)$.

To obtain $p_{13}(\omega)$ we solve (3.1) with the boundary conditions:

$$\sigma(u)\nu\cdot\nu = -\Delta P, \quad (x_1,x_3) \in \Gamma^T \cup \Gamma^R, \tag{3.11}$$
$$\sigma(u)\nu\cdot\chi = 0, \quad (x_1,x_3) \in \Gamma, \tag{3.12}$$
$$u\cdot\nu = 0, \quad (x_1,x_3) \in \Gamma^L \cup \Gamma^B. \tag{3.13}$$

Thus, in this experiment $\epsilon_{22} = 0$, and from (2.5) and (2.7) we get:

$$\begin{aligned} \tau_{11} &= p_{11}\epsilon_{11} + p_{13}\epsilon_{33}, \\ \tau_{33} &= p_{13}\epsilon_{11} + p_{33}\epsilon_{33}, \end{aligned} \tag{3.14}$$

where ϵ_{11} and ϵ_{33} are the strain components at the right lateral side and top side of the sample, respectively. Then from (3.14) and the fact that $\tau_{11} = \tau_{33} = -\Delta P$ (c.f. (3.11)) we obtain $p_{13}(\omega)$ as

$$p_{13}(\omega) = \frac{p_{11}\epsilon_{11} - p_{33}\epsilon_{33}}{\epsilon_{11} - \epsilon_{33}}. \tag{3.15}$$

In order to compute $p_{55}(\omega)$, we perform an oscillatory shear test by solving (3.1) with the boundary conditions:

$$-\sigma(u)\nu = g, \quad (x_1,x_3) \in \Gamma^T \cup \Gamma^L \cup \Gamma^R, \tag{3.16}$$
$$u = 0, \quad (x_1,x_3) \in \Gamma^B, \tag{3.17}$$

where

$$g = \begin{cases} (0,\Delta p), & (x_1,x_3) \in \Gamma^L, \\ (0,-\Delta p), & (x_1,x_3) \in \Gamma^R, \\ (-\Delta p,0), & (x_1,x_3) \in \Gamma^T. \end{cases}$$

Since in this experiment there is no volume change, $\epsilon_{11} = \epsilon_{33} = 0$, this experiment yields $p_{55}(\omega)$ measuring the change in shape of the sample using the relation:

$$\tan[\beta(\omega)] = \frac{\Delta p}{p_{55}(\omega)}, \tag{3.18}$$

where $\beta(\omega)$ is the departure angle between the original positions of the lateral boundaries and those after applying the shear stresses (see for example [12]). Eq. (3.18) holds in the quasi-static approximation. The horizontal displacements u_1 (x_1,L,ω) at the top boundary Γ^T can then be used to obtain an average horizontal displacement $u_1^T(\omega)$ at the boundary Γ^T. Then, $\tan[\beta(\omega)] \approx u_1^T(\omega)/L$ and $p_{55}(\omega)$ can be calculated from Eq. (3.18).

Finally, we should obtain $p_{66}(\omega)$. Since this stiffness is associated with shear waves traveling in the (x_1,x_2)-plane, we consider an homogeneous horizontal slab in the x_2-direction and an homogeneous sample $\Omega_2 = (0,L)^2$ in the (x_1,x_2) plane, with boundary $\Gamma_2 = \Gamma_2^L \cup \Gamma_2^B \cup \Gamma_2^R \cup \Gamma_2^T$, where

$$\Gamma_2^L = \{(x_1,x_2) \in \Gamma : x_1 = 0\}, \quad \Gamma_2^R = \{(x_1,x_2) \in \Gamma : x_1 = L\},$$
$$\Gamma_2^B = \{(x_1,x_2) \in \Gamma : x_2 = 0\}, \quad \Gamma_2^T = \{(x_1,x_2) \in \Gamma : x_2 = L\}.$$

Thus, we solve the equation:

$$-\omega^2 \rho u(x_1,x_2,\omega) - \nabla\cdot\sigma(u(x_1,x_2,\omega)) = 0, \quad \Omega_2, \tag{3.19}$$

with the boundary conditions:

$$-\sigma(u)\nu = g_2, \quad (x_1,x_2) \in \Gamma_2^T \cup \Gamma_2^L \cup \Gamma_2^R, \tag{3.20}$$
$$u = 0, \quad (x_1,x_2) \in \Gamma_2^B, \tag{3.21}$$

where

$$g_2 = \begin{cases} (0,\Delta p), & (x_1,x_2) \in \Gamma_2^L, \\ (0,-\Delta p), & (x_1,x_2) \in \Gamma_2^R, \\ (-\Delta p,0), & (x_1,x_2) \in \Gamma_2^T. \end{cases}$$

This problem is formally identical to that described for $p_{55}(\omega)$, with no volume change, where the only non-zero strain is $\epsilon_{12}(u(x_1,x_2))$. We obtain $p_{66}(\omega)$ by using (2.10).

Remark. Since the formulated boundary-value problems have boundary data in $L^2(\Omega)$, the corresponding weak solutions belong to the space $[H^{3/2}(\Omega)]^2$ [13]. This maximal regularity will be used later to derive our error estimates.

4. The variational formulations

In order to state a variational formulation for the boundary-value problems defined in the previous section we need to introduce some notation. For $X \subset \mathbb{R}^d$ with boundary ∂X, let $(\cdot,\cdot)_X$ and $\langle\cdot,\cdot\rangle_{\partial X}$ denote the complex $L^2(X)$ and $L^2(\partial X)$ inner products for scalar, vector, or matrix valued functions. Also, for $s \in \mathbb{R}$, $\|\cdot\|_{s,X}$ will denote the usual norm for the Sobolev space $H^s(X)$ [14]. In addition, if $X = \Omega$ or $X = \Gamma$, the subscript X may be omitted such that $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$ or $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_\Gamma$. Also, let us introduce the following closed subspaces of $[H^1(\Omega)]^2$ and $[H^1(\Omega_2)]^2$:

$$\mathcal{W}_{11}(\Omega) = \left\{ v \in \left[H^1(\Omega)\right]^2 : v \cdot \nu = 0 \quad \text{on } \Gamma^B \cup \Gamma^T, \quad v = 0 \quad \text{on } \Gamma^L \right\},$$

$$\mathcal{W}_{33}(\Omega) = \left\{ v \in \left[H^1(\Omega)\right]^2 : v \cdot \nu = 0 \quad \text{on } \quad \Gamma^L \cup \Gamma^R, \quad v = 0 \quad \text{on } \Gamma^B \right\},$$

$$\mathcal{W}_{13}(\Omega) = \left\{ v \in \left[H^1(\Omega)\right]^2 : v \cdot \nu = 0 \quad \text{on } \Gamma^L \cup \Gamma^B \right\},$$

$$\mathcal{W}_{55}(\Omega) = \left\{ v \in \left[H^1(\Omega)\right]^2 : v = 0 \quad \text{on } \Gamma^B \right\},$$

$$\mathcal{W}_{66}(\Omega_2) = \left\{ v \in \left[H^1(\Omega_2)\right]^2 : v = 0 \quad \text{on } \Gamma_2^B \right\}.$$

