

A model for wave propagation in a composite solid matrix saturated by a single-phase fluid

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This paper presents a theory to describe wave propagation in a porous medium composed of two solids saturated by a single-phase fluid for spatially variable porosity. This problem has been previously solved for constant porosity when one of the solids is ice or clay, but that model is not useful for most realistic situations. The equations for variable porosity are derived from the virtual work principle, where the generalized coordinates are identified as the displacements of the two solid phases and a new variable associated with the relative fluid flow, whose divergence is the change in fluid content. The generalized forces are the fluid pressure and combinations of the stress tensor of each solid phase and the fluid pressure. The Lagrangian equations of motion are derived for the isotropic case and a theorem on the existence and uniqueness of their solution is given. The plane wave analysis reveals the existence of three compressional and two shear waves. The theory is applied to wave propagation in shaley sandstones showing that phase velocities of the faster P and S waves agree very well with experimental data for varying porosity and clay content. A simulation through a plane interface separating two frozen sandstones of different ice contents is presented.

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I. INTRODUCTION

Wave propagation in composite porous materials has applications in many branches of science and technology, such as seismic methods in the presence of shaley sandstones,¹ permafrost,^{2,3} gas-hydrate concentration in ocean-bottom sediments,⁴ and evaluation of the freezing conditions of foods by ultrasonic techniques.⁵

Leclaire *et al.*⁶ have developed a theory for describing wave propagation in frozen porous media in which solid substrate, ice, and water coexist, under the assumption of the existence of a layer of unfrozen water around the solid particles isolating them from ice. This model, valid for uniform porosity, predicts the existence of three compressional and two shear waves; the verification that additional (slow) waves can be observed in laboratory experiments was published by Leclaire *et al.*⁷

Later, this theory was generalized by Carcione and Tinivella⁴ to include the interaction between the solid and ice particles and grain cementation with decreasing temperature. Also, Carcione *et al.*¹ have applied this theory to study the acoustic properties of shaley sandstones, assuming that sand and clay are *nonwelded* and form a continuous and interpenetrating porous composite skeleton.

Both frozen porous media and shaley sandstones are two examples of porous materials where the two solid phases are *weakly coupled* or *nonwelded*. Similar *weakly coupled* formulations have previously been proposed. For instance, McCoy⁸ explicitly assumed the weak coupling and phase connectivity conditions over macro-scale distances. He proposed a mixture theory appropriate for the combination of two *acoustic phases*.

This work generalizes the theory developed in Refs. 6 and 4 to the case of nonuniform porosity so that the differential equations can be used to perform numerical experiments or fit laboratory data related to heterogeneous media. The *nonwelding* condition between the two solid phases is assumed when the potential and kinetic energies are defined, with proper interaction terms among the solid and fluid phases. If the two solid phases would be welded, then additional slow waves would not be present.⁹ Our approach is based on the energy formulation used by Biot,¹⁰ rather than on volume averaging or homogenization methods used, for instance, by Burridge and Keller.¹¹

In this paper the virtual work principle for the composite material is stated and the strains in the two solid phases and the change in fluid content are identified as the state variables to represent the variation in strain energy $\delta\mathcal{W}$, consequently identifying the generalized forces. These are the total

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stress tensors in both solid phases, denoted by $\sigma^{1,T}$ and $\sigma^{3,T}$, respectively, and the fluid pressure p_f .

The kinetic energy density and the dissipation function are defined in terms of the generalized coordinates, which are the two solid displacement vectors and the new variable w , associated with the fluid flow relative to the solid composite matrix, whose divergence is the change in fluid content.

The Lagrangian formulation of the equations of motion is a generalization of the approach of Biot,¹⁰ Santos *et al.*,¹² and Carcione.¹³ It is shown that in the case of uniform porosity, the known theories^{6,4} are recovered. The plane wave analysis shows the existence of three compressional and two shear modes of propagation, in agreement with the original theory derived by Leclaire *et al.*⁶

An existence and uniqueness result for a general initial boundary value problem is given, showing that for each time t each component of the solid displacements belongs to the Sobolev space $H^1(\Omega)$, while the fluid displacement lies in the space $H(\text{div}, \Omega)$.

