GEOPHYSICS, VOL. 88, NO. 1 (JANUARY-FEBRUARY 2023); P. WA161–WA175, 8 FIGS., 1 TABLE.
10.1190/GEO2022-0271.1

Wave propagation in thermo-poroelasticity: A finite-element approach

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ABSTRACT

We have developed continuous and discrete-time finite-element (FE) methods to solve an initial boundary-value problem for the thermo-poroelasticity wave equation based on the combined Biot/Lord-Shulman (LS) theories to describe the porous and thermal effects, respectively. In particular, the LS model, which includes a Maxwell-Vernotte-Cattaneo relaxation term, leads to a hyperbolic heat equation, thus avoiding infinite signal velocities. The FE methods are formulated on a bounded domain with absorbing boundary conditions at the artificial boundaries. The dynamical equations predict four propagation modes, a fast P (P1) wave, a Biot slow (P2) wave, a thermal (T) wave, and a shear (S) wave. The spatial discretization uses globally continuous bilinear polynomials to represent solid displacements and temperature, whereas the vector part of the Raviart-Thomas-Nedelec of zero order is used to represent fluid displacements. First, a priori optimal error estimates are derived for the continuous-time FE method, and then an explicit conditionally stable discrete-time FE method is defined and analyzed. The explicit FE algorithm is implemented in one dimension to analyze the behavior of the P1, P2, and T waves. The algorithms can be useful for a better understanding of seismic waves in hydrocarbon reservoirs and crustal rocks, whose description is mainly based on the assumption of isothermal wave propagation.

INTRODUCTION

Thermoelasticity is the theory that couples the fields of deformation and temperature, where an elastic source gives rise to a temperature field and attenuation and a heat source induces anelastic deformations. The theory is useful in a variety of applications such as seismic attenuation in rocks and material science (Zener, 1938; Lifshitz and Roukes, 2000; Carcione et al., 2019a). The theory also might be relevant in low-temperature physics, theories of shocks and vibrations, and astrophysics.

The classical parabolic-type differential equations of thermoelasticity (nonporous) for the Fourier law of heat conduction are reported by Biot (1956), but his theory has unphysical solutions, such as discontinuities and infinite velocities at high frequencies. Subsequently, Lord and Shulman (1967) overcome these problems by formulating hyperbolic-type differential equations, introducing Maxwell-Vernotte-Cattaneo (MVC) relaxation times into the heat equation (Rudgers, 1990). The thermoelasticity theory predicts an S wave, two P waves, and a thermal wave. The fastest P wave and thermal wave have characteristics similar to the fast and slow P waves of poroelasticity, respectively (Carcione et al., 2019a; Carcione, 2022).

The work of Zener (1938) already contains the concept of mode conversion from a P wave to a thermal mode, e.g., he explains P-wave dissipation due to the presence of "microscopic stress inhomogeneities [which] arise from imperfections, such as cavities, and from the elastic anisotropy of the individual crystallites," in the same way that the White model (White et al., 1975) describes attenuation in porous media due to mesoscopic-scale inhomogeneities (as a P wave converted to Biot slow mode). Zener (1946) anticipates the concept of attenuation due to diffusion, where he mentions thermal, atomic, and magnetic diffusions as the causes. However, the Biot slow mode represents loss due to fluid-pressure diffusion. These attenuation mechanisms (and related velocity dispersion) are essential in forward modeling and inversion to honor the amplitude and phase of the wavefield.

Early works in geophysics worth mentioning in this sense were conducted by Treitel (1959) and Savage (1966), who obtain the P- and S-wave quality factors for empty round cavities or pores,

Manuscript received by the Editor 4 May 2022; revised manuscript received 25 August 2022; published ahead of production 16 September 2022; published online 6 December 2022.
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and Armstrong (1984), who considers a finely layered medium. The subject had been neglected in practice until recent works by Carcione and coworkers, who perform the first simulation of the thermal wave in the context of thermoelasticity and porothermoelasticity (Carcione et al., 2019a, 2019b, 2020; Wang et al., 2020, 2021; Wei et al., 2020). In these works, the numerical simulation was performed with a direct method to compute the spatial derivatives, namely, the Fourier pseudospectral differential operator (e.g., Carcione, 2022). The development of a new technique, based on the finite-element (FE) algorithm, will provide a more flexible approach to represent the heterogeneities of the medium and will provide a further crosscheck of algorithms and the physics of wave propagation.

Santos et al. (2021) prove the existence and uniqueness of the Biot/Lord-Shulman formulation in linear thermo-poroelastic isotropic media, with bounded domains under appropriate boundary and initial conditions. The analysis shows the existence of a unique solution, given in terms of displacements of the solid and fluid phases and temperature, and proves its regularity in the space and time variables. The FE spaces used for the spatial discretization of the initial boundary-value problem (IBVP) are as follows. The components of the solid displacement vector and the temperature are represented by globally continuous piecewise bilinear functions. For the fluid phase, we use the locally vector part of the Raviart-Thomas-Nedelec space of zero order. First, we derive a variational formulation of the continuous-time FE IBVP problem and show the existence and uniqueness of the continuous-time FE solution. Then, a priori error estimates are given, which are optimal for the FE spaces used and the assumed regularity of the solution. A novel explicit discrete-time FE algorithm is defined, and the conditional stability of the explicit FE procedure is analyzed. Finally, the implementation of the explicit FE algorithm is illustrated for the 1D case, with numerical experiments showing the behavior of all waves when using this nonisothermal model.

MODEL EQUATIONS

We consider a porous medium saturated by a single phase and compressible viscous fluid and assume that the whole aggregate is isotropic. Let $\mathbf{u}^s = (u_i^s)$ and $\mathbf{u}^f = (u_i^f)$ denote the average displacement vectors of the solid and relative fluid phases, respectively, and set $\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f)$. Let $\boldsymbol{\varepsilon}(\mathbf{u}^s) = (\varepsilon_{ij}(\mathbf{u}^s))$ be the strain tensor of the solid. Also, let $\boldsymbol{\sigma}(\mathbf{u}, \theta) = (\sigma_{ij}(\mathbf{u}, \theta))$ and $p_f = p_f(\mathbf{u}, \theta)$ denote the stress tensor of the bulk material and the fluid pressure, respectively, with θ being the increment of the temperature above a reference absolute temperature θ_0 for the state of zero stress and strain. The stress-strain relations are

$$\sigma_{ij}(\mathbf{u}, \theta) = 2\mu\varepsilon_{ij}(u^s) + \delta_{ij}(\lambda_u \nabla \cdot \mathbf{u}^s + B\nabla \cdot \mathbf{u}^f - \beta\theta), \quad (1)$$

$$-p_f(\mathbf{u}, \theta) = B\nabla \cdot \mathbf{u}^s + M\nabla \cdot \mathbf{u}^f - \beta_f\theta, \quad (2)$$

where μ is the wet- or dry-rock shear modulus; $\lambda_u = \lambda + \alpha^2 M$; $\alpha = 1 - (K_m/K_s)$; $M = (((\alpha - \phi)/K_s) + (\phi/K_f))^{-1}$; ϕ is the porosity; and $B = \alpha M$, with λ_u being the Lamé coefficient of the fluid-saturated frame and K_s, K_m, and K_f denoting the bulk moduli of the grains, solid, and fluid, respectively. The positive coupling coefficients β and β_f are the coefficients of thermoelasticity of the bulk material and fluid, respectively.

Dynamical equations

Let $\rho_b = (1 - \phi)\rho_s + \phi\rho_f$ denote the mass density of the bulk material, with ρ_s and ρ_f being the mass densities of the grains and fluid, respectively. Let the positive definite matrix $\mathcal{P}$ and the nonnegative matrix $\mathcal{B}$ be defined by

$$\mathcal{P} = \begin{pmatrix} \rho_b I & \rho_f I \\ \rho_f I & gI \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0I & 0I \\ 0I & \frac{\eta}{\kappa} I \end{pmatrix}, \quad (3)$$

where I is the identity matrix in $R^{d\times d}$, with $d = 2, 3$, η is the fluid viscosity, κ is the permeability, and $g = (S\rho_f/\phi)$, where S is the tortuosity.

Let us define the differential operator $\mathcal{L}(\mathbf{u}, \theta) = (\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}, \theta), -\nabla p_f(\mathbf{u}, \theta))$. Then, Biot's dynamical equation taking into account temperature is

$$\mathcal{P}\ddot{\mathbf{u}} + \mathcal{B}\dot{\mathbf{u}}^f - \mathcal{L}(\mathbf{u}, \theta) = \mathbf{f}. \quad (4)$$

Following Sharma (2008) and Carcione et al. (2019a), the generalized heat equation is

$$\tau c\ddot{\theta} + c\dot{\theta} - \nabla \cdot (\gamma\nabla\theta) + \beta\theta_0\nabla \cdot \dot{\mathbf{u}}^s + \beta\theta_0\nabla \cdot \dot{\mathbf{u}}^f + \tau\beta\theta_0\nabla \cdot \ddot{\mathbf{u}}^s + \tau\beta\theta_0\nabla \cdot \ddot{\mathbf{u}}^f = -q. \quad (5)$$

In equations 4 and 5, $\mathbf{f} = (\mathbf{f}^s, \mathbf{f}^f)$ is an external force and q is a heat source. Also, $\gamma = (1 - \phi)\gamma_m + \phi\gamma_f$ is the bulk coefficient of heat conduction (or thermal conductivity), with γ_m and γ_f being the heat conduction of the frame and the fluid, respectively; $c = (1 - \phi)c_m + \phi c_f$ is the bulk specific heat of the unit volume in the absence of deformation; and τ is an MVC relaxation time. These equations assume thermal equilibrium between the solid and the fluid, i.e., the temperature in both phases is the same. Thermal equilibrium is valid when the interstitial heat transfer coefficient between the solid and fluid is very large and the ratio of pore surface area to pore volume is sufficiently high. Here, we consider β_m, β_f, γ, and c as strictly positive parameters, obtained from experiments or from a specific theoretical model.

IBVP

The IBVP is formulated in the 2D case (with obvious extension to the 3D case) for the case of thermal equilibrium in an open bounded domain Ω with piecewise smooth boundary and a time interval $J = (0, T)$ as follows: find $(\mathbf{u}, \theta)$ satisfying equations 4 and 5 with initial conditions

$$\mathbf{u}(x,0) = \mathbf{u}^0 = (\mathbf{u}^{0,s}, \mathbf{u}^{0,f}), \ \dot{\mathbf{u}}(x,0) = \mathbf{u}^1 = (\mathbf{u}^{1,s}, \mathbf{u}^{1,f}), \ x \in \Omega, \quad (6)$$

$$\theta(x, 0) = \theta^0, \quad \dot{\theta}(x, 0) = \theta^1, \quad x \in \Omega, \quad (7)$$

and absorbing boundary conditions

$$-\mathcal{G}_\Gamma(\mathbf{u}, \theta) = \mathcal{D}\mathcal{S}(\dot{\mathbf{u}}), -\gamma\nabla\theta \cdot \nu = \tau c v_\theta \dot{\theta}, \quad x \in \Gamma, \quad t \in J, \quad (8)$$

where

$$\mathcal{G}(\mathbf{u},\theta) = (\boldsymbol{\sigma\nu}\cdot\boldsymbol{\nu}, \boldsymbol{\sigma\nu}\cdot\boldsymbol{\chi}, -p_f)(\mathbf{u},\theta),$$
$$\mathcal{S}(\dot{\mathbf{u}}) = (\dot{\mathbf{u}}^s\cdot\boldsymbol{\nu}, \dot{\mathbf{u}}^s\cdot\boldsymbol{\chi}, \dot{\mathbf{u}}^f\cdot\boldsymbol{\nu}). \quad (9)$$

In equations 8 and 9, $\boldsymbol{\nu}$ and $\boldsymbol{\chi}$ are the unit vector outer normal and unit vector tangent on Γ oriented counterclockwise. The absorbing boundary condition (equation 8) is derived in Santos et al. (1988), with the matrix $\mathcal{D}$ being positive definite. Also, $v_\theta = \sqrt{\gamma/(\tau c)}$ is the heat speed (e.g., Carcione et al., 2020).

In the 3D case, the formulation of the IVP (equations 6–9) remains valid if two tangents (i.e., $\boldsymbol{\chi}_1, \boldsymbol{\chi}_2$) are used in equation 9.

An existence and uniqueness result for the solution of equations 4–7 with different boundary conditions than those in equation 8 are given in Santos et al. (2021).

A VARIATIONAL FORMULATION

To obtain a variational formulation, we need to introduce some notation. For $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$, let $(\cdot,\cdot)_\Omega$ and $\langle\cdot,\cdot\rangle_\Gamma$ denote the $L^2(\Omega)$ and $L^2(\Gamma)$ inner products, respectively, for scalar, vector, or matrix-valued functions. Also, for $s \in \mathbb{R}$, $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Gamma}$ will denote the usual norms for the Sobolev space $H^s(\Omega)$ and $H^s(\Gamma)$, respectively (Adams and Fournier, 2003). If $X = \Omega$ or $X = \Gamma$, the subscript X may be omitted such that $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_\Gamma$, or $|\cdot|_s = |\cdot|_{s,\Gamma}$. Let

$$H(\mathrm{div};\Omega) = \{\mathbf{v}\in[L^2(\Omega)]^2 : \nabla\cdot\mathbf{v}\in L^2(\Omega)\}, \quad (10)$$

provided with the norm $\|\mathbf{v}\|_{H(\mathrm{div};\Omega)} = [\|\mathbf{v}\|_0^2 + \|\nabla\cdot\mathbf{v}\|_0^2]^{1/2}$. We also will refer to the space:

$$H^1(\mathrm{div};\Omega) = \{\mathbf{v}\in[H^1(\Omega)]^2 : \nabla\cdot\mathbf{v}\in H^1(\Omega)\}. \quad (11)$$

The following known results will be used (Girault and Raviart, 1981):

$$|\mathbf{v}\cdot\boldsymbol{\nu}|_{-1/2,\Gamma} \le C\|\mathbf{v}\|_{H(\mathrm{div};\Omega)}, \quad (12)$$

$$|\mathbf{v}|_{0,\Gamma} \le C\|\mathbf{v}\|_{0,\Omega}^{1/2}\|\mathbf{v}\|_{1,\Omega}^{1/2} \le C\|\mathbf{v}\|_{1,\Omega}. \quad (13)$$

Here, and in what follows, C denotes a generic constant that may take different values at different places. Also recall Korn's second inequality (Duvaut and Lions, 1976):

$$\int_\Omega \left[\sum_{i,j}(\varepsilon_{ij}(\mathbf{v}))^2\right] d\Omega + \|\mathbf{v}\|_0^2 \ge C\|\mathbf{v}\|_1^2. \quad (14)$$

Next, we introduce the space $\mathcal{V} = [H^1(\Omega)]^2 \times H(\mathrm{div};\Omega)$, provided with the natural norm:

$$\|\mathbf{v}\|_\mathcal{V} = (\|\mathbf{v}^s\|_1^2 + \|\mathbf{v}^f\|_{H(\mathrm{div};\Omega)}^2)^{1/2},$$
$$\mathbf{v}^s \in [H^1(\Omega)]^2, \quad \mathbf{v}^f \in H(\mathrm{div};\Omega). \quad (15)$$

Also, for any Banach space Y, let

$$L^2(J,Y) = \left\{f : J\to Y : \|f\|_{J,Y}^2 = \int_0^T \|f(t)\|_Y^2 dt < \infty\right\}, \quad (16)$$

$$L^\infty(J,Y) = \{f : J\to Y : \|f\|_{J,Y}^\infty = \mathrm{ess.\,sup}_{t\in J}\|f(t)\|_Y < \infty\}. \quad (17)$$