Set:

$$\Lambda(u,v) = -\omega^2(u,v) + \sum_{l,m=1,3} (\sigma_{lm}(u), \varepsilon_{lm}(v)),$$
$$\forall u, v \in \left[H^1(\Omega)\right]^2. \tag{4.1}$$

Note that the term $\sum_{l,m=1,3}(\sigma_{lm}(u), \varepsilon_{lm}(v))$ in (4.1) can be written in the form:

$$\sum_{l,m=1,3} (\sigma_{lm}(u), \varepsilon_{lm}(v)) = (\mathbf{M}(\omega)\tilde{\epsilon}(u), \tilde{\epsilon}(v)) = (\mathbf{M}_R(\omega)\tilde{\epsilon}(u), \tilde{\epsilon}(v)) + i(\mathbf{M}_I(\omega)\tilde{\epsilon}(u), \tilde{\epsilon}(v)), \tag{4.2}$$

where $\mathbf{M}(\omega) = \mathbf{M}_R(\omega) + i\mathbf{M}_I(\omega)$ is a complex matrix and

$$\tilde{\epsilon}(u) = \begin{pmatrix} \varepsilon_{11}(u) \\ \varepsilon_{33}(u) \\ \varepsilon_{13}(u) \end{pmatrix}.$$

Furthermore, we assume that the real part $\mathbf{M}_R(\omega)$ is positive definite since in the elastic limit it is associated with the strain energy density. On the other hand, the imaginary part $\mathbf{M}_I(\omega)$ is assumed to be positive definite because of the restriction imposed on our system by the first and second laws of thermodynamics. See for example [15] and the appendix in [16] for the proof of the validity of these assumptions.

In the case that the medium is composed of viscoelastic isotropic horizontal layers, the matrix $\mathbf{M}(\omega)$ has the form:

$$\mathbf{M}(\omega) = \begin{pmatrix} \lambda(\omega) + 2\mu(\omega) & \lambda(\omega) & 0 \\ \lambda(\omega) & \lambda(\omega) + 2\mu(\omega) & 0 \\ 0 & 0 & 4\mu(\omega) \end{pmatrix}. \tag{4.3}$$

Next, multiply Eq. (3.1) by $v \in \mathcal{W}_{33}(\Omega)$, use integration by parts and apply the boundary conditions (3.2), (3.3) to obtain the following variational formulation associated with the coefficient $p_{33}(\omega)$: find $u^{(33)} \in \mathcal{W}_{33}(\Omega)$ such that:

$$\Lambda\left(u^{(33)}, v\right) = -\langle \Delta P, v \cdot \nu \rangle_{\Gamma^T}, \quad \forall v \in \mathcal{W}_{33}(\Omega). \tag{4.4}$$

Proceeding in a similar fashion, variational formulations to determine $p_{11}(\omega)$, $p_{55}(\omega)$ and $p_{66}(\omega)$ can be stated as follows: find $u^{(11)} \in \mathcal{W}_{11}(\Omega), u^{(}55) \in \mathcal{W}_{55}(\Omega)$ and $u^{(66)} \in \mathcal{W}_{66}(\Omega_2)$ satisfying:

$$\Lambda\left(u^{(11)}, v\right) = -\langle \Delta P, v \cdot \nu \rangle_{\Gamma^R}, \quad \forall v \in \mathcal{W}_{11}(\Omega), \tag{4.5}$$

$$\Lambda\left(u^{(55)}, v\right) = -\langle g, v \rangle_{\Gamma^L \cup \Gamma^R \cup \Gamma^T}, \quad \forall v \in \mathcal{W}_{55}(\Omega), \tag{4.6}$$

and

$$\Lambda\left(u^{(66)}, v\right) = -\langle g, v \rangle_{\Gamma_2^L \cup \Gamma_2^R \cup \Gamma_2^T}, \quad \forall v \in \mathcal{W}_{66}(\Omega_2). \tag{4.7}$$

To analyze the uniqueness of the solution $u^{(33)}$ of (4.4), set $\Delta P = 0$ and choose $v = u^{(33)}$ in (4.4) to obtain:

$$-\omega^2\left(\rho u^{(33)}, u^{(33)}\right) + \left(\mathbf{M}_R \tilde{\epsilon}\left(u^{(33)}\right), \tilde{\epsilon}\left(u^{(33)}\right)\right) + \left(\mathbf{M}_I \tilde{\epsilon}\left(u^{(33)}\right), \tilde{\epsilon}\left(u^{(33)}\right)\right) = 0. \tag{4.8}$$

Taking the imaginary part in (4.8) and using that $\mathbf{M}_I$ is positive definite we conclude that:

$$\tilde{\epsilon}(u^{(33)}) = 0, \quad \text{in } L^2(\Omega). \tag{4.9}$$

Next, recall Korn's second inequality [17]:

$$\sum_{l,m=1,3} \|\epsilon_{lm}(v)\|_0^2 + \|v\|_0^2 \geqslant C_1\|v\|_1^2, \quad \forall\ v \in \left[H^1(\Omega)\right]^2, \tag{4.10}$$

and that for any $v \in [H^1(\Omega)]^2$ vanishing on a subset of positive measure of Γ, using (4.10) it can be shown that [18]:

$$|||v||| = \left(\sum_{l,m=1,3} \|\epsilon_{lm}(v)\|_0^2\right)^{1/2}, \tag{4.11}$$

defines a norm for v equivalent to the H^1-norm. Thus, for some positive constants C_2, C_3:

$$C_2\|v\|_1 \leqslant |||v||| \leqslant C_3\|v\|_1, \quad \forall v \in \mathcal{W}_{33}(\Omega). \tag{4.12}$$

Consequently, (4.9) and (4.12) imply that:

$$\|u^{(33)}\|_1 = 0, \tag{4.13}$$

and we have uniqueness for the solution of (4.4).