The theory is applied to wave propagation in shaley sandstones showing that the phase velocities of the faster waves (the seismic P and S waves) agree very well with experimental data for varying porosity and clay content. Moreover, in a simulation of waves travelling through a plane interface separating two frozen sandstones of different porosity is performed. The numerical solver is a modification of the pseudospectral modeling algorithm used by Carcione and Seriani³ to model propagation in frozen porous media with uniform porosity.

II. THE STRAIN ENERGY OF THE COMPOSITE SYSTEM

Let Ω be an elementary cube of porous material composed of two solid phases, referred to by the subscripts or superscripts 1 and 3, saturated by a fluid phase indicated by the subscript or superscript 2. Thus, $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$. Let V_i denote the volume of the phase Ω_i and V_b and V_{sm} the bulk volume of Ω and the solid matrix $\Omega_{sm} = \Omega_1 \cup \Omega_3$, so that

$$V_{sm} = V_1 + V_3, \quad V_b = V_1 + V_2 + V_3.$$

Let $S_1 = V_1/V_{sm}$ and $S_3 = V_3/V_{sm}$ denote the two solid fractions of the composite matrix and define the effective porosity as $\phi = V_2/V_b$. Let $u^{(1)}$, $u^{(2)}$, and $u^{(3)}$ be the averaged solid and fluid displacements over the bulk material. Here $u^{(2)}$ is defined such that on any face F of the cube Ω ,

$$\int_F \phi u^{(2)} \nu d\sigma$$

is the amount of fluid displaced through F , while

$$\int_F S_1 u^{(1)} \nu d\sigma, \quad \int_F S_3 u^{(3)} \nu d\sigma$$

represent the displacements in the two solid parts of F , respectively. Here $\nu = (\nu_j)$ denotes the unit outward normal to F and $d\sigma$ the surface measure on F .

Let $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(3)}$ denote the stress tensors in Ω_1 and Ω_3 averaged over the bulk material Ω , respectively, and let p_f

denote the fluid pressure. These quantities describe small changes with respect to reference values corresponding to an initial equilibrium state. Let us also introduce the tensors

$$\sigma_{ij}^{(1,T)} = \sigma_{ij}^{(1)} - S_1 \phi p_f \delta_{ij}, \quad \sigma_{ij}^{(3,T)} = \sigma_{ij}^{(3)} - S_3 \phi p_f \delta_{ij}, \quad (2.1)$$

associated with the total stresses in Ω_1 and Ω_3 , respectively, so that the total stress tensor in the bulk material Ω is given by

$$\sigma_{ij} = \sigma_{ij}^{(1,T)} + \sigma_{ij}^{(3,T)}. \quad (2.2)$$

Next the stress–strain relations for our system shall be derived using the virtual work principle, following the derivation of Biot¹⁰ for the case of a single solid phase. In what follows the Einstein convention is used, i.e., sum on repeated indices. Let \mathcal{W} be the strain energy density and \mathcal{V} the total potential energy. Also let \mathcal{V}_d denote the total potential energy density. Then, if $f_i^{(1)}$, $f_i^{(3)}$, $f_i^{(2)}$ represent the surface forces acting on the solid and fluid parts of the boundary of Ω , denoted by $\partial\Omega$, we have that

$$\mathcal{V} = \int_{\Omega} \mathcal{V}_d dx = \int_{\Omega} \mathcal{W} dx - \int_{\partial\Omega} (f_i^{(1)} u_i^{(1)} + f_i^{(2)} u_i^{(2)} + f_i^{(3)} u_i^{(3)}) d\sigma, \quad (2.3)$$

and the virtual work principle for the composite fluid–solid system can be stated in the form:

$$\delta\mathcal{V} = 0 = \int_{\Omega} \delta\mathcal{W} dx - \int_{\partial\Omega} (f_i^{(1)} \delta(u_i^{(1)}) + f_i^{(2)} \delta(u_i^{(2)}) + f_i^{(3)} (\delta u_i^{(3)})) d\sigma, \quad (2.4)$$

where

$$f_i^{(1)} = \sigma_{ij}^{(1)} \nu_j, \quad f_i^{(3)} = \sigma_{ij}^{(3)} \nu_j, \quad f_i^{(2)} = -\phi p_f \delta_{ij} \nu_j, \quad (2.5)$$

and δ denotes virtual changes in the different quantities. Using Eqs. (2.1) and (2.5) in Eq. (2.4) yields