To obtain a variational formulation of the IBVP (equations 4–8), multiply equation 4 by $\mathbf{v}^s$ and equation 5 by $\mathbf{v}^f$ such that $\mathbf{v} = (\mathbf{v}^s, \mathbf{v}^f) \in \mathcal{V}$; we use integration by parts and the boundary conditions equation 8 to obtain

$$(\mathcal{P}\ddot{\mathbf{u}}(x),\mathbf{v}) + \left(\frac{\eta}{\kappa}\dot{u}^f, \mathbf{v}^f\right) + \Lambda(\mathbf{u},\mathbf{v}) - (\beta\theta, \nabla\cdot\mathbf{v}^s) - (\beta_f\theta, \nabla\cdot\mathbf{v}^f)$$
$$+ (\tau c\ddot{\theta}, w) + (c\dot{\theta}, w) + (\gamma\nabla\theta, \nabla w) + (\beta\theta_0\nabla\cdot\dot{\mathbf{u}}^s, w)$$
$$+ (\beta\theta_0\nabla\cdot\dot{\mathbf{u}}^f, w) + (\tau\beta\theta_0\nabla\cdot\ddot{\mathbf{u}}^s, w) + (\tau\beta\theta_0\nabla\cdot\ddot{\mathbf{u}}^f, w)$$
$$+ \langle\mathcal{D}\mathcal{S}(\dot{\mathbf{u}}), \mathcal{S}(\mathbf{v})\rangle + \langle\tau c v_\theta\dot{\theta}, w\rangle$$
$$= (\mathbf{f},\mathbf{v}) - (q,w), \quad \mathbf{v} = (\mathbf{v}^s, \mathbf{v}^f, w) \in \mathcal{V}\times H^1(\Omega), \quad t\in J, \quad (18)$$

where $\Lambda(\mathbf{u},\mathbf{v})$ is the bilinear form

$$\Lambda(\mathbf{u},\mathbf{v}) = (\mathcal{E}\tilde{\varepsilon}(\mathbf{u}), \tilde{\varepsilon}(\mathbf{v})). \quad (19)$$

In equation 19, the matrix $\mathcal{E}$ and the column vector $\tilde{\varepsilon}(\mathbf{u})$ are defined by

$$\mathcal{E} = \begin{pmatrix} \lambda_u+2\mu & \lambda_u & B & 0 \\ \lambda_u & \lambda_u+2\mu & B & 0 \\ B & B & M & 0 \\ 0 & 0 & 0 & 4\mu \end{pmatrix}, \quad \tilde{\varepsilon}(\mathbf{u}) = \begin{pmatrix} \varepsilon_{11}(\mathbf{u}^s) \\ \varepsilon_{33}(\mathbf{u}^s) \\ \nabla\cdot\mathbf{u}^f \\ \varepsilon_{13}(\mathbf{u}^s) \end{pmatrix}. \quad (20)$$

The term $(\mathcal{E}\tilde{\varepsilon}(\mathbf{u}), \tilde{\varepsilon}(\mathbf{v}))$ in equation 19 is associated with the strain energy of the system, so that the symmetric matrix $\mathcal{E}$ must be positive definite. Furthermore, $\Lambda(\mathbf{u},\mathbf{v}) \le C\|\mathbf{u}\|_\mathcal{V}\|\mathbf{v}\|_\mathcal{V}$.

Also, note that, using equation 14, if $\xi_*^\mathcal{E}$ is the minimum eigenvalue of $\mathcal{E}$, the following Gärding inequality holds:

$$\Lambda(\mathbf{v},\mathbf{v}) \ge C_1\|\mathbf{v}\|_\mathcal{V}^2 - \xi_*^\mathcal{E}\|\mathbf{v}\|_0^2. \quad (21)$$

FE FORMULATIONS

We will find an FE solution of equation 18 as follows. Let $\mathcal{T}^h(\Omega)$ be a quasiregular nonoverlapping partition of Ω into rectangles Ω_j of diameter bounded by h such that $\overline{\Omega} = \cup_j^J \overline{\Omega}_j$. Let us denote by $\mathcal{W}^h(\Omega)$ the space of globally continuous piecewise bilinear polynomials to be used to approximate each component of the solid displacement $\mathbf{u}^s$ and the temperature θ. Also, let $\mathcal{V}^h(\Omega)$ be the vector part of the Raviart-Thomas-Nedelec space of zero order (Raviart and Thomas, 1977; Nedelec, 1980) used to approximate the fluid displacement vector $\mathbf{u}^f$. Then, let

$$\mathcal{Z}^h(\Omega) = \mathcal{W}^h(\Omega)\times\mathcal{W}^h(\Omega)\times\mathcal{V}^h(\Omega)\times\mathcal{W}^h(\Omega). \quad (22)$$

Next, let $\Pi : H^2(\Omega)\to\mathcal{W}^h(\Omega)$ be the interpolant operators associated with the space $\mathcal{W}^h$ and set $\Pi^{(2)} \equiv \Pi\times\Pi \to [\mathcal{W}^h(\Omega)]^2$. Let $Q : H^1(\mathrm{div};\Omega)\to\mathcal{V}^h(\Omega)$ be the projection defined by

$$\langle(Q\psi - \psi)\cdot\boldsymbol{\nu}, 1\rangle_B = 0, \quad B = \Gamma_{jk} \text{ or } B = \Gamma_j. \quad (23)$$

The approximating properties of Π and Q are (Raviart and Thomas, 1977; Nedelec, 1980)

$$\|\boldsymbol{\varphi}-\Pi\boldsymbol{\varphi}\|_0+h\|\boldsymbol{\varphi}-\Pi\varphi\|_1\leq Ch^2\|\boldsymbol{\varphi}\|_2,\ \varphi\in H^2(\Omega), \quad (24)$$

$$\|\boldsymbol{\varphi}-(\Pi)^{(2)}\boldsymbol{\varphi}\|_0+h\|\boldsymbol{\varphi}-(\Pi)^{(2)}\varphi\|_1\leq Ch^2\|\boldsymbol{\varphi}\|_2,\ \varphi\in[H^2(\Omega)]^2, \quad (25)$$

$$\|\boldsymbol{\psi}-Q\boldsymbol{\psi}\|_0\leq Ch\|\boldsymbol{\psi}\|_1, \quad (26)$$

$$\|\nabla\cdot(\boldsymbol{\psi}-Q\boldsymbol{\psi})\|_0\leq Ch(\|\boldsymbol{\psi}\|_1+\|\nabla\cdot\boldsymbol{\psi}\|_1),\ \ \boldsymbol{\psi}\in H^1(\mathrm{div};\Omega). \quad (27)$$

Continuous-time FE procedure

We find $(\mathbf{U}(t),\Theta(t))\in\mathcal{Z}^h(\Omega)$ such that

$$\begin{aligned}&(\mathcal{P}\ddot{\mathbf{U}},\mathbf{v})+\left(\frac{\eta}{\kappa}\dot{\mathbf{U}}^f,\mathbf{v}^f\right)+\Lambda(\mathbf{U},\mathbf{v})-(\beta\Theta,\nabla\cdot\mathbf{v}^s)-(\beta_f\Theta,\nabla\cdot\mathbf{v}^f)\\&+(\tau c\ddot{\Theta},w)+(c\dot{\Theta},w)+(\gamma\nabla\Theta,\nabla w)+(\beta\theta_0\nabla\cdot\dot{\mathbf{U}}^s,w)\\&+(\beta\theta_0\nabla\cdot\dot{\mathbf{U}}^f,w)+(\tau\beta\theta_0\nabla\cdot\ddot{\mathbf{U}}^s,w)+(\tau\beta\theta_0\nabla\cdot\ddot{\mathbf{U}}^f,w)\\&+\langle\mathcal{DS}(\dot{\mathbf{U}}),\mathcal{S}(\mathbf{v})\rangle+\langle\tau c v_\theta\dot{\Theta},w\rangle\\&=(\mathbf{f},\mathbf{v})-(q,w),\ \ \mathbf{v}=(\mathbf{v}^s,\mathbf{v}^f,w)\in\mathcal{Z}^h(\Omega),\ \ t\in J. \quad (28)\end{aligned}$$

Next, we state Theorem 1 where the existence and uniqueness of the solution of the problem (equation 28) are demonstrated, and Theorem 2 where a priori error estimates for this FE procedure are presented. Their proofs are given in Appendix A.

Theorem 1

Assume that the matrices $\mathcal{P}$ and $\mathcal{B}$ in equation 3 are positive definite and semidefinite, respectively, and that the matrix $\mathcal{E}$ in equation 20 is positive definite. Also, assume that the coefficients τ, c, γ, β, and β_f in equation 5 are bounded above and below by positive constants.

Then, there exists a unique solution $(\mathbf{U},\Theta)\in\mathcal{Z}^h$ of the continuous-time FE procedure (equation 28) that satisfies the inequality:

$$\begin{aligned}&\|\mathbf{U}(t)\|^2_{L^\infty(J,\mathcal{V})}+\|\dot{\mathbf{U}}(t)\|^2_{L^\infty(J,\mathcal{V})}+\|\ddot{\mathbf{U}}(t)\|^2_{L^2(J,[L^2(\Omega)]^4)}\\&+\|\Theta(t)\|^2_{L^2(J,L^2(\Omega))}+\|\dot{\Theta}(t)\|^2_{L^2(J,H^1(\Omega))}\\&\leq C(\|\mathbf{U}(0)\|^2_{\mathcal{V}}+\|\dot{\mathbf{U}}(0)\|^2_{\mathcal{V}}+\|\dot{\mathbf{U}}(0)\|^2_0+\|\ddot{\mathbf{U}}(0)\|^2_0\\&+\|\dot{\Theta}(0)\|^2_0+\|\Theta(0)\|^2_1)\\&\leq C(\|\mathbf{f}\|^2_{L^2(J,[L^2(\Omega)]^4)}+\|\dot{\mathbf{f}}\|^2_{L^2(J,[L^2(\Omega)]^4)}+\|q(s)\|^2_{L^2(J,L^2(\Omega))}). \quad (29)\end{aligned}$$

Theorem 2

Assume that the matrices $\mathcal{P}$ and $\mathcal{B}$ in equation 3 are positive definite and semidefinite, respectively, and that the matrix $\mathcal{E}$ in equation 20 is positive definite. Also, assume that the coefficients τ, c, γ, β, and β_f in equation 5 are bounded above and below by positive constants. Then, the solution $(\mathbf{U},\Theta)\in\mathcal{Z}^h$ of the FE procedure (equation 28) satisfies the a priori error estimate:

$$\begin{aligned}&\|\mathbf{E}\dot{\mathbf{u}}\|_{L^\infty(J,[L^2(\Omega)]^4)}+\|\mathbf{E}\ddot{\mathbf{u}}\|_{L^\infty(J,[L^2(\Omega)]^4)}\\&+\|\mathbf{E}\mathbf{u}\|_{L^\infty(J,\mathcal{V})}+\|\mathbf{E}\dot{\mathbf{u}}\|_{L^\infty(J,\mathcal{V})}\\&+\|\mathbf{E}\theta\|_{L^\infty(J,H^1(\Omega))}+\|\mathbf{E}\dot{\theta}\|_{L^\infty(J,L^2(\Omega))}\\&\leq Ch(\|\mathbf{u}^{0,s}\|_2+\|\mathbf{u}^{0,f}\|_1+\|\nabla\cdot\mathbf{u}^{0,f}\|_1+\|\mathbf{u}^{1,s}\|_2\\&+\|\mathbf{u}^{1,f}\|_1+\|\nabla\cdot\mathbf{u}^{1,f}\|_1\\&+\|\theta^0\|_2+|\theta^1\|_2+\|\ddot{f}(0)\|_0+\|\ddot{q}(0)\|_0\\&+\|\dot{\mathbf{u}}^s\|_{L^2(J,[H^2(\Omega)]^2)}+\|\dot{\mathbf{u}}^f\|_{L^\infty(J,[H^1(\Omega)]^2)}\\&+\|\dot{\mathbf{u}}^f\|_{L^2(J,[H^{3/2}(\Omega)]^2)}+\|\nabla\dot{\mathbf{u}}^f\|_{L^2(J,H^1(\Omega))}\\&+\|\ddot{\mathbf{u}}^s\|_{L^2(J,[H^2(\Omega)]^2)}+\|\ddot{\mathbf{u}}^f\|_{L^2(J,[H^1(\Omega)]^2)}+\|\dot{\theta}\|_{L^\infty(J,H^2(\Omega))}\\&+\|\ddot{\theta}\|_{L^2(J,H^2(\Omega))}). \quad (30)\end{aligned}$$

TIME-STEPPING PROCEDURE

Let

$$\partial^2\mathbf{U}^n=\frac{\mathbf{U}^{n+1}-2\mathbf{U}^n+\mathbf{U}^{n-1}}{\Delta t^2},\quad \partial\mathbf{U}^n=\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n-1}}{2\Delta t},\quad D_t\mathbf{U}^n=\frac{\mathbf{U}^{n+1}-\mathbf{U}^n}{\Delta t}. \quad (31)$$

An explicit time discretization of equation 18 can be stated as follows: we find $(\mathbf{U}^n,\Theta^n)\in\mathcal{Z}^h$ such that

$$\begin{aligned}&(\mathcal{P}\partial^2\mathbf{U}^n,\mathbf{v})+\left(\frac{\eta}{\kappa}\partial\mathbf{U}^{f,n},\mathbf{v}^f\right)+\Lambda(\mathbf{U}^n,\mathbf{v})-(\beta\Theta^n,\nabla\cdot\mathbf{v}^s)\\&-(\beta_f\Theta^n,\nabla\cdot\mathbf{v}^f)\\&+(\tau c\partial^2\Theta^n,w)+(c\partial\Theta^n,w)+(\gamma\nabla\Theta^n,\nabla w)+(\beta\theta_0\nabla\cdot\partial\mathbf{U}^{s,n},w)\\&+(\beta\theta_0\nabla\cdot\partial\mathbf{U}^{f,n},w)+(\tau\beta\theta_0\nabla\cdot\partial^2\mathbf{U}^{s,n},w)+(\tau\beta\theta_0\nabla\cdot\partial^2\mathbf{U}^{f,n},w)\\&+\langle\mathcal{DS}(\partial\mathbf{U}^n),\mathcal{S}(\mathbf{v})\rangle+\langle\tau c v_\theta\partial\Theta^n,w\rangle\\&=(\mathbf{f}^n,\mathbf{v})-(q^n,w),\ \mathbf{v}=(\mathbf{v}^s,\mathbf{v}^f,w)\in\mathcal{Z}^h(\Omega),\ n=1,2,\cdots,M. \quad (32)\end{aligned}$$

Conditional stability of the discrete FE procedure

We choose $\mathbf{v}=\partial\mathbf{U}^n=(\partial\mathbf{U}^{s,n},\partial\mathbf{U}^{f,n})$ and $w=\partial\Theta^n$ in equation 32 to obtain

$$\begin{aligned}&(\mathcal{P}\partial^2\mathbf{U}^n,\partial\mathbf{U}^n)+\left(\frac{\eta}{\kappa}\partial\mathbf{U}^{f,n},\partial\mathbf{U}^{f,n}\right)+\Lambda(\mathbf{U}^n,\partial\mathbf{U}^n)\\&-(\beta\Theta^n,\nabla\cdot\partial\mathbf{U}^{s,n})-(\beta_f\Theta^n,\nabla\cdot\partial\mathbf{U}^{f,n})\\&+(\tau c\partial^2\Theta^n,\partial\Theta^n)+(c\partial\Theta^n,\partial\Theta^n)+(\gamma\nabla\Theta^n,\nabla\partial\Theta^n)\\&+(\beta\theta_0\nabla\cdot\partial\mathbf{U}^{s,n},\partial\Theta^n)\\&+(\beta\theta_0\nabla\cdot\partial\mathbf{U}^{f,n},\partial\Theta^n)+(\tau\beta\theta_0\nabla\cdot\partial^2\mathbf{U}^{s,n},\partial\Theta^n)\\&+(\tau\beta\theta_0\nabla\cdot\partial^2\mathbf{U}^{f,n},\partial\Theta^n)\\&+\langle\mathcal{DS}(\partial\mathbf{U}^n),\mathcal{S}(\partial\mathbf{U}^n)\rangle+\langle\tau c v_\theta\partial\Theta^n,\partial\Theta^n\rangle\\&=(\mathbf{f}^n,\partial\mathbf{U}^n)-(q^n,\partial\Theta^n),\quad n=1,2,\cdots,M. \quad (33)\end{aligned}$$