To show existence, note that if $L_*(A)$ and $L^*(A)$ denote the minimum and maximum eigenvalues of the positive definite matrix A, using (4.12) we conclude that $\Lambda(v,v)$, satisfies the Garding inequality:

$$\mathrm{Re}(\Lambda(v,v)) \geqslant C_4(\omega)\|v\|_1^2 - C_5(\omega)\|v\|_0^2, \quad \forall\ v \in \mathcal{W}_{33}(\Omega), \tag{4.14}$$

where

$$C_4(\omega) = C_2 \frac{L_*(\mathbf{M}_R)(\omega)}{2}, \quad C_5(\omega) = \omega^2\rho^*,$$

and ρ^* denotes the maximum value of $\rho(x_1,x_3)$ in Ω. Since uniqueness holds for the solution of the adjoint problem to (4.4), existence follows from (4.14) applying the Fredholm alternative [19].

Existence and uniqueness for the solution of (4.5), (4.6) and (4.7) follows with the same argument than that given for (4.4). Thus we have the validity of the following theorem.

Theorem 1. *Assume that the matrices $\mathbf{M}_R(\omega)$ and $\mathbf{M}_I(\omega)$ are positive definite. Then existence and uniqueness holds for problems 4.4, 4.5, 4.6, 4.7.*

In order to obtain a weak formulation to determine the coefficient $p_{13}(\omega)$, multiply (3.1) by $v \in \mathcal{W}_{13}(\Omega)$, use integration by parts and apply the boundary conditions (3.11), (3.12) to get the following variational problem: find $u^{(13)} \in \mathcal{W}_{13}(\Omega)$ such that:

$$\Lambda\left(u^{(13)}, v\right) = -\langle \Delta P, v \cdot \nu \rangle_{\Gamma^T \cup \Gamma^R}, \quad \forall v \in \mathcal{W}_{13}(\Omega). \tag{4.15}$$

To analyze the uniqueness of the solution of (4.15), set $\Delta P = 0$ and choose $v = u^{(13)}$ in (4.15) to obtain the equation:

$$-\omega^2\left(\rho u^{(13)}, u^{(13)}\right) + \left(\mathbf{M}_R \tilde{\epsilon}(u^{(13)}), \tilde{\epsilon}(u^{(13)})\right) + i\left(\mathbf{M}_I \tilde{\epsilon}(u^{(13)}), \tilde{\epsilon}(u^{(13)})\right) = 0. \tag{4.16}$$

Taking the imaginary part in (4.16) and using that $\mathbf{M}_I$ is positive definite we conclude that:

$$\epsilon_{11}\left(u^{(13)}\right) = 0, \quad \text{in } L^2(\Omega), \tag{4.17}$$
$$\epsilon_{33}\left(u^{(13)}\right) = 0, \quad \text{in } L^2(\Omega), \tag{4.18}$$
$$\epsilon_{13}\left(u^{(13)}\right) = 0, \quad \text{in } L^2(\Omega). \tag{4.19}$$

In particular:

$$\epsilon_{11}\left(u^{(13)}\right) = \frac{\partial u_1^{(13)}(x_1, x_3)}{\partial x_1} = 0, \quad \epsilon_{33}\left(u^{(13)}\right) = \frac{\partial u_3^{(13)}(x_1, x_3)}{\partial x_3} = 0, \quad \text{a.e.} \quad \text{in } \Omega,$$

so that:

$$u_1^{(13)}(x_1, x_3) = f(x_3), \quad u_3^{(13)}(x_1, x_3) = g(x_1) \quad \text{a.e.} \quad \text{in } \Omega. \tag{4.20}$$

Thus from (4.19) and (4.20) have:

$$2\epsilon_{13}(u^{(13)}) = \frac{\partial f(x_3)}{\partial x_3} + \frac{\partial g(x_1)}{\partial x_1} = 0, \quad \text{a.e.} \quad \text{in } \Omega, \tag{4.21}$$

which in turn implies:

$$\frac{\partial f(x_3)}{\partial x_3} = -\frac{\partial g(x_1)}{\partial x_1} = C = \text{constant} \quad \text{a.e.} \quad \text{in } \Omega. \tag{4.22}$$

Hence:

$$g(x_1) = -Cx_1 + A, \quad f(x_3) = Cx_3 + B, \quad \text{a.e.} \quad \text{in } \Omega. \tag{4.23}$$

Next, by the Sobolev embedding theorem [14]:

$$H^{3/2}(\Omega) \to C^0(\overline{\Omega}), \tag{4.24}$$

so that $u_1^{(13)}, u_3^{(13)}$ are uniformly continuous functions on Ω. Consequently (4.20) holds for all $(x_1,x_3) \in \Omega$ as uniformly continuous functions, and $u_1^{(13)}, u_3^{(13)}$ have unique extensions to Γ. Hence:

$$u_1^{(13)}(x_1, x_3) = f(x_3), \quad u_3^{(13)}(x_1, x_3) = g(x_1) \quad \forall\ (x_1, x_3) \quad \text{in } \overline{\Omega}. \tag{4.25}$$

On the other hand, the boundary condition (3.13) tells us that the normal components of the traces of $u^{(13)}$ vanish on $\Gamma^B \cup \Gamma^L$, so that:

$$u_1^{(13)}(0, x_3) = 0, \quad u_3^{(13)}(x_1, 0) = 0. \tag{4.26}$$

Thus (4.25) and (4.26) imply that:

$$u_1^{(13)}(x_1, x_3) = u_3^{(13)}(x_1, x_3) = 0. \tag{4.27}$$

and we have uniqueness.

For existence, note that using (4.10) we get the Garding inequality:

$$\mathrm{Re}(\Lambda(v, v)) \geqslant C_6(\omega)\frac{L_*(\mathbf{M}_R)}{2}\|v\|_1^2 - C_7(\omega)\|v\|_0^2, \quad \forall\ v \in \mathcal{W}_{13}(\Omega), \tag{4.28}$$

where

$$C_6(\omega) = C_1\frac{L_*(\mathbf{M}_R)}{2} \quad C_7(\omega) = \omega^2\rho^* + \frac{L_*(\mathbf{M}_R)}{2}.$$

Since uniqueness holds for the dual problem of (3.1) with the boundary conditions (3.11)–(3.13), the Fredholm alternative yields uniqueness. The result is summarized in the following theorem :

Theorem 2. *Assume that the matrices $\mathbf{M}_R(\omega)$ and $\mathbf{M}_I(\omega)$ are positive definite. Then there exists a unique solution of problems (4.15).*

Remark. The existence and uniqueness results in Theorems 1 and 2 are valid for general TIV media, since the proofs use the positive definiteness of the matrices $\mathbf{M}_R$ and $\mathbf{M}_I$, a property valid for this type of materials.