$$\delta\mathcal{V} = 0 = \int_{\Omega} \delta\mathcal{W} dx - \int_{\partial\Omega} (\sigma_{ij}^{(1,T)} \nu_j \delta(u_i^{(1)}) + \sigma_{ij}^{(3,T)} \nu_j \delta(u_i^{(3)}) - p_f \delta_{ij} \nu_j \delta w_i) d\sigma. \quad (2.6)$$

where

$$w_i = \phi(u_i^{(2)} - S_1 u_i^{(1)} - S_3 u_i^{(3)}). \quad (2.7)$$

Then, transforming the surface integral in Eq. (2.6) into a volume integral gives

$$\delta\mathcal{V} = 0 = \int_{\Omega} \delta W dx - \int_{\Omega} \frac{\partial}{\partial x_j} [\sigma_{ij}^{(1,T)} \delta(u_i^{(1)}) + \sigma_{ij}^{(3,T)} \delta(u_i^{(3)}) - p_f \delta_{ij} \delta w_i] dx. \quad (2.8)$$

Since Ω remains in equilibrium under the action of the virtual displacements, it follows that

$$\frac{\partial}{\partial x_j} \sigma_{ij}^{(1,T)} = \frac{\partial}{\partial x_j} \sigma_{ij}^{(3,T)} = \frac{\partial}{\partial x_j} p_f \delta_{ij} = 0,$$

and consequently, using the symmetry of the stress tensors $\sigma_{ij}^{(1,T)}$ and $\sigma_{ij}^{(3,T)}$ we obtain

$$\begin{aligned} \delta \mathcal{V} = 0 = & \int_{\Omega} \delta \mathcal{W} - \int_{\Omega} (\sigma_{ij}^{(1,T)} \delta(\epsilon_{ij}(u^{(1)})) \\ & + \sigma_{ij}^{(3,T)} \delta(\epsilon_{ij}(u^{(3)})) + p_f \delta \zeta) dx, \end{aligned} \quad (2.9)$$

where $\zeta = -\nabla \cdot \mathbf{w}$ represents the change in fluid content and

$$\epsilon_{ij}(u^{(m)}) = \frac{1}{2} \left(\frac{\partial u_i^{(m)}}{\partial x_j} + \frac{\partial u_j^{(m)}}{\partial x_i} \right), \quad m = 1, 3,$$

denotes the strain tensor in Ω_m with linear invariant $\theta_m = \epsilon_{ii}(u^{(m)})$.

Thus from Eqs. (2.1) and (2.9) the following expression for the variation in strain energy density $\delta \mathcal{W}$ is finally obtained:

$$\begin{aligned} \delta \mathcal{W} = & (\sigma_{ij}^{(1)} - S_1 \phi p_f \delta_{ij}) \delta(\epsilon_{ij}(u^{(1)})) \\ & + (\sigma_{ij}^{(3)} - S_3 \phi p_f \delta_{ij}) \delta(\epsilon_{ij}(u^{(3)})) + p_f \delta \zeta. \end{aligned} \quad (2.10)$$

Consequently, since $\delta \mathcal{W}$ is an exact differential of the variables $\epsilon_{ij}(u^{(1)})$, $\epsilon_{ij}(u^{(3)})$, and ζ we have that

$$\frac{\partial \mathcal{W}}{\partial \epsilon_{ij}(u^{(m)})} = \sigma_{ij}^{(m)} - S_m \phi p_f \delta_{ij}, \quad m = 1, 3, \quad \frac{\partial \mathcal{W}}{\partial \zeta} = p_f. \quad (2.11)$$

Also, it follows from Eq. (2.10) that

$$\begin{aligned} \mathcal{W} = & \frac{1}{2} [(\sigma_{ij}^{(1)} - S_1 \phi p_f \delta_{ij})(\epsilon_{ij}(u^{(1)})) \\ & + (\sigma_{ij}^{(3)} - S_3 \phi p_f \delta_{ij})(\epsilon_{ij}(u^{(3)})) + p_f \zeta]. \end{aligned} \quad (2.12)$$