Next, we use the identities

$$2\Delta t\Lambda(\mathbf{U}^n,\partial\mathbf{U}^n)=\frac{1}{2}[\Lambda(\mathbf{U}^{n+1},\mathbf{U}^{n+1})-\Lambda(\mathbf{U}^{n-1},\mathbf{U}^{n-1}) + \Lambda(\mathbf{U}^n-\mathbf{U}^{n-1},\mathbf{U}^n-\mathbf{U}^{n-1}) - \Lambda(\mathbf{U}^{n+1}-\mathbf{U}^n,\mathbf{U}^{n+1}-\mathbf{U}^n)], \quad (34)$$

$$\Delta t(\gamma\nabla\Theta^n,\nabla\partial\Theta^n)=\frac{1}{2}[(\gamma\nabla\Theta^{n+1},\nabla\Theta^{n+1})-(\gamma\nabla\Theta^{n-1},\nabla\Theta^{n-1}) +(\gamma\nabla(\Theta^n-\Theta^{n-1}),\nabla(\Theta^n-\Theta^{n-1}))-(\gamma\nabla(\Theta^{n+1}-\Theta^n), \nabla(\Theta^{n+1}-\Theta^n)], \quad (35)$$

and add to equation 33 the inequalities

$$\begin{aligned}&\frac{\zeta}{4\Delta t}[\|\mathbf{U}^{n+1}\|_0^2-\|\mathbf{U}^{n-1}\|_0^2]\le\frac{\zeta}{4}(\|\mathbf{U}^{n+1}\|_0^2+\|\mathbf{U}^{n-1}\|_0^2 +\|D_t\mathbf{U}^n\|_0^2+\|D_t\mathbf{U}^{n-1}\|_0^2),\\ &\frac{1}{4\Delta t}(\gamma\Theta^{n+1},\Theta^{n+1})-(\gamma\Theta^{n-1},\Theta^{n-1}) \le C(\|\Theta^{n+1}\|_0^2+\|\Theta^{n-1}\|_0^2+\|D_t\Theta^n\|_0^2+\|D_t\Theta^{n-1}\|_0^2),\end{aligned} \quad (36)$$

to obtain

$$\begin{aligned}&\frac{1}{2\Delta t}[(\mathcal{P}D_t\mathbf{U}^n,D_t\mathbf{U}^n)-(\mathcal{P}D_t\mathbf{U}^{n-1},D_t\mathbf{U}^{n-1})]\\ &+\frac{1}{4\Delta t}[\Lambda_\zeta(\mathbf{U}^{n+1},\mathbf{U}^{n+1})-\Lambda_\zeta(\mathbf{U}^{n-1},\mathbf{U}^{n-1})]\\ &+\frac{1}{4\Delta t}[\Lambda(\mathbf{U}^n-\mathbf{U}^{n-1},\mathbf{U}^n-\mathbf{U}^{n-1})-\Lambda(\mathbf{U}^{n+1}-\mathbf{U}^n,\mathbf{U}^{n+1}-\mathbf{U}^n)]\\ &+\frac{1}{2\Delta t}[(\tau cD_t\Theta^n,D_t\Theta^n)-(\tau cD_t\Theta^{n-1},D_t\Theta^{n-1})]\\ &+\frac{1}{4\Delta t}(\|\gamma^{1/2}\Theta^{n+1}\|_1^2-\|\gamma^{1/2}\Theta^{n-1}\|_1^2)\\ &+\frac{1}{4\Delta t}[(\gamma\nabla(\Theta^n-\Theta^{n-1}),\nabla(\Theta^n-\Theta^{n-1}))-(\gamma\nabla(\Theta^{n+1}-\Theta^n), \nabla(\Theta^{n+1}-\Theta^n)]\\ &+\left(\frac{\eta}{\kappa}\partial\mathbf{U}^{f,n},\partial\mathbf{U}^{f,n}\right)+(c\partial\Theta^n,\partial\Theta^n)+\langle\mathcal{DS}(\partial\mathbf{U}^n),\mathcal{S}(\partial\mathbf{U}^n)\rangle\\ &+\langle\tau c v_\theta\partial\Theta^n,\partial\Theta^n\rangle\\ &+(\beta\theta_0\nabla\cdot\partial\mathbf{U}^{s,n},\partial\Theta^n)+(\beta\theta_0\nabla\cdot\partial\mathbf{U}^{f,n},\partial\Theta^n)\\ &+(\tau\beta\theta_0\nabla\cdot\partial^2\mathbf{U}^{s,n},\partial\Theta^n)+(\tau\beta\theta_0\nabla\cdot\partial^2\mathbf{U}^{f,n},\partial\Theta^n)\\ &\le C(\|\mathbf{U}^{n+1}\|_0^2+\|\mathbf{U}^{n-1}\|_0^2+\|D_t\mathbf{U}^n\|_0^2+\|D_t\mathbf{U}^{n-1}\|_0^2\\ &+\|\Theta^{n+1}\|_0^2+\|\Theta^{n-1}\|_0^2+\|D_t\Theta^n\|_0^2+\|D_t\Theta^{n-1}\|_0^2)\\ &+(\mathbf{f}^n,\partial\mathbf{U}^n)-(q^n,\partial\Theta^n)+(\beta\Theta^n,\nabla\cdot\partial\mathbf{U}^{s,n})\\ &+(\beta_f\Theta^n,\nabla\cdot\partial\mathbf{U}^{f,n}),\ \ n=1,\cdots M.\end{aligned} \quad (37)$$

To obtain estimates for the last two terms on the left side of equation 37, we use the following discrete-time form of equation A-11 in Appendix A:

$$\begin{aligned}&(\tau\beta\theta_0\nabla\cdot\partial^2\mathbf{U}^{s,n},\partial\Theta^n)+(\tau\beta\theta_0\nabla\cdot\partial^2\mathbf{U}^{f,n},\partial\Theta^n)\\ &\ge\frac{C_\tau}{2\Delta t}\frac{1}{2}[(\mathcal{P}\partial^2\mathbf{U}^{n+1},\partial^2\mathbf{U}^{n+1})-(\mathcal{P}\partial^2\mathbf{U}^{n-1},\partial^2\mathbf{U}^{n-1})]\\ &+\frac{C_\tau}{2\Delta t}[\Lambda(D_t\mathbf{U}^n,D_t\mathbf{U}^n)-\Lambda(D_t\mathbf{U}^{n-1},D_t\mathbf{U}^{n-1})]\\ &+C_\tau\left[\left(\frac{\eta}{\kappa}\partial^2\mathbf{U}^{f,n},\partial^2\mathbf{U}^{f,n}\right)+\langle\mathcal{DS}(\partial^2\mathbf{U}^n),\mathcal{S}(\partial^2\mathbf{U}^n)\rangle\right.\\ &\left.-(\partial\mathbf{f}^{s,n},\partial^2\mathbf{U}^{s,n})-(\partial\mathbf{f}^{f,n},\partial^2\mathbf{U}^{s,n})\right],\ \ n=1,2,\cdots,M.\end{aligned} \quad (38)$$

Next, note that

$$\frac{\zeta}{4\Delta t}[\|D_t\mathbf{U}^n\|_0^2-\|D_t\mathbf{U}^{n-1}\|_0^2] \le C(\|D_t\mathbf{U}^n\|_0^2+\|D_t\mathbf{U}^{n-1}\|_0^2+\|\partial^2\mathbf{U}^n\|_0^2). \quad (39)$$

Then, we use equation 38 in equation 37 and add equation 39 to the resulting inequality to obtain

$$\begin{aligned}&\frac{1}{2\Delta t}[(\mathcal{P}D_t\mathbf{U}^n,D_t\mathbf{U}^n)-(\mathcal{P}D_t\mathbf{U}^{n-1},D_t\mathbf{U}^{n-1})]\\ &+\frac{C_\tau}{4\Delta t}[(\mathcal{P}\partial^2\mathbf{U}^{n+1},\partial^2\mathbf{U}^{n+1})-(\mathcal{P}\partial^2\mathbf{U}^{n-1},\partial^2\mathbf{U}^{n-1})]\\ &+\frac{1}{4\Delta t}[\Lambda_\zeta(\mathbf{U}^{n+1},\mathbf{U}^{n+1})-\Lambda_\zeta(\mathbf{U}^{n-1},\mathbf{U}^{n-1})]\\ &+\frac{1}{4\Delta t}[\Lambda(\mathbf{U}^n-\mathbf{U}^{n-1},\mathbf{U}^n-\mathbf{U}^{n-1})-\Lambda(\mathbf{U}^{n+1}-\mathbf{U}^n,\mathbf{U}^{n+1}-\mathbf{U}^n)]\\ &+\frac{C_\tau}{2\Delta t}[\Lambda_\zeta(D_t\mathbf{U}^n,Dt\mathbf{U}^n)-\Lambda_\zeta(D_t\mathbf{U}^{n-1},Dt\mathbf{U}^{n-1})]\\ &+\frac{1}{2\Delta t}[(\tau cD_t\Theta^n,D_t\Theta^n)-(\tau cD_t\Theta^{n-1},D_t\Theta^{n-1})]\\ &+\frac{1}{4\Delta t}(\|\gamma^{1/2}\Theta^{n+1}\|_1^2-\|\gamma^{1/2}\Theta^{n-1}\|_1^2)\\ &+\frac{1}{4\Delta t}[(\gamma\nabla(\Theta^n-\Theta^{n-1}),\nabla(\Theta^n-\Theta^{n-1}))-(\gamma\nabla(\Theta^{n+1}-\Theta^n), \nabla(\Theta^{n+1}-\Theta^n))]\\ &+\left(\frac{\eta}{\kappa}\partial\mathbf{U}^{f,n},\partial\mathbf{U}^{f,n}\right)+(c\partial\Theta^n,\partial\Theta^n)+\langle\mathcal{DS}(\partial\mathbf{U}^n),\mathcal{S}(\partial\mathbf{U}^n)\rangle\\ &+\langle\tau c v_\theta\partial\Theta^n,\partial\Theta^n\rangle\\ &+C_\tau\left[\left(\frac{\eta}{\kappa}\partial^2\mathbf{U}^{f,n},\partial^2\mathbf{U}^{f,n}\right)+\langle\mathcal{DS}(\partial^2\mathbf{U}^n),\mathcal{S}(\partial^2\mathbf{U}^n)\rangle\right]\\ &\le C(\|\mathbf{U}^{n+1}\|_0^2+\|\mathbf{U}^{n-1}\|_0^2+\|D_t\mathbf{U}^n\|_0^2+\|D_t\mathbf{U}^{n-1}\|_0^2+\|\partial^2\mathbf{U}^n\|_0^2\\ &+\|\Theta^{n+1}\|_0^2+\|\Theta^{n-1}\|_0^2+\|D_t\Theta^n\|_0^2+\|D_t\Theta^{n-1}\|_0^2)\\ &+(\mathbf{f}^n,\partial\mathbf{U}^n)-(q^n,\partial\Theta^n)+C_\tau[(\partial\mathbf{f}^{s,n},p^2\mathbf{U}^{s,n})+(\partial\mathbf{f}^{f,n},p^2\mathbf{U}^{s,n})]\\ &+(\beta\Theta^n,\nabla\cdot\partial\mathbf{U}^{s,n})+(\beta_f\Theta^n,\nabla\cdot\partial\mathbf{U}^{f,n})\\ &-(\beta\theta_0\nabla\cdot\partial\mathbf{U}^{s,n},\partial\Theta^n)-(\beta\theta_0\nabla\cdot\partial\mathbf{U}^{f,n},\partial\Theta^n)n=1,\cdots M.\end{aligned} \quad (40)$$

We multiply equation 40 by Δt and sum from $n=1$ to $n=N$, $N\le M$. Because

$$\frac{1}{4}\sum_{n=1}^{n=N}[\Lambda(\mathbf{U}^n-\mathbf{U}^{n-1},\mathbf{U}^n-\mathbf{U}^{n-1})-\Lambda(\mathbf{U}^{n+1}-\mathbf{U}^n,\mathbf{U}^{n+1}-\mathbf{U}^n)]$$
$$=-\frac{1}{4}(\Delta t)^2\Lambda(D_t\mathbf{U}^N,D_t\mathbf{U}^N)+\frac{1}{4}(\Delta t)^2\Lambda(D_t\mathbf{U}^0,D_t\mathbf{U}^0), \quad (41)$$

we obtain

$$\begin{aligned}
&\frac{1}{2}[(\mathcal{P}D_t\mathbf{U}^N,D_t\mathbf{U}^N)-(\mathcal{P}D_t\mathbf{U}^0,D_t\mathbf{U}^0)]\\
&+\frac{C_\tau}{4}[(\mathcal{P}\partial^2\mathbf{U}^{N+1},\partial^2\mathbf{U}^{N+1})+(\mathcal{P}\partial^2\mathbf{U}^N,\partial^2\mathbf{U}^N)\\
&-(\mathcal{P}\partial^2\mathbf{U}^0,\partial^2\mathbf{U}^0)-(\mathcal{P}\partial^2\mathbf{U}^1,\partial^2\mathbf{U}^1)]\\
&+\frac{1}{4}[\Lambda_\zeta(\mathbf{U}^{N+1},\mathbf{U}^{N+1})+\Lambda_\zeta(\mathbf{U}^N,\mathbf{U}^N)-\Lambda_\zeta(\mathbf{U}^1,\mathbf{U}^1)-\Lambda_\zeta(\mathbf{U}^0,\mathbf{U}^0)]\\
&+\frac{C_\tau}{4}[\Lambda_\zeta(D_t\mathbf{U}^N,D_t\mathbf{U}^N)-\Lambda_\zeta(D_t\mathbf{U}^0,D_t\mathbf{U}^0)]\\
&+\frac{1}{4}[\Lambda(\mathbf{U}^{N+1},\mathbf{U}^{N+1})+\Lambda(\mathbf{U}^N,\mathbf{U}^N)-\Lambda(\mathbf{U}^1,\mathbf{U}^1)-\Lambda(\mathbf{U}^0,\mathbf{U}^0)]\\
&-\frac{(\Delta t)^2}{4}\Lambda(D_t\mathbf{U}^N,D_t\mathbf{U}^N)+\frac{(\Delta t)^2}{4}\Lambda(D_t\mathbf{U}^0,D_t\mathbf{U}^0)\\
&+\frac{1}{2}[(\tau cD_t\Theta^N,D_t\Theta^N)-(\tau cD_t\Theta^0,D_t\Theta^0)]\\
&+\frac{1}{4}(\|\gamma^{1/2}\Theta^{N+1}\|_1^2+\|\gamma^{1/2}\Theta^N\|_1^2-\|\gamma^{1/2}\Theta^0\|_1^2-\|\gamma^{1/2}\Theta^1\|_1^2)\\
&-\frac{(\Delta t)^2}{4}\|\gamma^{1/2}\nabla D_t\Theta^N\|_0^2+\frac{(\Delta t)^2}{4}\|\gamma^{1/2}\nabla D_t\Theta^0\|_0^2\\
&+\sum_{n=1}^{n=N}\left(\frac{\eta}{\kappa}\partial\mathbf{U}^{f,n},\mathbf{U}^{f,n}\right)\Delta t+\sum_{n=1}^{n=N}(c\partial\Theta^n,\partial\Theta^n)\Delta t\\
&+C_\tau\sum_{n=1}^{n=N}\langle\mathcal{D}\mathcal{S}(\partial^2\mathbf{U}^n),\mathcal{S}(\partial^2\mathbf{U}^n)\rangle\Delta t\\
&+\sum_{n=1}^{n=N}C_\tau\left(\frac{\eta}{\kappa}\partial^2\mathbf{U}^{f,n},\partial^2\mathbf{U}^{f,n}\right)+\sum_{n=1}^{n=N}[\langle\mathcal{D}\mathcal{S}(\partial\mathbf{U}^n),\mathcal{S}(\partial\mathbf{U}^n)\rangle\\
&+\langle\tau c v_\theta\partial\Theta^n,\partial\Theta^n\rangle]\Delta t\\
&\le C\sum_{n=1}^{n=N}(\|\mathbf{f}^n\|_0^2+\|\partial f^n\|_0^2+\|q^n\|_0^2+\|\mathbf{U}^{n+1}\|_0^2\\
&+\|\mathbf{U}^{n-1}\|_0^2+\|D_t\mathbf{U}^n\|_0^2\\
&+\|D_t\mathbf{U}^{n-1}\|_0^2+\|\partial^2\mathbf{U}^n\|_0^2+\|\Theta^{n+1}\|_0^2+\|\Theta^{n-1}\|_0^2+\|D_t\Theta^n\|_0^2\\
&+\|D_t\Theta^{n-1}\|_0^2)\Delta t\\
&+\sum_{n=1}^{n=N}(\beta\Theta^n,\nabla\cdot\partial\mathbf{U}^{s,n})\Delta t+\sum_{n=1}^{n=N}(\beta_f\Theta^n,\nabla\cdot\partial\mathbf{U}^{f,n})\Delta t\\
&-\sum_{n=1}^{n=N}(\beta\theta_0\nabla\cdot\partial\mathbf{U}^{s,n},\partial\Theta^n)\Delta t-\sum_{n=1}^{n=N}(\beta\theta_0\nabla\cdot\partial\mathbf{U}^{f,n},\partial\Theta^n)\Delta t. \quad (42)
\end{aligned}$$