5. The finite element procedure

Let $\mathcal{T}^h(\Omega)$ and $\mathcal{T}^h(\Omega_2)$ be non-overlapping partitions of Ω and Ω_2, respectively, into rectangles Ω_j and $\Omega_{2,j}$ of diameter bounded by h such that $\overline{\Omega} = \cup_{j=1}^J \overline{\Omega}_j$, $\overline{\Omega}_2 = \cup_{j=1}^J \overline{\Omega}_{j,2}$. Let us introduce the following finite element spaces:

$$\mathcal{W}_{11}^h(\Omega) = \left\{v : v|_{\Omega_j} \in P_{1,1} \times P_{1,1}, v \cdot \nu = 0 \quad \text{on } \Gamma^B \cup \Gamma^T, v = 0 \quad \text{on } \Gamma^L\right\} \cap \left[C^0(\overline{\Omega})\right]^2,$$

$$\mathcal{W}_{33}^h(\Omega) = \left\{v : v|_{\Omega_j} \in P_{1,1} \times P_{1,1}, v \cdot \nu = 0 \quad \text{on } \Gamma^L \cup \Gamma^R, v = 0 \quad \text{on } \Gamma^B\right\} \cap \left[C^0(\overline{\Omega})\right]^2,$$

$$\mathcal{W}_{13}^h(\Omega) = \left\{v : v|_{\Omega_j} \in P_{1,1} \times P_{1,1}, v \cdot \nu = 0 \quad \text{on } \Gamma^L \cup \Gamma^B\right\} \cap \left[C^0(\overline{\Omega})\right]^2,$$

$$\mathcal{W}_{55}^h(\Omega) = \left\{v : v|_{\Omega_j} \in P_{1,1} \times P_{1,1}, v = 0 \quad \text{on } \Gamma^B\right\} \cap \left[C^0(\overline{\Omega})\right]^2,$$

$$\mathcal{W}_{66}^h(\Omega_2) = \left\{v : v|_{\Omega_{j,2}} \in P_{1,1} \times P_{1,1}, v = 0 \quad \text{on } \Gamma_2^B\right\} \cap \left[C^0(\overline{\Omega}_2)\right]^2,$$

where $P_{1,1}$ denotes the polynomials of degree not greater than 1 on each variable. Let:

$$\mathbf{\Pi}_{h,33} : [H^s(\Omega) \cap \mathcal{W}_{33}(\Omega)]^2 \to \mathcal{W}_{33}^h(\Omega), \quad 1 < s \leqslant 2, \tag{5.1}$$

be the interpolant operator associated with the space $\mathcal{W}_{33}^h(\Omega)$. More specifically, the degrees of freedom associated with $\mathbf{\Pi}_{h,33}\varphi$ are the vertices of the rectangles Ω_j and if b is a common node of the adjacent rectangles Ω_j and Ω_k then $(\mathbf{\Pi}_{h,33}\ \varphi)_j(b) = (\mathbf{\Pi}_{h,33}\varphi)_k(b)$, where $(\mathbf{\Pi}_{h,33}\varphi)_j$ denotes the restriction of the interpolant $\mathbf{\Pi}_{h,33}\varphi$ of φ to Ω_j.

The interpolants $\mathbf{\Pi}_{h,11}$, $\mathbf{\Pi}_{h,13}$, $\mathbf{\Pi}_{h,55}$ and $\mathbf{\Pi}_{h,66}$ are defined in a similar fashion.

It is well known that for all $\varphi \in [H^s(\Omega)]^2 \cap \mathcal{W}_{33}(\Omega), 1 < s \leqslant 2$, the interpolant $\mathbf{\Pi}_{h,33}$ satisfies the approximating properties:

$$\left\|\varphi - \mathbf{\Pi}_{h,33}\varphi\right\|_0 + h\left\|\varphi - \mathbf{\Pi}_{h,33}\varphi\right\|_1 \leqslant Ch^s\|\varphi\|_s, \tag{5.2}$$

and (5.2) holds as well for the other interpolants.

The FEM procedures to compute the approximate solution of (4.4), (4.5), (4.6) and (4.7) are defined as follows: find $u^{(h,33)} \in \mathcal{W}_{33}^h(\Omega)$, $u^{(h,11)} \in \mathcal{W}_{11}^h(\Omega)$, $u^{(h,13)} \in \mathcal{W}_{13}^h(\Omega)$, $u^{(h,55)} \in \mathcal{W}_{55}^h(\Omega)$ and $u^{(h,66)} \in \mathcal{W}_{66}^h(\Omega_2)$ such that:

$$\Lambda\left(u^{(h,33)}, v\right) = -\langle \Delta P, v\cdot \nu\rangle_{\Gamma^T}, \quad v \in \mathcal{W}_{33}^h(\Omega), \tag{5.3}$$
$$\Lambda\left(u^{(h,11)}, v\right) = -\langle \Delta P, v\cdot \nu\rangle_{\Gamma^R}, \quad v \in \mathcal{W}_{11}^h(\Omega), \tag{5.4}$$
$$\Lambda\left(u^{(h,13)}, v\right) = -\langle \Delta P, v\cdot \nu\rangle_{\Gamma^T\cup\Gamma^R}, \quad v \in \mathcal{W}_{13}^h(\Omega), \tag{5.5}$$
$$\Lambda\left(u^{(h,55)}, v\right) = -\langle g, v\rangle_{\Gamma^L\cup\Gamma^R\cup\Gamma^T}, \quad v \in \mathcal{W}_{55}^h(\Omega), \tag{5.6}$$

and

$$\Lambda\left(u^{(h,66)}, v\right) = -\langle g, v\rangle_{\Gamma_2^L\cup\Gamma_2^R\cup\Gamma_2^T}, \qquad v \in \mathcal{W}_{66}^h(\Omega_2). \tag{5.7}$$

Uniqueness for the FEM procedures (5.3)–(5.7) can be shown with the same argument than for the continuous case. Existence follows from finite dimensionality.