Next, to obtain the expression for the *potential energy* of our system, let us consider perturbations of the system from the equilibrium state. Using Eq. (2.3), the argument leading to Eq. (2.8) and expression (2.10) for $\delta \mathcal{V}$ yields

$$\begin{aligned} \delta \mathcal{V}_d = & - \frac{\partial}{\partial x_j} [\sigma_{ij}^{(1)} - S_1 \phi p_f \delta_{ij}] \delta u_i^{(1)} \\ & - \frac{\partial}{\partial x_j} [\sigma_{ij}^{(3)} - S_3 \phi p_f \delta_{ij}] \delta u_i^{(3)} + \frac{\partial}{\partial x_i} p_f \delta w_i. \end{aligned} \quad (2.13)$$

If $u_i^{(1)}$, $u_i^{(3)}$, w_i are chosen as generalized coordinates to describe our composite system, since the system is assumed to be conservative it follows that

$$\begin{aligned} \frac{\partial \mathcal{V}_d}{\partial u_i^{(m)}} = & - \frac{\partial}{\partial x_j} [\sigma_{ij}^{(m)} - S_m \phi p_f \delta_{ij}], \\ m = 1, 3, \quad \frac{\partial \mathcal{V}_d}{\partial w_i} = & \frac{\partial}{\partial x_i} p_f. \end{aligned} \quad (2.14)$$

III. THE LINEAR ISOTROPIC STRESS–STRAIN RELATIONS

Let us denote the deviatoric strain tensor in Ω_m as

$$d_{ij}^{(m)} = \epsilon_{ij}(u^{(m)}) - \frac{1}{3} \theta_m \delta_{ij}, \quad m = 1, 3.$$

In the linear isotropic case the strain energy density \mathcal{W} in Eq. (2.12) is a quadratic positive definite form in the invariants θ_1 , θ_3 , ζ , $(d_1)^2 = d_{ij}^{(1)} d_{ij}^{(1)}$, $(d_3)^2 = d_{ij}^{(3)} d_{ij}^{(3)}$ and $d_{1,3} = d_{ij}^{(1)} d_{ij}^{(3)}$. Note that

$$(d_m)^2 = \epsilon_{ij}(u^{(m)}) \epsilon_{ij}(u^{(m)}) - \frac{1}{3} (\theta_m)^2, \quad m = 1, 3,$$

$$d_{1,3} = \epsilon_{ij}(u^{(1)}) \epsilon_{ij}(u^{(3)}) - \frac{1}{3} \theta_1 \theta_3.$$

Then,

$$\begin{aligned} \mathcal{W} = & \frac{1}{2} H_1 (\theta_1)^2 + \mu_1 (d_1)^2 + \frac{1}{2} H_3 (\theta_3)^2 + \mu_3 (d_3)^2 - B_1 \theta_1 \zeta \\ & - B_2 \theta_3 \zeta + B_3 \theta_1 \theta_3 + \frac{1}{2} K_{av} (\zeta)^2 + \mu_{1,3} d_{1,3}. \end{aligned} \quad (3.1)$$

Remark: The B_3 and μ_{13} terms represent elastic interaction between the two solid phases.

Thus,

$$\begin{aligned} \frac{\partial \mathcal{W}}{\partial \epsilon_{ij}(u^{(1)})} = & \sigma_{ij}^{(1)} - S_1 \phi p_f \delta_{ij} \\ = & [H_1 \theta_1 - B_1 \zeta + B_3 \theta_3] \delta_{ij} + 2 \mu_1 d_{ij}^{(1)} + \mu_{1,3} d_{ij}^{(3)}, \end{aligned} \quad (3.2)$$

$$\begin{aligned} \frac{\partial \mathcal{W}}{\partial \epsilon_{ij}(u^{(3)})} = & \sigma_{ij}^{(3)} - S_3 \phi p_f \delta_{ij} \\ = & [H_3 \theta_3 - B_2 \zeta + B_3 \theta_1] \delta_{ij} + 2 \mu_3 d_{ij}^{(3)} + \mu_{1,3} d_{ij}^{(1)}, \end{aligned} \quad (3.3)$$

$$\frac{\partial \mathcal{W}}{\partial \zeta} = p_f = -B_1 \theta_1 - B_2 \theta_3 + K_{av} \zeta. \quad (3.4)$$

Equations (3.2)–(3.4) express the generalized stresses $\sigma_{ij}^{(1)} - S_1 \phi p_f \delta_{ij}$, $\sigma_{ij}^{(3)} - S_3 \phi p_f \delta_{ij}$ and p_f in terms of the strains $\epsilon_{ij}(u^{(1)})$, $\epsilon_{ij}(u^{(3)})$, and ζ .