The last four terms in the right side of equation 42 can be bounded as follows:

$$\begin{aligned}
&\left|\sum_{n=1}^{n=N}(\beta\Theta^n,\nabla\cdot\partial\mathbf{U}^{s,n})\Delta t\right|+\left|\sum_{n=1}^{n=N}(\beta_f\Theta^n,\nabla\cdot\partial\mathbf{U}^{f,n})\Delta t\right|\\
&\left|\sum_{n=1}^{n=N}(\beta\theta_0\nabla\cdot\partial\mathbf{U}^{s,n},\partial\Theta^n)\Delta t\right|+\left|\sum_{n=1}^{n=N}(\beta\theta_0\nabla\cdot\partial\mathbf{U}^{f,n},\partial\Theta^n)\Delta t\right|\\
&\le C\sum_{n=1}^{n=N}(\|\Theta^n\|_0^2+\|\nabla\cdot\partial\mathbf{U}^{s,n}\|_0^2+\|\nabla\cdot\partial\mathbf{U}^n\|_0^2)\Delta t\\
&\le C\sum_{n=1}^{n=N}(\|\Theta^n\|_0^2+\|\partial\mathbf{U}^n\|_{\mathcal{V}}^2)\Delta t. \quad (43)
\end{aligned}$$

We use the bound equation 43 in equation 42 to obtain the estimate:

$$\begin{aligned}
&(\mathcal{P}D_t\mathbf{U}^N,D_t\mathbf{U}^N)+\frac{C_\tau}{4}[(\mathcal{P}\partial^2\mathbf{U}^{N+1},\partial^2\mathbf{U}^{N+1})+(\mathcal{P}\partial^2\mathbf{U}^N,\partial^2\mathbf{U}^N)]\\
&+\Lambda_\zeta(\mathbf{U}^{N+1},\mathbf{U}^{N+1})+\Lambda_\zeta(\mathbf{U}^N,\mathbf{U}^N)+C_\tau\Lambda_\zeta(Dt\mathbf{U}^N,Dt\mathbf{U}^N)\\
&+\Lambda(\mathbf{U}^{N+1},\mathbf{U}^{N+1})+\Lambda(\mathbf{U}^N,\mathbf{U}^N)-\frac{(\Delta t)^2}{4}\Lambda(D_t\mathbf{U}^N,D_t\mathbf{U}^N)\\
&+\frac{(\Delta t)^2}{4}\Lambda(D_t\mathbf{U}^0,D_t\mathbf{U}^0)+(\tau cD_t\Theta^N,D_t\Theta^N)\\
&+\|\gamma^{1/2}\Theta^{N+1}\|_1^2+\|\gamma^{1/2}\Theta^N\|_1^2\\
&-\frac{(\Delta t)^2}{4}\|\gamma^{1/2}D_t\Theta^N\|_1^2+\frac{(\Delta t)^2}{4}\|\gamma^{1/2}D_t\Theta^0\|_1^2\\
&+\sum_{n=1}^{n=N}\left(\frac{\eta}{\kappa}\partial\mathbf{U}^{f,n},\partial\mathbf{U}^{f,n}\right)\Delta t\\
&+\sum_{n=1}^{n=N}(c\partial\Theta^n,\partial\Theta^n)\Delta t+C_\tau\sum_{n=1}^{n=N}\langle\mathcal{D}\mathcal{S}(\partial^2\mathbf{U}^n),\mathcal{S}(\partial^2\mathbf{U}^n)\rangle\Delta t\\
&+C_\tau\sum_{n=1}^{n=N}\left(\frac{\eta}{\kappa}\partial^2\mathbf{U}^{f,n},\partial^2\mathbf{U}^{f,n}\right)\Delta t+\sum_{n=1}^{n=N}[\langle\mathcal{D}\mathcal{S}(\partial\mathbf{U}^n),\mathcal{S}(\partial\mathbf{U}^n)\rangle\\
&+\langle\tau c v_\theta\partial\Theta^n,\partial\Theta^n\rangle]\Delta t\\
&\le C(\|\mathbf{U}^0\|_{\mathcal{V}}^2+\|\mathbf{U}^1\|_{\mathcal{V}}^2+\|D_t\mathbf{U}^0\|_{\mathcal{V}}^2+\|\partial^2\mathbf{U}^0\|_0^2+\|\partial^2\mathbf{U}^1\|_0^2\\
&+\|\Theta^0\|_1^2+\|\Theta^1\|_1^2+\|D_t\Theta^0\|_0^2+\sum_{n=1}^{n=N}(\|\mathbf{f}^n\|_0^2+\|\partial f^n\|_0^2+\|q^n\|_0^2)\Delta t\\
&+C\sum_{n=1}^{n=N}(\|\mathbf{U}^{n+1}\|_0^2+\|\mathbf{U}^{n-1}\|_0^2+\|D_t\mathbf{U}^n\|_0^2+\|D_t\mathbf{U}^{n-1}\|_0^2+\|\partial^2\mathbf{U}^n\|_0^2\\
&+\|\Theta^{n+1}\|_0^2+\|\Theta^n\|_0^2+\|\Theta^{n-1}\|_0^2+\|D_t\Theta^n\|_0^2+\|D_t\Theta^{n-1}\|_0^2)\Delta t.
\end{aligned} \quad (44)$$

Next, note that there exist positive constants C_8, C_9 independent of h such that the following inverse hypothesis holds:

$$\begin{aligned}
&\Lambda(\mathbf{U}^N,\mathbf{U}^N)\le\xi^*(\mathcal{E})\|\tilde{\varepsilon}(D_t\mathbf{U}^N)\|_0^2\le C_8^2h^{-2}\|(D_t\mathbf{U}^N)\|_0^2,\\
&\|\gamma^{1/2}D_t\Theta^N\|_1^2\le\gamma^*C_9^2h^{-2}\|(D_t\Theta^N)\|_0^2. \quad (45)
\end{aligned}$$

In equation 45, the constants C_8 and C_9 have a factor that measures the quasiuniformity of $\mathcal{T}^h$ and $\xi^*(\mathcal{E})$ and γ^* denote the maximum eigenvalue of $\mathcal{E}$ and the maximum value of γ, respectively. Let $\xi_*(\mathcal{P})$ and $(\tau c)_*$ be the minimum eigenvalue of $\mathcal{P}$ and the minimum value of (τc), respectively. Hence,

$$\|\mathcal{P}^{1/2}D_t\mathbf{U}^N\|_0^2-\frac{(\Delta t)^2}{4}\Lambda(D_t\mathbf{U}^N,D_t\mathbf{U}^N)$$
$$\geq(\xi_*(\mathcal{P})-C_8^2h^{-2}\frac{(\Delta t)^2}{4}\xi^*(\mathcal{E}))\|D_t\mathbf{U}^N\|_0^2\geq\frac{1}{2}\xi_*(\mathcal{P})\|D_t\mathbf{U}^N\|_0^2,$$
$$\|(\tau c)^{1/2}D_t\Theta^N\|_0^2-\frac{(\Delta t)^2}{4}\|\gamma^{1/2}\Theta^N\|_1^2$$
$$\geq((\tau c)_*-\frac{(\Delta t)^2}{4}C_9^2\gamma^*h^{-2})\|D_t\Theta^N\|_0^2\geq\frac{1}{2}(\tau c)_*\|D_t\Theta^N\|_0^2, \quad (46)$$

provided that Δt and h satisfy the stability constraint:

$$\Delta t\leq\min\left(h\frac{\sqrt{2}}{C_8}\left(\frac{\xi_*(\mathcal{P})}{\xi^*(\mathcal{E})}\right)^{1/2},h\frac{\sqrt{2}}{C_9}\left(\frac{(\tau c)_*}{\gamma^*}\right)^{1/2}\right). \quad (47)$$

Hence, for Δt and h as in equation 47, from equation 44 and the fact that $\mathcal{P}$ is positive definite and Λ_ζ is $\mathcal{V}$-coercive, we obtain the inequality:

$$\frac{\xi_*(\mathcal{P})}{2}\|D_t\mathbf{U}^N\|_0^2+\|\partial^2\mathbf{U}^N\|_0^2+\|\mathbf{U}^N\|_{\mathcal{V}}^2$$
$$+\|D_t\mathbf{U}^N\|_{\mathcal{V}}^2+\frac{(\tau c)_*}{2}\|D_t\Theta^n\|_0^2+\|\Theta^n\|_1^2$$
$$\leq C(\|\mathbf{U}^0\|_{\mathcal{V}}^2+\|\mathbf{U}^1\|_{\mathcal{V}}^2+\|D_t\mathbf{U}^0\|_{\mathcal{V}}^2+\|\partial^2\mathbf{U}^0\|_0^2+\|\partial^2\mathbf{U}^1\|_0^2$$
$$+\|\Theta^0\|_1^2+\|\Theta^1\|_1^2+\|D_t\Theta^0\|_0^2$$
$$+\sum_{n=1}^{n=N}(\|\mathbf{f}^n\|_0^2+\|\partial f^n\|_0^2+\|q^n\|_0^2)\Delta t)$$
$$+C\sum_{n=1}^{n=N}(\|\mathbf{U}^{n+1}\|_0^2+\|\mathbf{U}^{n-1}\|_0^2+\|D_t\mathbf{U}^n\|_0^2$$
$$+\|D_t\mathbf{U}^{n-1}\|_0^2+\|\partial^2\mathbf{U}^n\|_0^2$$
$$+\|\Theta^{n+1}\|_0^2+\|\Theta^n\|_0^2+\|\Theta^{n-1}\|_0^2+\|D_t\Theta^n\|_0^2+\|D_t\Theta^{n-1}\|_0^2)\Delta t. \quad (48)$$

Finally, we apply Gronwall's lemma in equation 48 to conclude the validity of Theorem 3.

Theorem 3

Assume that the matrices $\mathcal{P}$ and $\mathcal{B}$ in equation 3 are positive definite and semidefinite, respectively, and that the matrix $\mathcal{E}$ in equation 20 is positive definite. Also, assume that the coefficients τ, c, γ, β, and β_f in equation 5 are bounded above and below by positive constants and that Δt and h satisfy the stability constraint (equation 47). Then, there exists a unique solution $(\mathbf{U}^n,\Theta^n)\in\mathcal{Z}^h$ of the discrete-time explicit FE procedure (equation 32), which satisfies the estimate:

$$\max_{1\leq N\leq M}(\|D_t\mathbf{U}^N\|_{\mathcal{V}}^2+\|\partial^2\mathbf{U}^N\|_0^2\|\mathbf{U}^N\|_{\mathcal{V}}^2+\|D_t\Theta^N\|_0^2+\|\Theta^N\|_1^2)$$
$$\leq C(\|\mathbf{U}^0\|_{\mathcal{V}}^2+\|\mathbf{U}^1\|_{\mathcal{V}}^2+\|D_t\mathbf{U}^0\|_{\mathcal{V}}^2+\|\partial^2\mathbf{U}^0\|_0^2+\|\partial^2\mathbf{U}^1\|_0^2$$
$$+\|\Theta^0\|_1^2+\|\Theta^1\|_1^2+\|D_t\Theta^0\|_0^2$$
$$+\sum_{n=1}^{n=N}(\|\mathbf{f}^n\|_0^2+\|\partial\mathbf{f}^n\|_0^2+\|q^n\|_0^2)\Delta t)\Delta t. \quad (49)$$

Remark

Note that the first and second time derivatives in the formulation of the time-discrete explicit FE procedure (equation 32) are discretized with errors on the order of $(\Delta t)^2$. Thus, the arguments for obtaining a priori error estimates for the time-continuous FE procedure (equation 28) can be used to conclude that the a priori errors associated with those discrete-time FE methods are on the order of $(\Delta t)^2+h$.

NUMERICAL EXPERIMENTS

The FE explicit procedure (equation 32) is implemented for the 1D case in an interval $\Omega=(0,L)$, where L = 116 m. Thus, the FE spaces for the solid, fluid, and temperature spatial representation are C^0 piecewise linear polynomials over a partition of Ω into subintervals of size $h=0.175$m. The time step is $dt=7.95\times10^{-3}$ ms.

In the 2D (3D) case, a partition $\mathcal{T}^h$ of the computational domain Ω consists of rectangular (parallelepipeds) elements of diameter bounded by h. The lowest order conforming FE spaces over $\mathcal{T}^h$ are C^0 piecewise continuous polynomials to represent the temperature and each component of the particle displacement vector, with the local degrees of freedom (DOFs) being the values at the vertices of the elements. However, the fluid displacement is represented using the vector part of the Raviart-Thomas-Nedelec space of zero order (Raviart and Thomas, 1977), with local DOF being the values at the midpoints of the edges (faces) of the elements.

Among the advantages of the FE method to simulate wave propagation in these types of media are the ability to fit complex subsurface geometries using variable mesh size as well as providing a natural way to include absorbing boundary conditions at artificial boundaries of the computational domain to eliminate spurious reflections.

The point source (f^s,f^f,q) located at $x_{\rm sour}=1$ m is defined as

$$f^j(x,t)=\frac{d}{dx}\delta_{x-x_{\rm sour}}g(t), \quad j=s,f, \quad (50)$$

$$q=\delta_{x-x_{\rm sour}}g(t), \quad (51)$$

with $g(t)$ being the waveform

$$g(t)=\cos[2\pi f_0(t-1.5/f_0)]\exp[-2f_0^2(t-1.5/f_0)^2] \quad (52)$$

and f_0 being the dominant frequency. However, values of the frame and fluid displacements and temperature are recorded at $x_r=59$ m. Table 1 shows the thermoporoelastic material properties (Carcione et al., 2019a).

The experiments analyzed the coupled and uncoupled cases considering the coupling coefficients β and β_f nonzero (coupled case) or null (uncoupled case).

The results of the plane-wave analysis presented in Carcione et al. (2019a) predict, at the dominant frequency of 150 Hz, the approximate values of the phase velocities of the P1, P2, and T waves: 2568, 827, and 420 m/s for the coupled case, and 2216, 665, and 604 m/s for the uncoupled case, respectively.

In the following figures, the labels P1, P2, and T indicate fast, slow, and temperature waves, respectively.

Figure 1 displays snapshots of temperature at 48 ms for the coupled and uncoupled cases and nonzero viscosity. In the uncoupled case, as expected, only a T wave is observed, whereas, in the coupled case, two waves are clearly seen, a T wave with a much larger amplitude than in the uncoupled case and a P1 wave due to the coupling effects. The P2 wave is not observed due to its diffusive behavior.

Figure 2 exhibits a frame snapshot at 48 ms, where now a T wave is only observable for the coupled case due to its very small amplitude. However, two P1 wavefronts can be seen, the one for the coupled case traveling at a faster speed than for the uncoupled one.

Time histories of the particle displacement of the frame for the uncoupled and coupled cases are shown in Figures 3 and 4 for null and nonzero viscosity, respectively.

In Figure 3 (null viscosity), two wave arrivals can be seen for the uncoupled case, which correspond to the classical P1 and P2 Biot waves, whereas the coupled case exhibits an additional T wave. Notice, for the coupled case, the earlier arrival times of the P1 and P2 waves as compared with those of the uncoupled case. This behavior is in agreement with the one presented in the plane-wave analysis of Carcione et al. (2019a), with the values of the measured arrival times being very close to those predicted by the theory.

Table 1. Material properties.