Let us analyze the error associated with the procedure (5.3). Set:

$$e^{(h,33)} = u^{(33)} - u^{(h,33)},$$

and note that from (4.4) and (5.3) we get:

$$\Lambda\left(e^{(h,33)}, v\right) = 0, \quad v \in \mathcal{W}_{33}^h(\Omega). \tag{5.8}$$

Choose $v = e^{(h,33)} + \mathbf{\Pi}_{h,33}u^{(33)} - u^{(33)}$ in (5.8), take the imaginary part in the resulting equation and use the positive definiteness of $\mathbf{M}_I$, (4.12), the H^1-continuity of $\Lambda(u,v)$, the fact that $u^{(33)} \in [H^{3/2}(\Omega)]^2$ and the approximating properties (5.2) to get:

$$\begin{aligned}
\frac{C_2 L_*(\mathbf{M}_I)}{2}\left\|e^{(h,33)}\right\|_1^2 &\leqslant \frac{L_*(\mathbf{M}_I)}{2}\sum_{l,m=1,3}\left\|\epsilon_{lm}\left(e^{(h,33)}\right)\right\|_0^2\\
&\leqslant \mathrm{Im}\left(\Lambda\left(e^{(h,33)}, e^{(h,33)}\right)\right)\\
&\leqslant \left|\Lambda\left(e^{(h,33)}, u^{(33)} - \mathbf{\Pi}_{h,33}u^{(33)}\right)\right|\\
&\leqslant \omega^2\rho^*\left\|e^{(h,33)}\right\|_0\left\|u^{(33)} - \mathbf{\Pi}_{h,33}u^{(33)}\right\|_0\\
&\quad + C_8(\omega)\left\|e^{(h,33)}\right\|_1\left\|u^{(33)} - \mathbf{\Pi}_{h,33}u^{(33)}\right\|_1\\
&\leqslant \left[\omega^2\rho^*\left\|e^{(h,33)}\right\|_1 h^{3/2} + C_8(\omega)\left\|e^{(h,33)}\right\|_1 h^{1/2}\right]\left\|u^{(33)}\right\|_{3/2},
\end{aligned} \tag{5.9}$$

where

$$C_8(\omega) = 2\max\left(L^*(\mathbf{M}_R(\omega)), L^*(\mathbf{M}_I(\omega))\right).$$

Hence:

$$\left\|e^{(h,33)}\right\|_1 \leqslant h^{1/2}C_9(\omega)\left\|u^{(33)}\right\|_{3/2}, \tag{5.10}$$

with

$$C_9(\omega) = \left(\max\left(\omega^2\rho^*, C_8(\omega)\right)\right)\left(C_2 L_*(\mathbf{M}_I)(\omega)/2\right)^{-1}.$$

To estimate $\|e^{(h,33)}\|_0$ we employ a duality argument. Let ψ be the solution of the dual problem:

$$-\omega^2\rho\psi(x_1,x_3,\omega) - \nabla\cdot\sigma^*(\psi(x_1,x_3,\omega)) = e^{(h,33)}, \quad \Omega, \tag{5.11}$$
$$\sigma^*(\psi)\nu\cdot\nu = 0, \quad (x_1,x_3)\in\Gamma^T, \tag{5.12}$$
$$\sigma^*(\psi)\nu\cdot\chi = 0, \quad (x_1,x_3)\in\Gamma^L\cup\Gamma^R\cup\Gamma^T, \tag{5.13}$$
$$\psi\cdot\nu = 0, \quad (x_1,x_3)\in\Gamma^L\cup\Gamma^R, \tag{5.14}$$
$$\psi = 0, \quad (x_1,x_3)\in\Gamma^B, \tag{5.15}$$

where $\sigma^*(\psi)$ is defined as in (2.2) but using the complex conjugates of the coefficients. By elliptic regularity, we have the estimate:

$$\|\psi\|_2 \leqslant C_{10}(\omega)\left\|e^{(h,33)}\right\|_0. \tag{5.16}$$

Testing (5.11) against $v \in \mathcal{W}_{33}(\Omega)$ we see that:

$$\Lambda(v,\psi) = \left(v, e^{(h,33)}\right), \quad v \in \mathcal{W}_{33}(\Omega). \tag{5.17}$$

Choose $v = e^{(h,33)}$ in (5.17) and use (5.8) to get:

$$\left\|e^{(h,33)}\right\|_0^2 = \Lambda\left(e^{(h,33)},\psi\right) = \Lambda\left(e^{(h,33)}, \psi - \mathbf{\Pi}_{h,33}\psi\right). \tag{5.18}$$

Thus from (5.18) and (5.2) we obtain the estimate:

$$\begin{aligned}
\left\|e^{(h,33)}\right\|_0^2 &\leqslant \omega^2\rho^*\left\|e^{(h,33)}\right\|_0\left\|\psi - \mathbf{\Pi}_{h,33}\psi\right\|_0\\
&\quad + C_8(\omega)\left\|e^{(h,33)}\right\|_1\left\|\psi - \mathbf{\Pi}_{h,33}\psi\right\|_1\\
&\leqslant \left(\omega^2\rho^*h^2\left\|e^{(h,33)}\right\|_0 + C_8(\omega)h\left\|e^{(h,33)}\right\|_1\right)\|\psi\|_2\\
&\leqslant h^2 C_{10}(\omega)\omega^2\rho^*\left\|e^{(h,33)}\right\|_0^2\\
&\quad + C_8(\omega)C_{10}(\omega)h\left\|e^{(h,33)}\right\|_0\left\|e^{(h,33)}\right\|_1.
\end{aligned}$$

Hence, for h small:

$$\left\|e^{(h,33)}\right\|_0 \leqslant C_{11}(\omega)h\left\|e^{(h,33)}\right\|_1, \tag{5.19}$$

and using the H^1-estimate (5.10) in (5.19), we conclude that:

$$\left\|e^{(h,33)}\right\|_0 \leqslant C_{12}(\omega)h^{3/2}\left\|u^{(33)}\right\|_{3/2}. \tag{5.20}$$

The results are summarized in the following theorem.