IV. DETERMINATION OF THE COEFFICIENTS IN THE STRESS–STRAIN RELATIONS

This section presents a procedure to determine the coefficients in the stress–strain relations (3.2)–(3.4) for the variable porosity case. It is assumed that the moduli for the case of uniform porosity can be obtained from known expressions given in a previous formulations.⁴ First note that setting $\theta_2 = \nabla \cdot \mathbf{u}^{(2)}$, for the case of uniform porosity we have that

$$\zeta = \phi (S_1 \theta_1 + S_3 \theta_3 - \theta_2) \quad (4.1)$$

and consequently, from Eq. (3.4) we get

$$\begin{aligned} -\phi p_f = & (B_1 \phi - K_{av} \phi^2 S_1) \theta_1 + (B_2 \phi - K_{av} \phi^2 S_3) \theta_3 \\ & + K_{av} \phi^2 \theta_2. \end{aligned} \quad (4.2)$$

Next, combining Eqs. (3.2), (4.1), and (4.2) yields

$$\begin{aligned} \sigma_{ij}^{(1)} = & \{ [H_1 + (S_1\phi)^2 K_{av} - 2S_1\phi B_1] \theta_1 + [S_1 S_3 \phi^2 K_{av} \\ & - S_3\phi B_1 - S_1\phi B_2 + B_3] \theta_3 - (\phi^2 S_1 K_{av} \\ & - \phi B_1) \theta_2 \} \delta_{ij} + 2\mu_1 d_{ij}^{(1)} + \mu_{1,3} d_{ij}^{(3)}. \end{aligned} \quad (4.3)$$

Also, it follows from Eqs. (3.3), (4.1), and (4.2) that

$$\begin{aligned} \sigma_{ij}^{(3)} = & \{ [H_3 + (S_3\phi)^2 K_{av} - 2S_3\phi B_2] \theta_3 + [S_3 S_1 \phi^2 K_{av} \\ & - S_3\phi B_1 - S_1\phi B_2 + B_3] \theta_1 - (\phi^2 S_3 K_{av} \\ & - \phi B_2) \theta_2 \} \delta_{ij} + 2\mu_3 d_{ij}^{(3)} + \mu_{1,3} d_{ij}^{(1)}. \end{aligned} \quad (4.4)$$

Set

$$\begin{aligned} K_2 &= K_{av} \phi^2, \quad C_{12} = B_1 \phi - K_{av} \phi^2 S_1, \\ C_{23} &= B_2 \phi - K_{av} \phi^2 S_3, \\ K_1 &= H_1 + (S_1 \phi)^2 K_{av} - 2S_1 \phi B_1, \\ K_3 &= H_3 + (S_3 \phi)^2 K_{av} - 2S_3 \phi B_2, \\ C_{13} &= S_1 S_3 \phi^2 K_{av} - S_3 \phi B_1 - S_1 \phi B_2 + B_3. \end{aligned} \quad (4.5)$$

Then, the stress-strain relations (4.2)–(4.4) for constant porosity can be stated as follows:

$$\sigma_{ij}^{(1)} = (K_1 \theta_1 + C_{13} \theta_3 + C_{12} \theta_2) \delta_{ij} + 2\mu_1 d_{ij}^{(1)} + \mu_{1,3} d_{ij}^{(3)}, \quad (4.6)$$

$$\sigma_{ij}^{(3)} = (K_3 \theta_3 + C_{13} \theta_1 + C_{23} \theta_2) \delta_{ij} + 2\mu_3 d_{ij}^{(3)} + \mu_{1,3} d_{ij}^{(1)}, \quad (4.7)$$

$$-\phi p_f = C_{12} \theta_1 + C_{23} \theta_3 + K_2 \theta_2. \quad (4.8)$$

Relations similar to Eqs. (4.6)–(4.8) were derived by Leclaire *et al.*⁶ for the case of uniform porosity and when one of the solid phases is ice; it is also assumed in that paper that there is no contact between the solid and ice phases. This situation corresponds to the particular case in which $C_{13} = \mu_{1,3} = 0$. Carcione and Tinivella⁴ generalized the model of Leclaire *et al.*⁶ to include interaction between the solid and ice phases and grain cementation with temperature and obtained stress-strain relations in the form given in Eqs. (4.6)–(4.8).