Grain bulk modulus (K_s)	35 GPa
Density (ρ_s)	2650 kg/m^3
Frame bulk modulus (K_m)	1.7 GPa
Shear modulus (μ_m)	1.885 GPa
Porosity (ϕ)	0.3
Permeability (κ)	1 D
Fluid bulk modulus (K_f)	2.4 GPa
Density (ρ_f)	1000 kg/m^3
Viscosity (η_f)	0.001 Pa · s
Thermoelasticity coefficient (β_f)	50,000 kg/(m s^2 K)
Bulk specific heat (c)	820 kg/(m s^2 K)
Thermoelasticity coefficient (β)	90,000 kg/(ms^2 K)
Absolute temperature (T_0)	300 K
Thermal conductivity (γ)	4.5×10^6 kg/m^3
Relaxation time (τ)	1.5×10^{-2} s

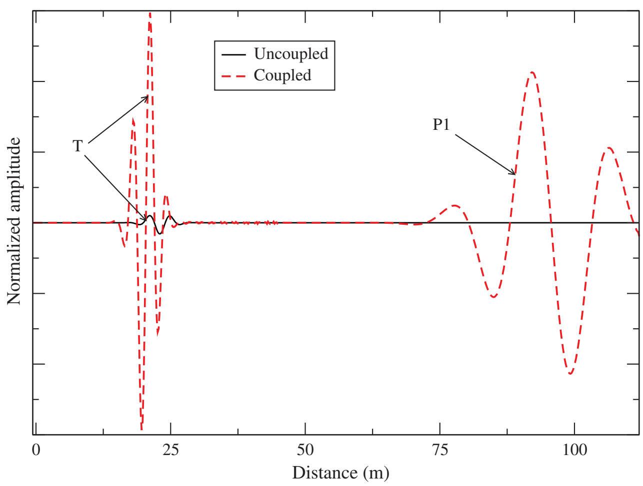

Figure 1. Snapshot of the temperature field at 48 ms for the uncoupled and coupled cases with nonzero viscosity.

Figure 4 (nonzero viscosity) shows only a P1 arrival for the uncoupled case, whereas, for the coupled case, three waves are seen to arrive at the receiver, corresponding to P1, P2, and T waves. First, as shown in Figure 3, the P1 wave arrives earlier in the coupled case as compared with the uncoupled one. A P2 arrival also is observed, which would not be present in the uncoupled case because of its diffusive behavior as a classical P2 Biot wave.

The next example considers a uniform medium stiffer and less permeable than the one in Table 1, with $K_m = 5.1$ GPa, $\mu = 5.565$, and $\kappa = 0.5$ D whereas the other properties are the same.

Figures 5 and 6 display frame snapshots for nonzero viscosity and uncoupled and coupled cases, respectively, for the medium in Table 1 as well as the stiffer and less permeable medium. Although Figure 5 only shows P1 wavefronts for both media, Figure 6 displays the P1 and T waves, which travel faster and with lower amplitude compared with the signals related to the medium defined in Table 1.

Next, we consider an inhomogeneous medium representing an interface using two intervals $I_1 = (0, I)$ and $I_2 = (I, L)$, with

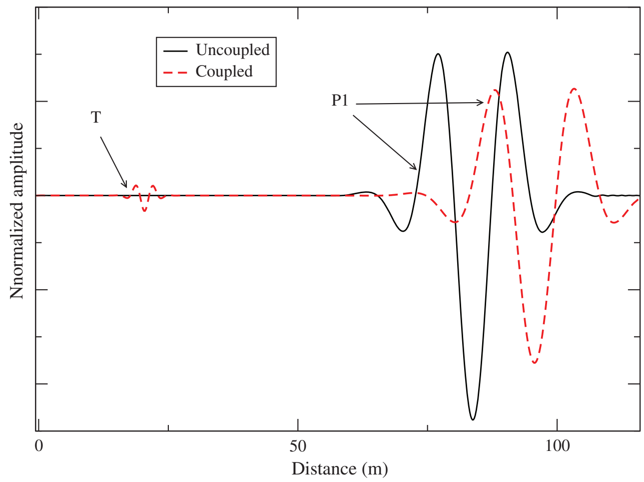

Figure 2. Snapshot of the particle displacement of the frame at 48 ms for the uncoupled and coupled cases with nonzero viscosity.

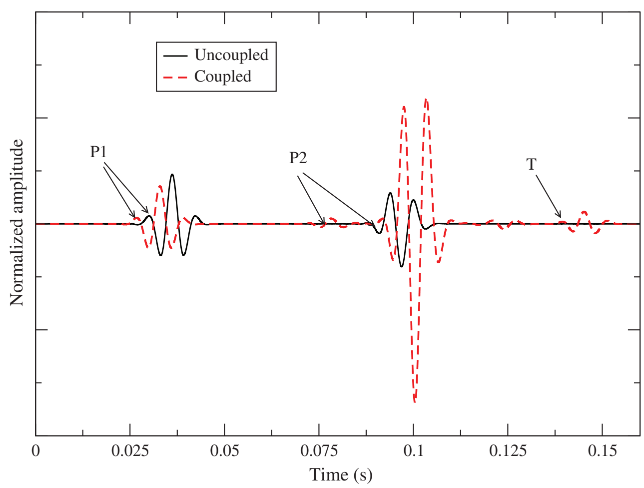

Figure 3. Time history of the particle displacement of the frame for null viscosity and uncoupled and coupled cases.

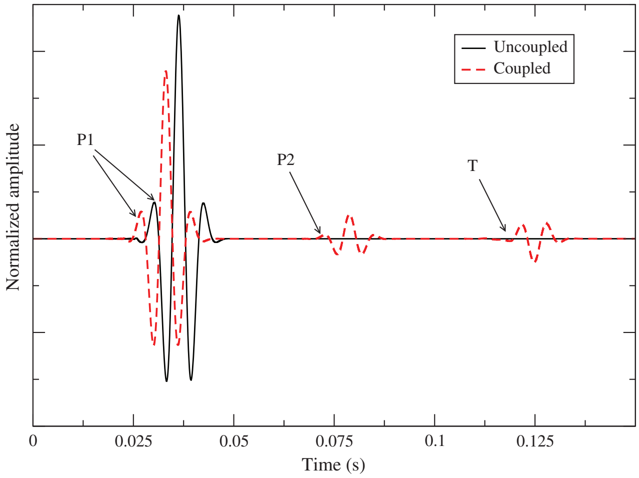

Figure 4. Time history of the particle displacement of the frame for nonzero viscosity and uncoupled and coupled cases.

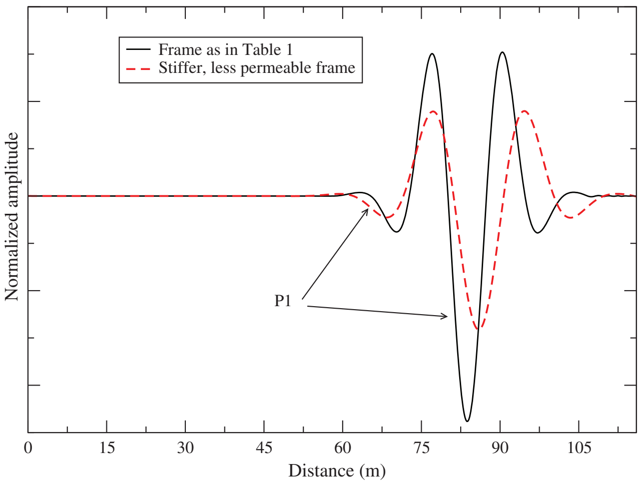

Figure 5. Frame snapshots for the uniform material in Table 1 and the harder and less permeable one. Uncoupled case and nonzero viscosity.

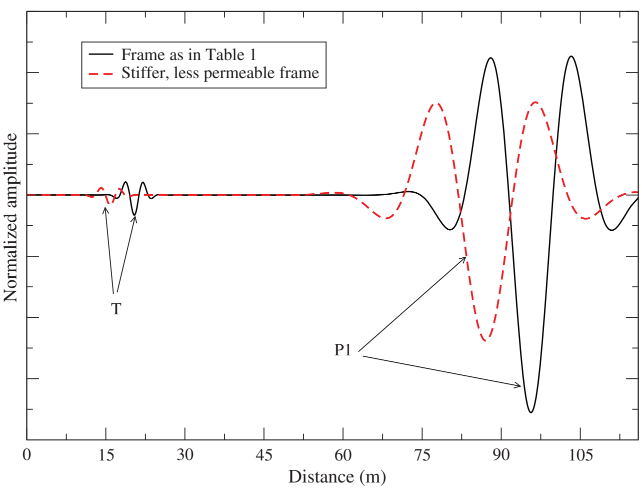

Figure 6. Frame snapshots for the uniform material in Table 1 and the harder and less permeable one. Coupled case and nonzero viscosity.

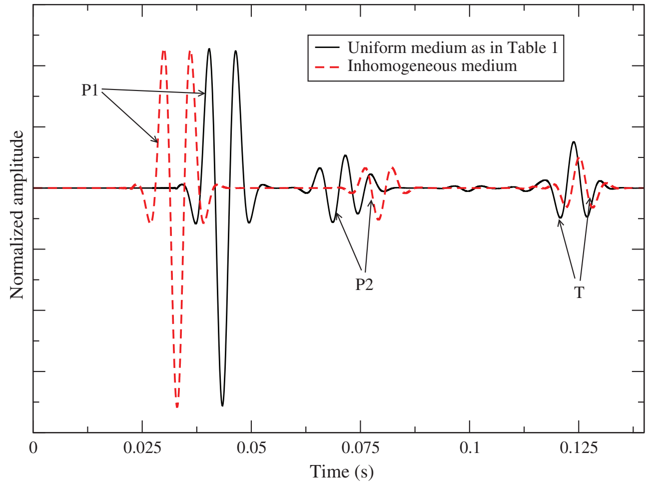

Figure 7. Time history of the particle displacement of the frame for the uniform medium in Table 1 and the inhomogeneous one representing an interface. Coupled case and nonzero viscosity.

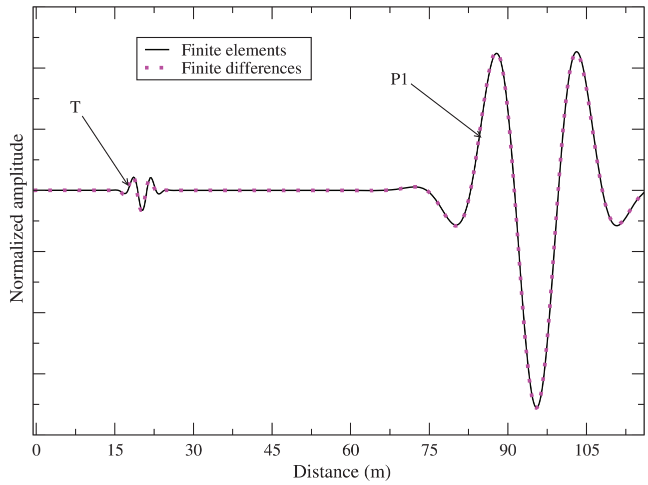

Figure 8. Comparison between frame snapshots of the FE procedure and the finite-differences algorithm for the uniform frame in Table 1. Coupled case and nonzero viscosity.

$I = 38$ m and $L = 116$ m. In the interval I_1, the material properties are those in Table 1, whereas, in interval I_2, the properties are those of the stiffer medium. Figure 7 displays time histories recorded at 84 m from the source, which are compared with those corresponding to the medium of Table 1. It is observed that the P1 wave arrives earlier in the inhomogeneous case, whereas the opposite occurs for the P2 and T waves.

Finally, Figure 8 compares the results of the FE procedure with those computed with a finite-difference algorithm. A very good agreement can be observed.

CONCLUSION

We solve the IBVP associated with the thermo-poroelasticity wave equation by applying continuous and discrete-time FE methods. A priori error estimates are derived, which are optimal for the assumed regularity of the solution. Furthermore, we present an explicit discrete-time FE method, analyze its stability, and establish the stability constraint. The numerical experiments illustrate the implementation of the novel explicit FE algorithm and study the behavior of all waves for the coupled and uncoupled cases. The proposed algorithms overcome the limitations of isothermal wave propagation.

ACKNOWLEDGMENTS

J. E. Santos is grateful to Alberto González Domínguez for his help. This work was partially funded by ANPCyT, Argentina (PICT 2015 1909) and the Universidad de Buenos Aires (UBACyT 20020190100236BA). The authors are grateful to the National Natural Science Foundation of China (grant no. 41974123), the Jiangsu Innovation and Entrepreneurship Plan, and the Jiangsu Province Science Fund for distinguished young scholars (grant no. BK20200021).

DATA AND MATERIALS AVAILABILITY

Data associated with this research are available and can be obtained by contacting the corresponding author.

APPENDIX A

PROOF OF THEOREMS

Proof of Theorem 1

We choose $\mathbf{v} = \dot{\mathbf{U}}, w = \dot{\Theta}$ in equation 28 to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}[(\mathcal{P}\dot{\mathbf{U}},\dot{\mathbf{U}}) + \Lambda(\mathbf{U},\mathbf{U}) + (\tau c\dot{\Theta},\dot{\Theta}) + (\gamma\nabla\Theta,\nabla\Theta)]\\ &\quad + \left(\frac{\eta}{\kappa}\dot{\mathbf{U}}^f,\dot{\mathbf{U}}^f\right)\\ &\quad - (\beta\Theta,\nabla\cdot\dot{\mathbf{U}}^s) - (\beta_f\Theta,\nabla\cdot\dot{\mathbf{U}}^f) + (c\dot{\Theta},\dot{\Theta})\\ &\quad + (\beta\theta_0\nabla\cdot\dot{\mathbf{U}}^s,\dot{\Theta}) + (\beta\theta_0\nabla\cdot\dot{\mathbf{U}}^f,\dot{\Theta}) + \langle\mathcal{DS}(\dot{\mathbf{U}}),\mathcal{S}(\dot{\mathbf{U}})\rangle\\ &\quad + \langle\tau c v_\theta\dot{\Theta},\dot{\Theta}\rangle + (\tau\beta\theta_0\nabla\cdot\ddot{\mathbf{U}}^s,\dot{\Theta}) + (\tau\beta\theta_0\nabla\cdot\ddot{\mathbf{U}}^f,\dot{\Theta})\\ &\quad = (\mathbf{f}^s,\dot{\mathbf{U}}^s) + (\mathbf{f}^f,\dot{\mathbf{U}}^f) - (q,\dot{\Theta}),\quad t\in J. \qquad \text{(A-1)}\end{aligned}$$

To handle the last two terms on the left side of equation A-1, we take the time derivative in equation 28 and choose $w = 0$ to obtain

$$\begin{aligned}&(\mathcal{P}\ddot{\mathbf{U}}^s,\mathbf{v}) + \Lambda(\dot{\mathbf{U}},\mathbf{v}) - (\beta\dot{\Theta},\nabla\cdot\mathbf{v}^s) - (\beta_f\dot{\Theta},\nabla\cdot\mathbf{v}^f)\\ &+ \left(\frac{\eta}{\kappa}\ddot{\mathbf{U}}^f,\mathbf{v}^f\right) + \langle\mathcal{DS}(\ddot{\mathbf{U}}),\mathcal{S}(\mathbf{v})\rangle = (\dot{\mathbf{f}}^s,\mathbf{v}^s) + (\dot{\mathbf{f}}^f,\mathbf{v}^f). \qquad \text{(A-2)}\end{aligned}$$

We choose $\mathbf{v}^s = \ddot{\mathbf{U}}^s, \mathbf{v}^f = 0$ in equation A-2 to obtain

$$\begin{aligned}&(\mathcal{P}\ddot{\mathbf{U}},(\ddot{\mathbf{U}}^s,0)) + \Lambda(\dot{\mathbf{U}},(\ddot{\mathbf{U}}^s,0)) - (\beta\dot{\Theta},\nabla\cdot\ddot{\mathbf{U}}^s)\\ &+ \langle\mathcal{DS}(\ddot{\mathbf{U}}),\mathcal{S}(\ddot{\mathbf{U}}^s,0)\rangle = (\dot{\mathbf{f}}^s,\ddot{\mathbf{U}}^s). \qquad \text{(A-3)}\end{aligned}$$