Theorem 3. *Let $u^{(33)}$ and $u^{(h,33)}$ be the solutions of (4.4) and (5.3), respectively. Then for sufficiently small $h > 0$ the following error estimate hols:*

$$\begin{aligned}
&\left\|u^{(33)} - u^{(h,33)}\right\|_0 + h\left\|u^{(33)} - u^{(h,33)}\right\|_1\\
&\quad\leqslant C_{13}(\omega)h^{3/2}\left\|u^{(33)}\right\|_{3/2}.
\end{aligned} \tag{5.21}$$

Remark. An identical argument shows the validity of the error estimate given in Theorem 3 for the solution of the problems (5.4), (5.6) and (5.7).

Remark. The estimate in (5.21) is optimal given the maximal regularity of the solution of the continuous problem.

Let us proceed to analyze the error associated with the calculation of the coefficient p_{13} in the procedure (5.5). Since the imaginary part of the bilinear form $\Lambda(u,v)$ is not H^1-coercive in $\mathcal{W}_{13}(\Omega)$ but satisfies instead a Garding inequality as in (4.28), the error analysis for the procedure (5.5) needs an argument similar to that given by Schatz [20]. Thus, since $u^{(13)} \in [H^{3/2}(\Omega)]^2$, we first estimate $\|e^{(h,13)}\|_1$ in terms of $\|e^{(h,13)}\|_0$ and $\|u^{(13)}\|_{3/2}$ and then we combine duality with a boot-strapping argument to get the desired estimates in terms of $\|u^{(13)}\|_{3/2}$.

First, setting:

$$e^{(h,13)} = u^{(13)} - u^{(h,13)},$$

from (4.15) and (5.5), we get:

$$\Lambda\left(e^{(h,13)}, v\right) = 0, \quad v \in \mathcal{W}_{13}^h(\Omega). \tag{5.22}$$

Choose $v = e^{(h,13)} + \mathbf{\Pi}_{h,13}u^{(13)} - u^{(13)}$ in (5.22), take the imaginary part in the resulting equation, use the positive definiteness of $\mathbf{M}_I$, Korn's second inequality (4.10) and the approximating properties of the operator $\mathbf{\Pi}_{h,13}$ to get:

$$\begin{aligned}
&\frac{L_*(\mathbf{M}_I)}{2}\left(C_1\left\|e^{(h,13)}\right\|_1^2 - \left\|e^{(h,13)}\right\|_0^2\right)\\
&\leqslant \frac{L_*(\mathbf{M}_I)}{2}\sum_{l,m=1,3}\left\|\epsilon_{lm}\left(e^{(h,13)}\right)\right\|_0^2 \leqslant \mathrm{Im}\left(\Lambda\left(e^{(h,13)}, e^{(h,13)}\right)\right)\\
&\leqslant \left|\Lambda\left(e^{(h,13)}, u^{(13)} - \mathbf{\Pi}_{h,13}u^{(13)}\right)\right|\\
&\leqslant \omega^2\rho^*\left\|e^{(h,13)}\right\|_0\left\|u^{(13)} - \mathbf{\Pi}_{h,13}u^{(13)}\right\|_0\\
&\quad + C_8(\omega)\left\|e^{(h,13)}\right\|_1\left\|u^{(13)} - \mathbf{\Pi}_{h,13}u^{(13)}\right\|_1\\
&\leqslant \left[\omega^2\rho^*\left\|e^{(h,13)}\right\|_1 h^{3/2} + C_8(\omega)\left\|e^{(h,13)}\right\|_1 h^{1/2}\right]\left\|u^{(13)}\right\|_{3/2},
\end{aligned} \tag{5.23}$$

so that:

$$\left\|e^{(h,13)}\right\|_1 \leqslant C_1^{-1}\left\|e^{(h,13)}\right\|_0 + C_{14}(\omega)h^{1/2}\left\|u^{(13)}\right\|_{3/2}, \tag{5.24}$$

with

$$C_{14}(\omega) = \left(\max\left(\omega^2\rho^*, C_8(\omega)\right)\right)\left(C_1 L_*(\mathbf{M}_l)(\omega)/2\right)^{-1}$$

To estimate $\|e^{(h,13)}\|_0$ in (5.24), we solve the following dual problem. Let ψ be the solution of

$$-\omega^2\rho\psi(x_1,x_3,\omega) - \nabla\cdot\sigma^*(\psi(x_1,x_3,\omega)) = e^{(h,13)}, \quad \Omega, \tag{5.25}$$
$$\sigma^*(\psi)\nu\cdot\nu = 0, \quad (x_1,x_3)\in\Gamma^T\cup\Gamma^R, \tag{5.26}$$
$$\sigma^*(\psi)\nu\cdot\chi = 0, \quad (x_1,x_3)\in\Gamma, \tag{5.27}$$
$$\psi\cdot\nu = 0, \quad (x_1,x_3)\in\Gamma^L\cup\Gamma^B, \tag{5.28}$$

which has the regularity estimate:

$$\|\psi\|_2 \leqslant C_{15}(\omega)\left\|e^{(h,13)}\right\|_0. \tag{5.29}$$

Testing (5.25) against $v\in\mathcal{W}_{13}(\Omega)$ we have that:

$$\Lambda(v,\psi) = \left(v, e^{(h,13)}\right), \quad v\in\mathcal{W}_{13}(\Omega). \tag{5.30}$$

Choose $v = e^{(h,13)}$ in (5.30) and use (5.22) to get:

$$\left\|e^{(h,13)}\right\|_0^2 = \Lambda\left(e^{(h,13)},\psi\right) = \Lambda\left(e^{(h,13)},\psi - \mathbf{\Pi}_{h,13}\psi\right), \tag{5.31}$$

so that repeating the argument given to derive the estimate (5.19) for $\|e^{(h,33)}\|_0$ we see that for h small:

$$\left\|e^{(h,13)}\right\|_0 \leqslant C_{16}(\omega)h\left\|e^{(h,13)}\right\|_1. \tag{5.32}$$

Thus, use (5.32) in (5.24) to see that for h small:

$$\left\|e^{(h,13)}\right\|_1 \leqslant C_{17}(\omega)h^{1/2}\left\|u^{(13)}\right\|_{3/2}. \tag{5.33}$$

Finally use (5.33) in (5.32) to obtain the L^2-estimate:

$$\left\|e^{(h,13)}\right\|_0 \leqslant C_{18}(\omega)h^{3/2}\left\|u^{13}\right\|_{3/2}. \tag{5.34}$$

Then we have the validity of the following theorem:

Theorem 4. *Let $u^{(13)}$ and $u^{(h,13)}$ be the solutions of (4.15) and (5.5), respectively. Then for sufficiently small $h > 0$ the following error estimate hols:*

$$\left\|u^{(13)} - u^{(h,13)}\right\|_0 + h\left\|u^{(13)} - u^{(h,13)}\right\|_1 \leqslant C_{19}(\omega)h^{3/2}\left\|u^{(13)}\right\|_{3/2}. \tag{5.35}$$

6. Numerical Examples

Let us consider that each layer is isotropic and anelastic with complex Lamé constants given by

$$\lambda(\omega) = \rho\left(c_P^2 - \frac{4}{3}c_S^2\right)M_1(\omega) - \frac{2}{3}\rho c_S^2 M_2(\omega) \quad \text{and} \quad \mu(\omega) = \rho c_S^2 M_2(\omega), \tag{6.1}$$

where M_1 and M_2 are the dilatational and shear complex moduli, respectively, and c_P and c_S are the elastic high-frequency limit compressional- and shear-wave velocities. (In [4] the relaxed moduli correspond to the elastic limit.) The dilatational modulus is

$$K(\omega) = \lambda(\omega) + \frac{2}{3}\mu(\omega) = \rho\left(c_P^2 - \frac{4}{3}c_S^2\right)M_1(\omega), \tag{6.2}$$

and the P-wave modulus is given by

$$E(\omega) = K(\omega) + \frac{4}{3}\mu(\omega). \tag{6.3}$$

We assume constant quality factors over the frequency range of interest (until about 100 Hz), which can be modeled by a continuous distribution of relaxation mechanisms based on the standard linear solid [21,22]. The dimensionless dilatational and shear complex moduli for a specific frequency can be expressed as

$$M_j(\omega) = \left(1 + \frac{2}{\pi Q_{0j}}\ln\frac{1+\mathrm{i}\omega\tau_2}{1+\mathrm{i}\omega\tau_1}\right)^{-1}, \quad j = 1,2, \tag{6.4}$$

where τ_1 and τ_2 are time constants, with $\tau_2 < \tau_1$, and Q_{0j} defines the value of the quality factor which remains nearly constant over the selected frequency range.

The example considers an epoxy-glass periodic layered medium. The properties of the isotropic viscoelastic materials are given in Table 1, i.e., the low-frequency Lamé constants, wave velocities, densities and quality factors [23]. Let the time constants in Eq. (6.4) be τ_1 = 0.16 s and τ_2 = 0.3 ms, so that the quality factor of each single isotropic layer is nearly constant from about 10 to 100 Hz.

In the long-wavelength limit, the wave characteristics of the layered medium are defined by the averaging relations (2.11a), the phase velocities (2.14) and the quality factors (2.15). In order to validate the BC theory we perform the numerical compressibility and shear oscillatory tests described in the previous sections. A crucial parameter for the validation is the ratio between the P-wave dominant pulse wavelength and the spatial period of the layering, which depends on the contrast between the constituents.

An optimal ratio can be found for which the equivalence between a finely layered medium and a homogeneous transversely isotropic medium is valid. We have performed preliminary tests, at a dominant frequency of 30 Hz and at a propagation angle of θ = 60°, in order to find the optimal value of the spatial period to have a percentage error of about 1%. There are two type of tests, namely, (i) The size of the sample is L =50 cm and we vary the number of layers, and therefore the thickness of the layers; and (ii) The number of layers is 100 and we vary the size of the sample, so that this size depends on the thickness of the layers. We refer to these two tests as "constant size" and "variable size", respectively.

Fig. 1 shows the results of the tests, where the errors corresponding to the "variable size" approach are much lower than those corresponding to the "constant size" approach. This reflects the fact that in the latter case the number of layers is lower than in the former test, showing that a large number of layers is required to obtain reliable results. Therefore, considering valid only the "variable size" test, the optimal ratio is about 2000, which corresponds to a spatial period of about 7 cm. Notice that the error in $1/Q$ for this test is very small so that the corresponding curve almost can not be seen in the graphic. The $1/Q$ error value is about 0.5% a ratio of 2000. For the same ratio, the error in the qP-phase velocity is approximately 0.9%. The analysis performed for the qSV and SH waves yields a similar conclusion.

We validate the BC theory in the following simulations. The stratified medium is a square sample of side length L = 50 cm composed of 100 alternating plane layers of epoxy and glass of equal thickness. The spatial period of the layering is then 1 cm, i.e., less

Table 1
Material properties.

Medium	λ (GPa)	μ (GPa)	ρ (kg/m^3)	c_P (m/s)	c_S (m/s)	Q_{01}	Q_{02}
Epoxy	3.94	1.61	1120	2530	1200	30	10
Glass	26.2	25.7	2510	5560	3200	500	100

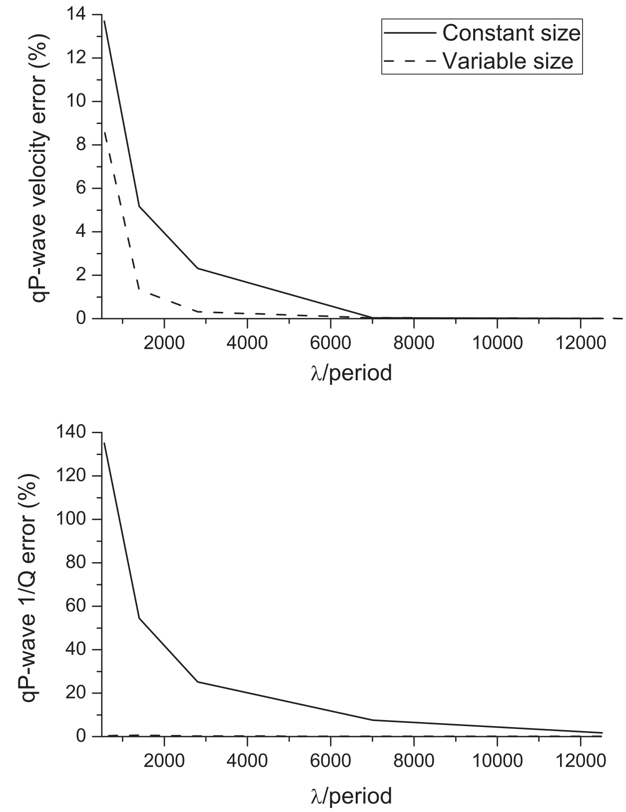

Fig. 1. Preliminar tests to determine the optimal ratio between the *P*-wave dominant pulse wavelength and the spatial period of the layering. The frequency is 30 Hz and the propagation angle is $\theta = 60°$. In the constant-size case we used a square sample of side length $L = 50$ cm, varying the number of layers. In the variable-size case, the number of layers is 100.