The nonsingular linear system of equations (4.5) yields the following expressions for the coefficients of the variable porosity formulation:

$$\begin{aligned} H_1 &= K_1 + (S_1)^2 K_2 + 2S_1 C_{12}, \\ H_3 &= K_3 + (S_3)^2 K_2 + 2S_3 C_{23}, \\ B_1 &= \frac{S_1 K_2 + C_{12}}{\phi}, \quad B_2 = \frac{S_3 K_2 + C_{23}}{\phi}, \\ B_3 &= (C_{13} + S_3 C_{12} + S_1 C_{23} + S_3 S_1 K_2), \quad K_{av} = \frac{K_2}{\phi^2}. \end{aligned} \quad (4.9)$$

In Appendix A the ideas presented in Refs. 6 and 4 are used to obtain formulas for the computation of the coefficients K_1 , K_2 , K_3 , C_{12} , C_{13} , C_{23} , μ_1 , μ_3 , $\mu_{1,3}$, which combined with Eq. (4.9) allows for the evaluation of the moduli H_1 , H_3 , B_1 , B_2 , B_3 , K_{av} , needed for this new formulation. However, it must be remarked that appropriate

theoretical (*gedanken*) experiments should be devised for a rigorous determination of these coefficients.

V. A LAGRANGIAN FORMULATION OF THE EQUATIONS OF MOTION

A. The kinetic energy density

Let \mathcal{T} denote the kinetic energy density on Ω and let ρ_m , $m = 1, 2, 3$ denote the mass density of each solid and fluid constituent in Ω . Also let $\phi_m = V_m/V_b$, $m = 1, 3$ be the fractions of the two solid phases in the bulk material.

Let us consider the kinetic energy density in the solid parts Ω_1 and Ω_3 . Here the argument follows the ideas presented in Ref. 4. Let us introduce the relative macrovelocity of each solid phase with respect to the other:

$$q_i^{(1,3)} = \phi_1(u_i^{(1)} - u_i^{(3)}), \quad q_i^{(3,1)} = \phi_3(u_i^{(3)} - u_i^{(1)}), \quad (5.1)$$

and denote $s_i^{(1,3)}$ and $s_i^{(3,1)}$ the corresponding relative microvelocity fields. Assuming that the relative flows of the solid phases are of laminar type, it follows that

$$s_i^{(1,3)} = \beta_{ij}^{(1,3)} \dot{q}_j^{(1,3)}, \quad s_i^{(3,1)} = \beta_{ij}^{(3,1)} \dot{q}_j^{(3,1)}. \quad (5.2)$$

Note that by their definition,

$$\frac{1}{V_b} \int_{\Omega_1} s_i^{(1,3)} dx = \dot{q}_i^{(1,3)}, \quad \frac{1}{V_b} \int_{\Omega_3} s_i^{(3,1)} dx = \dot{q}_i^{(3,1)}. \quad (5.3)$$

Then the kinetic energy densities \mathcal{T}_1 and \mathcal{T}_3 in Ω_1 and Ω_3 are given by

$$\begin{aligned} \mathcal{T}_1 &= \frac{1}{2} \frac{1}{V_b} \int_{\Omega_1} \rho_1 (\dot{u}_i^{(3)} + s_i^{(1,3)}) (\dot{u}_i^{(3)} + s_i^{(1,3)}) dx \\ &= \frac{1}{2} \rho_1 \phi_1 \dot{u}_i^{(3)} \dot{u}_i^{(3)} + \rho_1 \dot{u}_i^{(3)} \dot{q}_i^{(1,3)} + \frac{1}{2} n_{ij}^{(1,3)} \dot{q}_i^{(1,3)} \dot{q}_j^{(1,3)}, \end{aligned} \quad (5.4)$$

$$\begin{aligned} \mathcal{T}_3 &= \frac{1}{2} \frac{1}{V_b} \int_{\Omega_3} \rho_3 (\dot{u}_i^{(1)} + s_i^{(3,1)}) (\dot{u}_i^{(1)} + s_i^{(3,1)}) dx \\ &= \frac{1}{2} \rho_3 \phi_3 \dot{u}_i^{(1)} \dot{u}_i^{(1)} + \rho_3 \dot{u}_i^{(1)} \dot{q}_i^{(3,1)} + \frac{1}{2} n_{ij}^{(3,1)} \dot{q}_i^{(3,1)} \dot{q}_j^{(3,1)}, \end{aligned} \quad (5.5)$$

where

$$\begin{aligned} n_{ij}^{(1,3)} &= \rho_1 \frac{1}{V_b} \int_{\Omega_1} \beta_{ki}^{(1,3)} \beta_{kj}^{(1,3)} dx, \\ n_{ij}^{(3,1)} &= \rho_3 \frac{1}{V_b} \int_{\Omega_3} \beta_{ki}^{(3,1)} \beta_{kj}^{(3,1)} dx. \end{aligned}$$