Also, the choice $\mathbf{v}^s = 0$ and $\mathbf{v}^f = \ddot{\mathbf{U}}^f$ in equation A-2 yields

$$\begin{aligned}&(\mathcal{P}\ddot{\mathbf{U}},(0,\ddot{\mathbf{U}}^f)) + \Lambda(\dot{\mathbf{U}},(0,\ddot{\mathbf{U}}^f)) - (\beta_f\dot{\Theta},\nabla\cdot\ddot{\mathbf{U}}^f) + \left(\frac{\eta}{\kappa}\ddot{\mathbf{U}}^f,\ddot{\mathbf{U}}^f\right)\\ &+ \langle\mathcal{DS}(\ddot{\mathbf{U}}),\mathcal{S}(0,\ddot{\mathbf{U}}^f)\rangle = (\dot{\mathbf{f}}^f,\ddot{\mathbf{U}}^f). \qquad \text{(A-4)}\end{aligned}$$

Set

$$C_\tau = \inf_{x\in\Omega}(\tau\theta_0). \qquad \text{(A-5)}$$

Then, from equation A-3,

$$\begin{aligned}&(\tau\beta\theta_0\nabla\cdot\ddot{\mathbf{U}}^s,\dot{\Theta}) \geq C_\tau(\beta\dot{\Theta},\nabla\cdot\ddot{\mathbf{U}}^s)\\ &= C_\tau[(\mathcal{P}\ddot{\mathbf{U}},(\ddot{\mathbf{U}}^s,0)) + \Lambda(\dot{\mathbf{U}},(\ddot{\mathbf{U}}^s,0))\\ &+ \langle\mathcal{DS}(\ddot{\mathbf{U}}),\mathcal{S}(\ddot{\mathbf{U}}^s,0)\rangle - (\dot{\mathbf{f}}^s,\ddot{\mathbf{U}}^s)]. \qquad \text{(A-6)}\end{aligned}$$

Also, from equation A-4,

$$\begin{aligned}&(\tau\beta\theta_0\nabla\cdot\ddot{\mathbf{U}}^f,\dot{\Theta}) \geq C_\tau(\beta\dot{\Theta},\nabla\cdot\ddot{\mathbf{U}}^f)\\ &= C_\tau[(\mathcal{P}\ddot{\mathbf{U}},(0,\ddot{\mathbf{U}}^f)) + \Lambda(\dot{\mathbf{U}},(0,\ddot{\mathbf{U}}^f))\\ &+ \left(\frac{\eta}{\kappa}\ddot{\mathbf{U}}^f,\ddot{\mathbf{U}}^f\right) + \langle\mathcal{DS}(\ddot{\mathbf{U}}),\mathcal{S}(0,\ddot{\mathbf{U}}^f)\rangle - (\dot{\mathbf{f}}^f,\ddot{\mathbf{U}}^f)]. \qquad \text{(A-7)}\end{aligned}$$

Next, we use that

$$C_\tau(\mathcal{P}\ddot{\mathbf{U}},(\ddot{\mathbf{U}}^s,0)) + C_\tau(\mathcal{P}\ddot{\mathbf{U}},(0,\ddot{\mathbf{U}}^f)) = C_\tau(\mathcal{P}\ddot{\mathbf{U}},\ddot{\mathbf{U}}), \qquad \text{(A-8)}$$

$$C_\tau\Lambda(\dot{\mathbf{U}},(\ddot{\mathbf{U}}^s,0)) + C_\tau\Lambda(\dot{\mathbf{U}},(0,\ddot{\mathbf{U}}^f)) = C_\tau\Lambda(\dot{\mathbf{U}},\ddot{\mathbf{U}}), \qquad \text{(A-9)}$$

and

$$\begin{aligned}&C_\tau\langle\mathcal{DS}(\ddot{\mathbf{U}}),\mathcal{S}(\ddot{\mathbf{U}}^s,0)\rangle + C_\tau\langle\mathcal{DS}(\ddot{\mathbf{U}}),\mathcal{S}(0,\ddot{\mathbf{U}}^f)\rangle\\ &= C_\tau\langle\mathcal{DS}(\ddot{\mathbf{U}}),\mathcal{S}(\ddot{\mathbf{U}})\rangle. \qquad \text{(A-10)}\end{aligned}$$

Hence,

$$\begin{aligned}&(\tau\beta\theta_0\nabla\cdot\ddot{\mathbf{U}}^s,\dot{\Theta}) + (\tau\beta\theta_0\nabla\cdot\ddot{\mathbf{U}}^f,\dot{\Theta})\\ &\geq \frac{1}{2}C_\tau\left[\frac{d}{dt}[(\mathcal{P}\dot{\mathbf{U}},\dot{\mathbf{U}}) + \Lambda(\dot{\mathbf{U}},\dot{\mathbf{U}})] + \langle\mathcal{DS}(\ddot{\mathbf{U}}),\mathcal{S}(\ddot{\mathbf{U}})\rangle\right.\\ &\left.+ \left(\frac{\eta}{\kappa}\ddot{\mathbf{U}}^f,\ddot{\mathbf{U}}^f\right) - (\dot{\mathbf{f}}^s,\ddot{\mathbf{U}}^s) - (\dot{\mathbf{f}}^f,\ddot{\mathbf{U}}^f)\right]. \qquad \text{(A-11)}\end{aligned}$$

Using equation A-11 in equation A-1, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}[(\mathcal{P}\dot{\mathbf{U}},\dot{\mathbf{U}}) + \Lambda(\mathbf{U},\mathbf{U}) + (\tau c\dot{\Theta},\dot{\Theta}) + (\gamma\nabla\Theta,\nabla\Theta)] + \left(\frac{\eta}{\kappa}\dot{\mathbf{U}}^f,\dot{\mathbf{U}}^f\right)\\ &- (\beta\Theta,\nabla\cdot\dot{\mathbf{U}}^s) - (\beta_f\Theta,\nabla\cdot\dot{\mathbf{U}}^f) + (c\dot{\Theta},\dot{\Theta})\\ &+ (\beta\theta_0\nabla\cdot\dot{\mathbf{U}}^s,\dot{\Theta}) + (\beta\theta_0\nabla\cdot\dot{\mathbf{U}}^f,\dot{\Theta}) + \langle\mathcal{DS}(\dot{\mathbf{U}}),\mathcal{S}(\dot{\mathbf{U}})\rangle + \langle\tau c v_\theta\dot{\Theta},\dot{\Theta}\rangle\\ &+ \frac{C_\tau}{2}\frac{d}{dt}[(\mathcal{P}\dot{\mathbf{U}},\dot{\mathbf{U}}) + \Lambda(\dot{\mathbf{U}},\dot{\mathbf{U}})] + \frac{C_\tau}{2}\left[\langle\mathcal{DS}(\ddot{\mathbf{U}}),\mathcal{S}(\ddot{\mathbf{U}})\rangle + \left(\frac{\eta}{\kappa}\ddot{\mathbf{U}}^f,\ddot{\mathbf{U}}^f\right)\right.\\ &\left.- (\dot{\mathbf{f}}^s,\ddot{\mathbf{U}}^s) - (\dot{\mathbf{f}}^f,\ddot{\mathbf{U}}^f)\right]\\ &\leq (\mathbf{f}^s,\dot{\mathbf{U}}^s) + (\mathbf{f}^f,\dot{\mathbf{U}}^f) - (q,\dot{\Theta}),\ t\in J. \qquad \text{(A-12)}\end{aligned}$$

Next, note that, using the Gärding inequality (equation 21), we can choose ζ to define the bilinear form:

$$\Lambda_\zeta(\mathbf{v},\mathbf{v}) = \Lambda(\mathbf{v},\mathbf{v}) + \zeta(\mathbf{v},\mathbf{v}) \qquad \text{(A-13)}$$

such that Λ_ζ is $\mathcal{V}$-coercive, i.e.,

$$\Lambda_\zeta(\mathbf{v},\mathbf{v}) \geq C_2\|\mathbf{v}\|_{\mathcal{V}}^2. \tag{A-14}$$

Thus, adding to equation A-12 the inequalities

$$\zeta\frac{d}{dt}\|\mathbf{U}\|_0^2 \leq \zeta(\|\mathbf{U}\|_0^2+\|\dot{\mathbf{U}}\|_0^2),$$

$$\frac{d}{dt}\|\gamma^{1/2}\theta\|_0^2 \leq (\|\gamma^{1/2}\theta\|_0^2+\|\gamma^{1/2}\dot{\theta}\|_0^2), \tag{A-15}$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[(\mathcal{P}\dot{\mathbf{U}},\dot{\mathbf{U}})+C_\tau(\mathcal{P}\ddot{\mathbf{U}},\ddot{\mathbf{U}})+\Lambda_\zeta(\mathbf{U},\mathbf{U})+C_\tau\Lambda_\zeta(\dot{\mathbf{U}},\dot{\mathbf{U}})\\
&\quad+\|(\tau c)^{1/2}\dot{\Theta}\|_0^2+\|\gamma^{1/2}\Theta\|_1^2]\\
&\quad+\left(\frac{\eta}{\kappa}\dot{\mathbf{U}}^f,\dot{\mathbf{U}}^f\right)+(c\dot{\Theta},\dot{\Theta})+\langle\mathcal{D}\mathcal{S}(\dot{\mathbf{U}}),\mathcal{S}(\dot{\mathbf{U}})\rangle+\langle\tau c v_\theta\dot{\Theta},\dot{\Theta}\rangle\\
&\quad+C_\tau\left[\langle\mathcal{D}\mathcal{S}(\ddot{\mathbf{U}}),\mathcal{S}(\ddot{\mathbf{U}})\rangle+\left(\frac{\eta}{\kappa}\ddot{\mathbf{U}}^f,\ddot{\mathbf{U}}^f\right)\right]\\
&\leq C(\|\mathbf{f}\|_0^2+\|q\|_0^2+\|\mathbf{U}\|_0^2+\|\dot{\mathbf{U}}\|_0^2+\|\Theta\|_0^2+\|\dot{\Theta}\|_0^2)\\
&\quad+(\beta\Theta,\nabla\cdot\dot{\mathbf{U}}^s)+(\beta_f\Theta,\nabla\cdot\dot{\mathbf{U}}^f)\\
&\quad-(\beta\theta_0\nabla\cdot\dot{\mathbf{U}}^s,\dot{\Theta})-(\beta\theta_0\nabla\cdot\dot{\mathbf{U}}^f,\dot{\Theta})+C_\tau[(\dot{\mathbf{f}}^s,\ddot{\mathbf{U}}^s)\\
&\quad+(\dot{\mathbf{f}}^f,\ddot{\mathbf{U}}^f)],\quad t\in J.
\end{aligned} \tag{A-16}$$

Next, note that the integrals in time of the last six terms on the right side of equation A-16 can be bounded as follows:

$$\begin{aligned}
&\left|\int_0^t(\beta\Theta,\nabla\cdot\dot{\mathbf{U}}^s)(s)\right|+\left|\int_0^t(\beta_f\Theta,\nabla\cdot\dot{\mathbf{U}}^f)(s)ds\right|\\
&\leq C\int_0^t[\|\Theta(s)\|_0^2+\|\dot{\mathbf{U}}(s)\|_{\mathcal{V}}^2]ds,\\
&\left|\int_0^t(\beta\theta_0\nabla\cdot\dot{\mathbf{U}}^s,\dot{\Theta})(s)ds\right|+\left|\int_0^t(\beta\theta_0\nabla\cdot\dot{\mathbf{U}}^f,\dot{\Theta})(s)ds\right|\\
&\leq C\int_0^t[\|\dot{\Theta}(s)\|_0^2+\|\dot{\mathbf{U}}(s)\|_{\mathcal{V}}^2]ds,\\
&\left|\int_0^t(\dot{\mathbf{f}}^s,\ddot{\mathbf{U}}^s)(s)ds\right|+\left|\int_0^t(\dot{\mathbf{f}}^f,\ddot{\mathbf{U}}^f)(s)ds\right|\\
&\leq C\int_0^t[\|\dot{\mathbf{f}}(s)\|_0^2+\|\ddot{\mathbf{U}}(s)\|_0^2]ds.
\end{aligned} \tag{A-17}$$

Thus, integration in time of equations A-16 and A-14 yields

$$\begin{aligned}
&(\mathcal{P}\dot{\mathbf{U}},\dot{\mathbf{U}})(t)+(\mathcal{P}\ddot{\mathbf{U}},\ddot{\mathbf{U}})(t)+\|\mathbf{U}\|_{\mathcal{V}}^2+\|\dot{\mathbf{U}}\|_{\mathcal{V}}^2\\
&\quad+\|(\tau c)^{1/2}\dot{\Theta}(t)\|_0^2+\|\gamma^{1/2}\Theta(t)\|_1^2\\
&\quad+\int_0^t\left[\left(\frac{\eta}{\kappa}\dot{\mathbf{U}}^f,\dot{\mathbf{U}}^f\right)(s)+(c\dot{\Theta},\dot{\Theta})(s)+\langle\mathcal{D}\mathcal{S}(\dot{\mathbf{U}}),\mathcal{S}(\dot{\mathbf{U}})\rangle(s)\right.\\
&\quad+\langle\tau c v_\theta\dot{\Theta},\dot{\Theta}\rangle(s)+C_\tau\Big(\langle\mathcal{D}\mathcal{S}(\ddot{\mathbf{U}}),\mathcal{S}(\ddot{\mathbf{U}})\rangle(s)\\
&\quad\left.+\left(\frac{\eta}{\kappa}\ddot{\mathbf{U}}^f,\ddot{\mathbf{U}}^f\right)(s)\Big)\right]ds\\
&\leq C\int_0^t(\|\mathbf{f}(s)\|_0^2+\|q(s)\|_0^2)ds\\
&\quad+C(\|\mathbf{U}(0)\|_{\mathcal{V}}^2+\|\dot{\mathbf{U}}(0)\|_{\mathcal{V}}^2+\|\dot{\mathbf{U}}(0)\|_0^2+\|\ddot{\mathbf{U}}(0)\|_0^2\\
&\quad+\|\dot{\Theta}(0)\|_0^2+\|\Theta(0)\|_1^2)\\
&\quad+C\int_0^t(\|\mathbf{U}(s)\|_0^2+\|\dot{\mathbf{U}}(s)\|_0^2+\|\ddot{\mathbf{U}}(s)\|_0^2+\|\dot{\mathbf{U}}(s)\|_{\mathcal{V}}^2\\
&\quad+\|\Theta(s)\|_1^2+\|\dot{\Theta}(s)\|_0^2),\quad t\in J.
\end{aligned} \tag{A-18}$$

Because all integral terms on the left side of equation A-18 are nonnegative, the matrix $\mathcal{P}$ is positive definite and the coefficients τ, c and γ are bounded below by positive constants, apply Gronwall's lemma in equation A-16 to obtain the conclusion of Theorem 1.