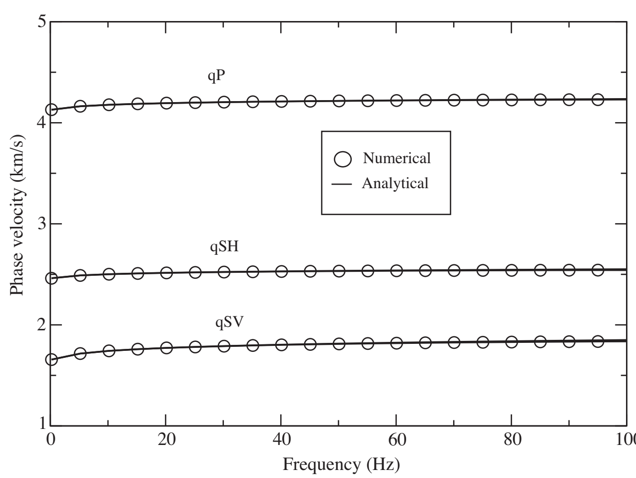

Fig. 2. Numerical (circles) and analytical (solid lines) qP, qSV and qSH phase velocities as function of frequency. The propagation angle is $\theta = 60°$.

that the value computed in the preliminar tests. The simulation uses a uniform partition $\mathcal{T}^h(\Omega)$ into 100×100 elements.

Figs. 2 and 3 show the phase velocities and quality factors as a function of frequency, with a propagation (phase) angle of $\theta = 60°$. Figs. 4 and 5 show the same quantities as a function of θ at a frequency of 30 Hz. The curves show the values obtained with the oscillatory tests (symbols), compared to the BC theory (solid line).

In particular, in Figs. 4 and 5 can be seen that attenuation anisotropy due to fine layering is more pronounced for qSV and

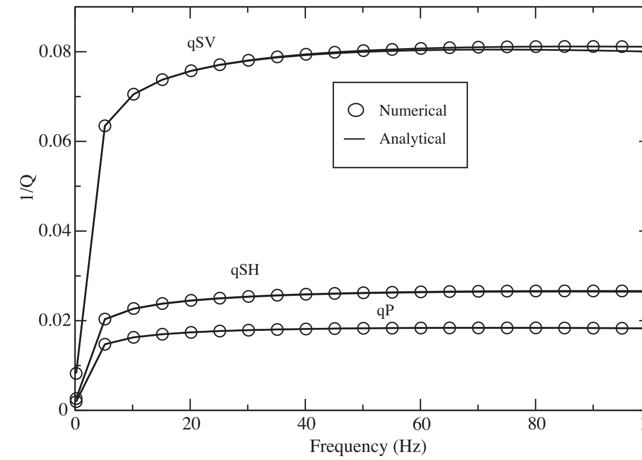

Fig. 3. Numerical (circles) and analytical (solid lines) qP, qSV and qSH inverse quality factors as function of frequency. The propagation angle is $\theta = 60°$.

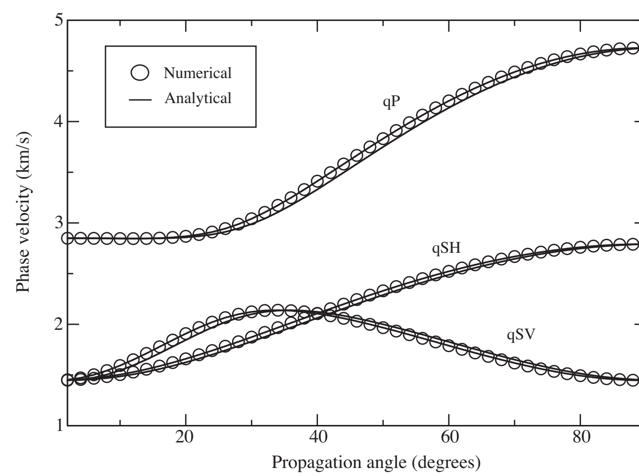

Fig. 4. Numerical (circles) and analytical (solid lines) qP, qSV and qSH phase velocities at 30 Hz as function of the propagation angle.

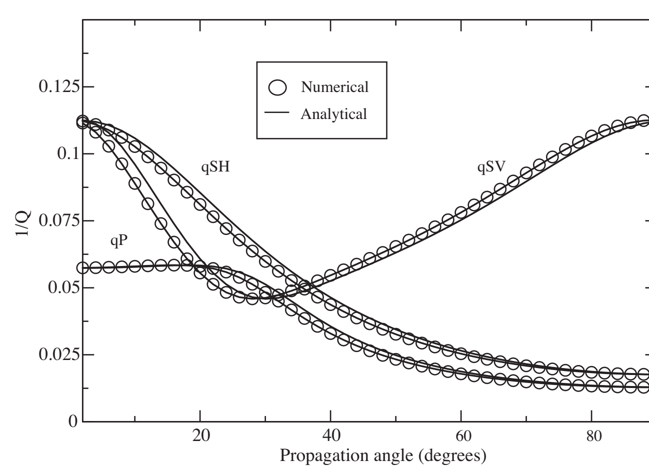

Fig. 5. Numerical (circles) and analytical (solid lines) qP, qSV and qSH inverse quality factors at 30 Hz as function of the propagation angle.

qSH waves than for qP waves. We observe an excellent agreement between the theoretical and numerical results, which validates the BC theory being tested. Similar results were obtained for other finely layered anelastic materials [24].

7. Conclusion

The Backus/Carcione theory yields the frequency-dependent effective stiffnesses and wave properties of finely-layered anelastic media at long wavelengths. In order to test the theory, we introduced a novel numerical procedure based on oscillatory experiments, which allows us to obtain the phase velocities and quality factors of homogeneous body waves as a function of frequency and propagation angle. The experiments are defined as boundary-value problems in the space-frequency domain, representing harmonic compressibility and shear tests which are performed by using a finite-element method. To illustrate the methodology, we applied the tests to a periodic sequence of thin epoxy and glass layers. The agreement between the numerical and theoretical results is excellent. The theory and numerical solver proposed in this work can also be applied and/or generalized to complex geological situations (lower symmetries, stochastic heterogeneities, fractures, etc.) and used in the processing and interpretation of real seismic data for characterization purposes.

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