Proof of Theorem 2

From equations 18 and 28, we see that $\mathbf{E}\mathbf{u}=(\mathbf{E}\mathbf{u}^s,\mathbf{E}\mathbf{u}^f)=(\mathbf{u}^s-\mathbf{U}^s,\mathbf{u}^f-\mathbf{U}^f)$ and $\mathbf{E}\theta=\theta-\Theta$ satisfy the equation:

$$\begin{aligned}
&(\mathcal{P}\mathbf{E}\ddot{\mathbf{u}},\mathbf{v})+\left(\frac{\eta}{\kappa}\mathbf{E}\dot{\mathbf{u}}^f,\mathbf{v}^f\right)+\Lambda(\mathbf{E}\mathbf{u},\mathbf{v})\\
&\quad-(\beta\mathbf{E}\theta,\nabla\cdot\mathbf{v}^s)-(\beta_f\mathbf{E}\theta,\nabla\cdot\mathbf{v}^f)\\
&\quad+(\tau c\mathbf{E}\ddot{\theta},w)+(c\mathbf{E}\dot{\theta},w)+(\gamma\nabla\mathbf{E}\theta,\nabla w)+(\beta\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s,w)\\
&\quad+(\beta\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f,w)+(\tau\beta\theta_0\nabla\cdot\mathbf{E}\ddot{\mathbf{u}}^s,w)+(\tau\beta\theta_0\nabla\cdot\mathbf{E}\ddot{\mathbf{u}}^f,w)\\
&\quad+\langle\mathcal{D}\mathcal{S}(\mathbf{E}\dot{\mathbf{u}}),\mathcal{S}(\mathbf{v})\rangle+\langle\tau c v_\theta\mathbf{E}\dot{\theta},w\rangle=0,\quad t\in J.
\end{aligned} \tag{A-19}$$

We choose $\mathbf{v}^s=\mathbf{E}\dot{\mathbf{u}}^s+\Pi^{(2)}\dot{\mathbf{u}}^s-\dot{\mathbf{u}}^s$, $\mathbf{v}^f=\mathbf{E}\dot{\mathbf{u}}^f+Q\dot{\mathbf{u}}^f-\dot{\mathbf{u}}^f$, $w=\mathbf{E}\dot{\theta}+\Pi\dot{\theta}-\dot{\theta}$ in equation A-19 to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[(\mathcal{P}\mathbf{E}\dot{\mathbf{u}},\mathbf{E}\dot{\mathbf{u}})+\Lambda(\mathbf{E}\mathbf{u},\mathbf{E}\mathbf{u})+(\tau c\mathbf{E}\dot{\theta},\mathbf{E}\dot{\theta})+(\gamma\nabla\mathbf{E}\theta,\nabla\mathbf{E}\theta)]\\
&\quad+\left(\frac{\eta}{\kappa}\mathbf{E}\dot{\mathbf{u}}^f,\mathbf{E}\dot{\mathbf{u}}^f\right)-(\beta\mathbf{E}\theta,\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s)-(\beta_f\mathbf{E}\theta,\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f)\\
&\quad+(c\mathbf{E}\dot{\theta},\mathbf{E}\dot{\theta})+\langle\mathcal{D}\mathcal{S}(\mathbf{E}\dot{\mathbf{u}}),\mathcal{S}(\mathbf{E}\dot{\mathbf{u}})\rangle+\langle\tau c v_\theta\mathbf{E}\dot{\theta},\mathbf{E}\dot{\theta}\rangle\\
&\quad+(\beta\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s,\mathbf{E}\dot{\theta})+(\beta\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f,\mathbf{E}\dot{\theta})+(\tau\beta\theta_0\nabla\cdot\mathbf{E}\ddot{\mathbf{u}}^s,\mathbf{E}\dot{\theta})+(\tau\beta\theta_0\nabla\cdot\mathbf{E}\ddot{\mathbf{u}}^f,\mathbf{E}\dot{\theta})\\
&=(\mathcal{P}\mathbf{E}\ddot{\mathbf{u}},(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f))+\left(\frac{\eta}{\kappa}\mathbf{E}\dot{\mathbf{u}}^f,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f\right)\\
&\quad+\Lambda(\mathbf{E}\mathbf{u},(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f))\\
&\quad-(\beta\mathbf{E}\theta,\nabla\cdot(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s))-(\beta_f\mathbf{E}\theta,\nabla\cdot(Q\dot{\mathbf{u}}^f-\dot{\mathbf{u}}^f))\\
&\quad+(\tau c\mathbf{E}\ddot{\theta},\Pi\dot{\theta}-\dot{\theta})+(c\mathbf{E}\dot{\theta},\dot{\theta}-\Pi\dot{\theta})+(\gamma\nabla\mathbf{E}\theta,\nabla(\Pi\dot{\theta}-\dot{\theta}))+(\beta\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s,\Pi\dot{\theta}-\dot{\theta})\\
&\quad+(\beta\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f,\Pi\dot{\theta}-\dot{\theta})+(\tau\beta\theta_0\nabla\cdot\mathbf{E}\ddot{\mathbf{u}}^s,\Pi\dot{\theta}-\dot{\theta})+(\tau\beta\theta_0\nabla\cdot\mathbf{E}\ddot{\mathbf{u}}^f,\Pi\dot{\theta}-\dot{\theta})\\
&\quad+\langle\mathcal{D}\mathcal{S}(\mathbf{E}\dot{\mathbf{u}}),\mathcal{S}(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f)\rangle+\langle\tau c v_\theta\mathbf{E}\dot{\theta},\Pi\dot{\theta}-\dot{\theta}\rangle,\quad t\in J.
\end{aligned} \tag{A-20}$$

The last two terms on the left side of equation A-20 can be handled by taking the time derivative in equation A-19 and choosing $w=0$, $\mathbf{v}^s=\mathbf{E}\ddot{\mathbf{u}}^s, \mathbf{v}^f=0$ and $\mathbf{v}^s=0, \mathbf{v}^f=\mathbf{E}\ddot{\mathbf{u}}^f$ in the resulting equation. Then, the argument leading to equation A-11 yields the inequality:

$$(\tau\beta\theta_0\nabla\cdot\mathbf{E}\ddot{\mathbf{u}}^s,\dot{\Theta})+(\tau\beta\theta_0\nabla\cdot\mathbf{E}\ddot{\mathbf{u}}^f,\dot{\Theta}) \geq \frac{1}{2}C_\tau\frac{d}{dt}\Big[(\mathcal{P}\mathbf{E}\ddot{\mathbf{u}},\mathbf{E}\ddot{\mathbf{u}})+\Lambda(\mathbf{E}\dot{\mathbf{u}},\mathbf{E}\dot{\mathbf{u}})+\langle\mathcal{D}\mathcal{S}(\mathbf{E}\ddot{\mathbf{u}}),\mathcal{S}(\mathbf{E}\ddot{\mathbf{u}})\rangle + \left(\frac{\eta}{\kappa}\mathbf{E}\ddot{\mathbf{u}}^f,\mathbf{E}\ddot{\mathbf{u}}^f\right)\Big]. \quad \text{(A-21)}$$

We use equation A-21 in equation A-20 and add to the resulting equation the inequalities

$$\zeta\frac{d}{dt}\|\mathbf{E}\mathbf{u}\|_0^2\leq\zeta(\|\mathbf{E}\mathbf{u}\|_0^2+\|\mathbf{E}\dot{\mathbf{u}}\|_0^2),$$
$$\frac{d}{dt}\|\gamma^{1/2}\mathbf{E}\theta\|_0^2\leq(\|\gamma^{1/2}\mathbf{E}\theta\|_0^2+\|\gamma^{1/2}\mathbf{E}\dot{\theta}\|_0^2) \quad \text{(A-22)}$$

to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}[(\mathcal{P}\mathbf{E}\dot{\mathbf{u}},\mathbf{E}\dot{\mathbf{u}})+C_\tau(\mathcal{P}\mathbf{E}\ddot{\mathbf{u}},\mathbf{E}\ddot{\mathbf{u}})+\Lambda_\zeta(\mathbf{E}\mathbf{u},\mathbf{E}\mathbf{u})+C_\tau\Lambda_\zeta(\mathbf{E}\dot{\mathbf{u}},\mathbf{E}\dot{\mathbf{u}})\\&+(\tau c\mathbf{E}\dot{\theta},\mathbf{E}\dot{\theta})+(\gamma\mathbf{E}\theta,\mathbf{E}\theta)+(\gamma\nabla\mathbf{E}\theta,\nabla\mathbf{E}\theta)]\\&+\left(\frac{\eta}{\kappa}\mathbf{E}\dot{\mathbf{u}}^f,\mathbf{E}\dot{\mathbf{u}}^f\right)+(c\mathbf{E}\dot{\theta},\mathbf{E}\dot{\theta})+\langle\mathcal{D}\mathcal{S}(\mathbf{E}\dot{\mathbf{u}}),\mathcal{S}(\mathbf{E}\dot{\mathbf{u}})\rangle+\langle\tau c v_\theta\mathbf{E}\dot{\theta},\mathbf{E}\dot{\theta}\rangle\\&+C_\tau\left[\langle\mathcal{D}\mathcal{S}(\mathbf{E}\ddot{\mathbf{u}}),\mathcal{S}(\mathbf{E}\ddot{\mathbf{u}})\rangle+\left(\frac{\eta}{\kappa}\mathbf{E}\ddot{\mathbf{u}}^f,\mathbf{E}\ddot{\mathbf{u}}^f\right)\right]\\&\leq C(\|\mathbf{E}\mathbf{u}\|_0^2+\|\mathbf{E}\dot{\mathbf{u}}\|_0^2+\|\mathbf{E}\theta\|_0^2+\|\mathbf{E}\dot{\theta}\|_0^2)+(\beta\mathbf{E}\theta,\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s)+(\beta_f\mathbf{E}\theta,\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f)\\&-(\tau\beta\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s,\mathbf{E}\dot{\theta})-(\tau\beta\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f,\mathbf{E}\dot{\theta})\\&+(\mathcal{P}\mathbf{E}\ddot{\mathbf{u}},(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f))+\left(\frac{\eta}{\kappa}\mathbf{E}\dot{\mathbf{u}}^f,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f\right)\\&+\Lambda(\mathbf{E}\mathbf{u},(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f))\\&-(\beta\mathbf{E}\theta,\nabla\cdot(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s))+(\beta_f\mathbf{E}\theta,\nabla\cdot(Q\dot{\mathbf{u}}^f-\dot{\mathbf{u}}^f))\\&+(\tau c\mathbf{E}\ddot{\theta},\Pi\dot{\theta}-\dot{\theta})+(c\mathbf{E}\dot{\theta},\dot{\theta}-\Pi\dot{\theta})+(\gamma\nabla\mathbf{E}\theta,\nabla(\Pi\dot{\theta}-\dot{\theta}))+(\beta\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s,\Pi\dot{\theta}-\dot{\theta})\\&+(\beta\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f,\Pi\dot{\theta}-\dot{\theta})+(\tau\beta\theta_0\nabla\cdot\mathbf{E}\ddot{\mathbf{u}}^s,\Pi\dot{\theta}-\dot{\theta})+(\tau\beta\theta_0\nabla\cdot\mathbf{E}\ddot{\mathbf{u}}^f,\Pi\dot{\theta}-\dot{\theta})\\&+\langle\mathcal{D}\mathcal{S}(\mathbf{E}\dot{\mathbf{u}}),\mathcal{S}(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f)\rangle+\langle\tau c v_\theta\mathbf{E}\dot{\theta},\Pi\dot{\theta}-\dot{\theta}\rangle,\ t\in J.\end{aligned} \quad \text{(A-23)}$$

Next, we obtain estimates for the time integrals of the terms on the right side of equation A-23. First,

$$\left|\int_0^t(\beta\mathbf{E}\theta),\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s)(s)ds\right|+\left|\int_0^t(\beta_f\mathbf{E}\theta),\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f)(s)ds\right| \leq C\left(\int_0^t\|\mathbf{E}\theta(s)\|_0^2ds+\int_0^t\|\mathbf{E}\dot{\mathbf{u}}(s)\|_{\mathcal{V}}^2ds\right) \quad \text{(A-24)}$$

and

$$\left|\int_0^t(\tau\beta\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s,\mathbf{E}\dot{\theta})(s)ds\right|+\left|\int_0^t(\tau\beta\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f,\mathbf{E}\dot{\theta})(s)ds\right| \leq C\left(\int_0^t\|\mathbf{E}\dot{\theta}(s)\|_0^2ds+\int_0^t\|\mathbf{E}\dot{\mathbf{u}}(s)\|_{\mathcal{V}}^2ds\right). \quad \text{(A-25)}$$

Next, using the approximating properties of Π in equation 24

$$\begin{aligned}&\left|\int_0^t(\beta\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s,\Pi\dot{\theta}-\dot{\theta})(s)ds\right|+\left|\int_0^t(\beta\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f,\Pi\dot{\theta}-\dot{\theta})(s)ds\right|\\&\leq C\left(\int_0^t\|\mathbf{E}\dot{\mathbf{u}}(s)\|_{\mathcal{V}}^2ds+h^4\|\dot{\theta}(s)\|_2^2ds\right)\\&\leq C\left(\int_0^t\|\mathbf{E}\dot{\mathbf{u}}(s)\|_{\mathcal{V}}^2ds+h^4\|\dot{\theta}\|_{L^2(J,H^2(\Omega))}^2\right).\end{aligned} \quad \text{(A-26)}$$

Also, using equation 26

$$\left|\int_0^t\left(\frac{\eta}{\kappa}\mathbf{E}\dot{\mathbf{u}}^f,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f\right)(s)ds\right| \leq C\left(\int_0^t\|\mathbf{E}\dot{\mathbf{u}}^f(s)\|_0^2ds+h^2\|\dot{\mathbf{u}}^f\|_{L^2(J,H^1(\Omega))}^2\right). \quad \text{(A-27)}$$

Next,

$$\begin{aligned}&\left|\int_0^t\Lambda(\mathbf{E}\mathbf{u},(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f))(s)ds\right|\\&\leq C(\|\mathbf{E}\mathbf{u}\|_{\mathcal{V}}^2+\|\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s\|_1^2+\|\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f\|_{H(\mathrm{div};\Omega)}^2)\\&\leq C\Big(\int_0^t\|\mathbf{E}\mathbf{u}(s)\|_{\mathcal{V}}^2ds+h^2(\|\dot{\mathbf{u}}^s\|_{L^2(J,[H^2(\Omega)]^2)}^2\\&+\|\dot{\mathbf{u}}^f\|_{L^2(J,[H^1(\Omega)]^2)}^2+\|\nabla\cdot\dot{\mathbf{u}}^f\|_{L^2(J,H^1(\Omega))}^2)\Big),\end{aligned} \quad \text{(A-28)}$$

$$\begin{aligned}&\left|\int_0^t(\beta\mathbf{E}\theta,\nabla\cdot(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s))(s)ds\right|\\&+\left|\int_0^t(\beta_f\mathbf{E}\theta,\nabla\cdot(Q\dot{\mathbf{u}}^f-\dot{\mathbf{u}}^f))(s)ds\right|\\&\leq C\Big(\int_0^t\|\mathbf{E}\theta(s)\|_0^2ds+h^2[\|\dot{\mathbf{u}}^s\|_{L^2(J,[H^2(\Omega)]^2)}^2\\&+\|\nabla\cdot\dot{\mathbf{u}}^f\|_{L^2(J,H^1(\Omega))}^2]\Big),\end{aligned} \quad \text{(A-29)}$$

$$\left|\int_0^t(c\mathbf{E}\dot{\theta},\dot{\theta}-\Pi\dot{\theta})(s)ds\right| \leq C\left(\int_0^t\|\mathbf{E}\dot{\theta}(s)\|_0^2ds+h^4\|\dot{\theta}\|_{L^2(J,H^2(\Omega))}^2\right), \quad \text{(A-30)}$$

and

$$\left|\int_0^t(\gamma\nabla\mathbf{E}\theta,\nabla(\Pi\dot{\theta}-\dot{\theta}))(s)ds\right| \leq C\left(\int_0^t\|\mathbf{E}\theta\|_1^2(s)ds+h^2\|\dot{\theta}\|_{L^2(J,H^2(\Omega))}\right). \quad \text{(A-31)}$$

The terms on the second time derivatives of $\mathbf{Eu}$ and $\mathbf{E}\theta$ on the right side of equation A-23 can be bounded using integration by parts in time as follows. First,

$$
\begin{aligned}
&|\int_0^t(\mathcal{P}\mathbf{E}\ddot{\mathbf{u}},(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f)(s)ds)|\\
&\quad=|(\mathcal{P}\mathbf{E}\dot{\mathbf{u}},(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f))(t)\\
&\quad-(\mathcal{P}\mathbf{E}\dot{\mathbf{u}},(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f))(0)\\
&\quad-\int_0^t(\mathcal{P}\mathbf{E}\dot{\mathbf{u}},(\ddot{\mathbf{u}}^s-\Pi^{(2)}\ddot{\mathbf{u}}^s,\ddot{\mathbf{u}}^f-Q\ddot{\mathbf{u}}^f))(s)ds|\\
&\quad\le\varepsilon\|\mathbf{E}\dot{\mathbf{u}}(t)\|_0^2+C\int_0^t\|\mathbf{E}\dot{\mathbf{u}}(s)\|_0^2ds+C(\|\mathbf{E}\dot{\mathbf{u}}(0)\|_0^2\\
&\quad+h^4\|\dot{\mathbf{u}}^s\|^2_{L^\infty(J,[H^2(\Omega)]^2)}\\
&\quad+h^2\|\dot{\mathbf{u}}^f\|^2_{L^\infty(J,[H^1(\Omega)]^2)}+h^4\|\ddot{\mathbf{u}}^s\|^2_{L^2(J,[H^2(\Omega)]^2)}\\
&\quad+h^2\|\ddot{\mathbf{u}}^f\|^2_{L^2(J,[H^1(\Omega)]^2)}).
\end{aligned}\tag{A-32}
$$

Also,

$$
\begin{aligned}
&|\int_0^t(\tau\beta\theta_0\nabla\cdot\mathbf{E}\ddot{\mathbf{u}}^s,\Pi\dot{\theta}-\dot{\theta})(s)ds|\\
&\quad=|\tau\beta\theta_0[(\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s,\Pi\dot{\theta}-\dot{\theta})(t)-(\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s,\Pi\dot{\theta}-\dot{\theta})(0)]\\
&\quad-\int_0^t(\tau\beta\theta_0\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s,\Pi\ddot{\theta}-\ddot{\theta})(s)ds|\le\varepsilon\|\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s(t)\|_0^2\\
&\quad+C\left(\|\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s(0)\|_0^2+\int_0^t\|\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^s\|_0^2(s)ds\right)\\
&\quad+Ch^4(\|\dot{\theta}\|^2_{L^\infty(J,H^2(\Omega))}+\|\ddot{\theta}\|^2_{L^2(J,H^2(\Omega))}).
\end{aligned}\tag{A-33}
$$

Proceeding similarly,

$$
\begin{aligned}
&|\int_0^t(\tau\beta\theta_0\nabla\cdot\mathbf{E}\ddot{\mathbf{u}}^f,\Pi\dot{\theta}-\dot{\theta})(s)ds|\le\varepsilon\|\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f(t)\|_0^2\\
&\quad+C\Bigg(\|\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f(0)\|_0^2+\int_0^t\|\nabla\cdot\mathbf{E}\dot{\mathbf{u}}^f(\tau)\|_0^2d\tau\\
&\quad+h^4[\|\dot{\theta}\|^2_{L^\infty(J,H^2(\Omega))}+\|\ddot{\theta}\|^2_{L^2(J,H^2(\Omega))}]\Bigg).
\end{aligned}\tag{A-34}
$$

Next,

$$
\begin{aligned}
&\left|\int_0^t(\tau c\mathbf{E}\ddot{\theta},\Pi\dot{\theta}-\dot{\theta})(s)ds\right|\\
&\quad=\Bigg|(\tau c\mathbf{E}\dot{\theta},\Pi\dot{\theta}-\dot{\theta})(t)-(\tau c\mathbf{E}\dot{\theta},\Pi\dot{\theta}-\dot{\theta})(0)\\
&\quad-\int_0^t(\tau c\mathbf{E}\dot{\theta},\Pi\ddot{\theta}-\ddot{\theta})(s)ds\Bigg|\\
&\quad\le\varepsilon\|\mathbf{E}\dot{\theta}\|_0^2(t)+C\Bigg(\|\mathbf{E}\dot{\theta}\|_0^2(0)+\int_0^t\|\mathbf{E}\dot{\theta}\|_0^2(s)ds\\
&\quad+h^4[\|\dot{\theta}\|^2_{L^\infty(J,H^2(\Omega))}+\|\ddot{\theta}\|^2_{L^2(J,H^2(\Omega))}]\Bigg).
\end{aligned}\tag{A-35}
$$

Also, using the approximating properties of the projections Π and Q in equations 24–26 and equation 13, we obtain

$$
\begin{aligned}
&\left|\int_0^t\langle\mathcal{D}\mathcal{S}(\mathbf{E}\dot{\mathbf{u}}),\mathcal{S}(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f)\rangle(s)ds\right|\\
&\le\varepsilon\int_0^t\langle\mathcal{D}\mathcal{S}(\mathbf{E}\dot{\mathbf{u}}),\mathcal{S}(\mathbf{E}\dot{\mathbf{u}})\rangle(s)ds\\
&+C\Bigg(\int_0^t\langle S(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f),\\
&S(\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s,\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f)\rangle(s)ds\Bigg)\\
&\le\varepsilon\int_0^t\langle\mathcal{D}\mathcal{S}(\mathbf{E}\dot{\mathbf{u}}),\mathcal{S}(\mathbf{E}\dot{\mathbf{u}})\rangle(s)ds\\
&+\sum_j\Bigg[\int_0^t|\dot{\mathbf{u}}^s-\Pi^{(2)}\dot{\mathbf{u}}^s|^2_{0,\Gamma\cap\partial\Omega_j}(s)ds\\
&+\int_0^t|(\dot{\mathbf{u}}^f-Q\dot{\mathbf{u}}^f)\cdot\boldsymbol{\nu}|^2_{0,\Gamma\cap\partial\Omega_j}(s)ds\Bigg]\\
&\le\varepsilon\int_0^t\langle\mathcal{D}\mathcal{S}(\mathbf{E}\dot{\mathbf{u}}),\mathcal{S}(\mathbf{E}\dot{\mathbf{u}})\rangle(s)ds\\
&+\sum_j\left[\int_0^t h^3\|\dot{\mathbf{u}}^s\|^2_{2,\Omega_j}(s)ds+h^2\int_0^t|\dot{\mathbf{u}}^f\cdot\boldsymbol{\nu}|^2_{1,\Gamma\cap\partial\Omega_j}(s)ds\right]\\
&\le\varepsilon\int_0^t\langle\mathcal{D}\mathcal{S}(\mathbf{E}\dot{\mathbf{u}}),\mathcal{S}(\mathbf{E}\dot{\mathbf{u}})\rangle(s)ds\\
&+\left[\int_0^t h^3\|\dot{\mathbf{u}}^s\|^2_{L^2(J,[H^2(\Omega)]^2)}+h^2\|\dot{\mathbf{u}}^f\|^2_{L^2(J,[H^{3/2}(\Omega)]^2)}\right].
\end{aligned}\tag{A-36}
$$

In a similar fashion,

$$
\begin{aligned}
&\left|\int_0^t\langle\tau c v_\theta\mathbf{E}\dot{\theta},\Pi\dot{\theta}-\dot{\theta}\rangle(s)ds\right|\le\varepsilon\int_0^t\langle\tau c v_\theta\mathbf{E}\dot{\theta},\mathbf{E}\dot{\theta}\rangle(s)ds\\
&\quad+C\int_0^t\sum_j|\Pi\dot{\theta}-\dot{\theta}|^2_{0,\Gamma\cap\partial\Omega_j}(s)ds\\
&\quad\le\varepsilon\int_0^t\langle\tau c v_\theta\mathbf{E}\dot{\theta},\mathbf{E}\dot{\theta}\rangle(s)ds+Ch^3\|\dot{\theta}\|^2_{L^2(J,H^2(\Omega))}.
\end{aligned}\tag{A-37}
$$

Thus, we integrate the inequality in equation A-23 in time and absorb the ε terms in equations A-24–A-37 on the left side of equation A-23. Then, we apply Gronwall's lemma in the resulting equation and use that $\mathcal{P}$ is positive definite, Λ is $\mathcal{V}$-coercive, and the coefficients τ, c, and γ are bounded below by positive constants to obtain

$$
\begin{aligned}
&\|\mathbf{E}\dot{\mathbf{u}}\|_{L^\infty(J,[L^2(\Omega)]^4)}+\|\mathbf{E}\ddot{\mathbf{u}}\|_{L^\infty(J,[L^2(\Omega)]^4)}+\|\mathbf{E}\mathbf{u}\|_{L^\infty(J,\mathcal{V})}+\|\mathbf{E}\dot{\mathbf{u}}\|_{L^\infty(J,\mathcal{V})}\\
&+\|\mathbf{E}\theta\|_{L^\infty(J,H^1(\Omega))}+\|\mathbf{E}\dot{\theta}\|_{L^\infty(J,L^2(\Omega))}\\
&\le C(\|\mathbf{E}\dot{\mathbf{u}}(0)\|_0^2+\|\mathbf{E}\ddot{\mathbf{u}}(0)\|_0^2+\|\mathbf{E}\mathbf{u}(0)\|_{\mathcal{V}}^2+\|\mathbf{E}\dot{\mathbf{u}}(0)\|_{\mathcal{V}}^2\\
&+\|\mathbf{E}\dot{\theta}(0)\|_0^2+\|\mathbf{E}\theta(0)\|_1^2\\
&+h[\|\dot{\mathbf{u}}^s\|_{L^2(J,[H^2(\Omega)]^2)}+\|\dot{\mathbf{u}}^f\|_{L^\infty(J,[H^1(\Omega)]^2)}+\|\dot{\mathbf{u}}^f\|_{L^2(J,[H^{3/2}(\Omega)]^2)}\\
&+\|\nabla\dot{\mathbf{u}}^f\|_{L^2(J,H^1(\Omega))}\\
&+\|\ddot{\mathbf{u}}^s\|_{L^2(J,[H^2(\Omega)]^2)}+\|\ddot{\mathbf{u}}^f\|_{L^2(J,[H^1(\Omega)]^2)}+\|\dot{\theta}\|_{L^\infty(J,H^2(\Omega))}\\
&+\|\ddot{\theta}\|_{L^2(J,H^2(\Omega))}]).
\end{aligned}\tag{A-38}
$$

The error at $t = 0$ in equation A-38 can be estimated by defining the FE initial conditions as follows. First, we take $\mathbf{U}(0), \dot{\mathbf{U}}(0) \in \mathcal{W}^h \times \mathcal{W}^h \times \mathcal{V}^h$ such that

$$\Lambda_\zeta(\mathbf{u}^0 - \mathbf{U}(0), \mathbf{v}) = \Lambda_\zeta(\mathbf{Eu}(0), \mathbf{v}) = 0, \ \ \mathbf{v} \in \mathcal{W}^h \times \mathcal{W}^h \times \mathcal{V}^h, \quad \text{(A-39)}$$

$$\Lambda_\zeta(\mathbf{u}^1 - \dot{\mathbf{U}}(0), \mathbf{v}) = \Lambda_\zeta(\mathbf{E}\dot{\mathbf{u}}(0), \mathbf{v}) = 0, \ \ \mathbf{v} \in \mathcal{W}^h \times \mathcal{W}^h \times \mathcal{V}^h. \quad \text{(A-40)}$$

We choose $\mathbf{v} = \mathbf{Eu}(0) + (\Pi^{(2)}\mathbf{u}^{0,s} - \mathbf{u}^{0,s}, Q\mathbf{u}^{0,f} - \mathbf{u}^{0,f})$ in equation A-39 and use the $\mathcal{V}$-coercivity of Λ_ζ and the approximating properties of $\Pi^{(2)}$ and Q in equations 24–27 to obtain

$$C_4\|\mathbf{Eu}(0)\|_{\mathcal{V}}^2 \le \Lambda_{\zeta_1}(\mathbf{Eu}(0), (\Pi^{(2)}\mathbf{u}^{0,s} - \mathbf{u}^{0,s}, Q\mathbf{u}^{0,f} - \mathbf{u}^{0,f})) \le C_5 h\|\mathbf{Eu}(0)\|_{\mathcal{V}}(\|\mathbf{u}^{0,s}\|_2 + \|\mathbf{u}^{0,f}\|_1 + \|\nabla\cdot\mathbf{u}^{0,f}\|_1). \quad \text{(A-41)}$$

Thus,

$$\|\mathbf{Eu}(0)\|_{\mathcal{V}} \le Ch(\|\mathbf{u}^{0,s}\|_2 + \|\mathbf{u}^{0,f}\|_1 + \|\nabla\cdot\mathbf{u}^{0,f}\|_1). \quad \text{(A-42)}$$

Similarly, by choosing $\mathbf{v} = \mathbf{E}\dot{\mathbf{u}}(0) + (\Pi^{(2)}\dot{\mathbf{u}}^{1,s} - \mathbf{u}^{1,s}, Q\mathbf{u}^{1,f} - \mathbf{u}^{1,f})$ in equation A-40, we obtain

$$\|\mathbf{E}\dot{\mathbf{u}}(0)\|_{\mathcal{V}} \le Ch(\|\mathbf{u}^{1,s}\|_2 + \|\mathbf{u}^{1,f}\|_1 + \|\nabla\cdot\mathbf{u}^{1,f}\|_1). \quad \text{(A-43)}$$

To obtain a bound for the term $\|\mathbf{E}\ddot{\mathbf{u}}(0)\|_0$ in equation A-38, we assume that the initial value problem (equations 4 and 5) with the initial conditions (equation 6) and the boundary condition (equation 8) satisfies the regularity inequality:

$$\|\mathbf{u}^s\|_2 + \|\mathbf{u}^f\|_1 + \|\nabla\cdot\mathbf{u}^f\|_1 + \|\theta\|_2 \le C(\|f\|_0 + \|q\|_0). \quad \text{(A-44)}$$

We also assume that equation A-44 holds for time derivatives of $\mathbf{u}$ and θ. Thus, at $t = 0$, we have

$$\|\ddot{\mathbf{u}}^s(0)\|_2 + \|\ddot{\mathbf{u}}^f(0)\|_1 + \|\nabla\cdot\ddot{\mathbf{u}}^f(0)\|_1 + \|\theta(0)\|_2 \le C(\|\ddot{f}(0)\|_0 + \|\ddot{q}(0)\|_0). \quad \text{(A-45)}$$

Hence, defining $\ddot{\mathbf{U}}(0)$ by the equation

$$\Lambda_\zeta(\ddot{\mathbf{u}}(0) - \ddot{\mathbf{U}}(0), \mathbf{v}) = \Lambda_\zeta(\mathbf{E}\ddot{\mathbf{u}}(0), \mathbf{v}) = 0, \ \ \mathbf{v} \in \mathcal{W}^h \times \mathcal{W}^h \times \mathcal{V}^h, \quad \text{(A-46)}$$

the choice $\mathbf{v} = \mathbf{E}\ddot{\mathbf{u}}(0) + (\Pi^{(2)}\ddot{\mathbf{u}}^s(0) - \ddot{\mathbf{u}}^s(0), Q\ddot{\mathbf{u}}^f(0) - \ddot{\mathbf{u}}^f(0))$ in equation A-46 yields the bound:

$$\|\mathbf{E}\dot{\mathbf{u}}(0)\|_{\mathcal{V}} \le Ch(\|\ddot{u}^s(0)\|_2 + \|\ddot{u}^f(0)\|_1 + \|\nabla\cdot\ddot{u}^f(0)\|_1 + \|\ddot{\theta}(0)\|_2) \le Ch((\|\ddot{f}(0)\|_0 + \|\ddot{q}(0)\|_0). \quad \text{(A-47)}$$

For the temperature variables, we take $\Theta(0), \dot{\Theta}(0) \in \mathcal{W}^h$ such that

$$\mathbf{E}\theta(0) = \theta^0 - \Theta(0), \quad \mathbf{E}\dot{\theta}(0) = \theta^1 - \dot{\Theta}(0) \quad \text{(A-48)}$$

satisfy the relations

$$(\gamma\mathbf{E}\theta(0), w) + (\gamma\nabla\mathbf{E}\theta(0), \nabla w) = 0, w \in \mathcal{W}^h, \quad \text{(A-49)}$$

$$(\gamma\mathbf{E}\dot{\theta}(0), w) + (\gamma\nabla\mathbf{E}\dot{\theta}(0), \nabla w) = 0, w \in \mathcal{W}^h. \quad \text{(A-50)}$$

Because

$$C_6\|\mathbf{E}\theta(0)\|_1^2 \le (\gamma\mathbf{E}\theta(0), \mathbf{E}\theta(0)) + (\gamma\nabla\mathbf{E}\theta(0), \nabla\mathbf{E}\theta(0)), \quad \text{(A-51)}$$

we choose $w = \mathbf{E}\theta(0) + \Pi\theta^0 - \theta^0$ in equation A-49 to obtain

$$C_5\|\mathbf{E}\theta(0)\|_1^2 \le (\gamma\mathbf{E}\theta(0), \theta^0 - \Pi\theta^0) + (\gamma\nabla\mathbf{E}\theta(0), \nabla(\theta^0 - \Pi\theta^0)) \le C_7 h\|\mathbf{E}\theta(0)\|_1\|\theta^0\|_2, \quad \text{(A-52)}$$

so that

$$\|\mathbf{E}\theta(0)\|_1 \le Ch\|\theta^0\|_2. \quad \text{(A-53)}$$

Similarly, the choice $w = \mathbf{E}\dot{\theta}(0) + \Pi\theta^1 - \theta^1$ in equation A-50 yields the inequality:

$$\|\mathbf{E}\dot{\theta}(0)\|_1 \le Ch\|\theta^1\|_2. \quad \text{(A-54)}$$

The bounds (equations A-42–A-54) in equation A-38 imply the validity of Theorem 2.

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Biographies and photographs of the authors are not available.