In terms of the original variables $u_i^{(1)}$, $u_i^{(3)}$ and for the case of statistical isotropy, (i.e., $n_{ij}^{(1,3)} = n^{(1,3)} \delta_{ij}$, $n_{ij}^{(3,1)} = n^{(3,1)} \delta_{ij}$), we can rewrite \mathcal{T}_1 and \mathcal{T}_3 in the form

$$\begin{aligned} \mathcal{T}_1 &= \frac{1}{2} ((\phi_1)^2 n^{(1,3)} - \rho_1 \phi_1) \dot{u}_i^{(3)} \dot{u}_i^{(3)} + (\rho_1 \phi_1 \\ &\quad - (\phi_1)^2 n^{(1,3)}) \dot{u}_i^{(1)} \dot{u}_i^{(3)} + \frac{1}{2} (\phi_1)^2 n^{(1,3)} \dot{u}_i^{(1)} \dot{u}_i^{(1)}, \end{aligned} \quad (5.6)$$

and

$$\mathcal{T}_3 = \frac{1}{2}((\phi_3)^2 n^{(3,1)} - \rho_3 \phi_3) \dot{u}_i^{(1)} \dot{u}_i^{(1)} + (\rho_3 \phi_3 - \frac{1}{2} \rho_2 \phi \dot{u}_i^{(2)} \dot{u}_i^{(2)}), \quad (5.10)$$

$$- (\phi_3)^2 n^{(3,1)} \dot{u}_i^{(1)} \dot{u}_i^{(3)} + \frac{1}{2} (\phi_3)^2 n^{(3,1)} \dot{u}_i^{(3)} \dot{u}_i^{(3)}. \quad (5.7)$$

Next, let us define the macroscopic relative velocities of the fluid with respect to the two solid phases:

$$w_i^{(m)} = \phi(u_i^{(2)} - u_i^{(m)}), \quad m = 1, 3. \quad (5.8)$$

Then if $v_i^{(1)}$ and $v_i^{(3)}$ denote the corresponding relative microvelocity fields, the assumption that the relative fluid flow is of laminar type yields

$$v_i^{(l)} = \alpha_{ij}^{(l,2)} \dot{w}_j^{(l)}, \quad l = 1, 3. \quad (5.9)$$

Since $(1/V_b) \int_{\Omega_2} v_i^{(l)} dx = \dot{w}_i^{(l)}$, $l = 1, 3$, the kinetic energy density \mathcal{T}_2 in the fluid part Ω_2 is given by

$$\begin{aligned} \mathcal{T}_2 &= \frac{1}{2} \frac{1}{V_b} \int_{\Omega_2} \rho_2 (\dot{u}_i^{(1)} + v_i^{(1)}) (\dot{u}_i^{(1)} + v_i^{(1)}) dx \\ &+ \frac{1}{2} \frac{1}{V_b} \int_{\Omega_2} \rho_2 ((\dot{u}_i^{(3)} + v_i^{(3)})) (\dot{u}_i^{(3)} + v_i^{(3)}) \\ &- \frac{1}{2} \rho_2 \phi \dot{u}_i^{(2)} \dot{u}_i^{(2)} \\ &= \frac{1}{2} \rho_2 \phi \dot{u}_i^{(1)} \dot{u}_i^{(1)} + \rho_2 \dot{u}_i^{(1)} \dot{w}_i^{(1)} + \frac{1}{2} m_{ij}^{(1,2)} \dot{w}_i^{(1)} \dot{w}_j^{(1)} \\ &+ \frac{1}{2} \rho_2 \phi \dot{u}_i^{(3)} \dot{u}_i^{(3)} + \rho_2 \dot{u}_i^{(3)} \dot{w}_i^{(3)} + \frac{1}{2} m_{ij}^{(3,2)} \dot{w}_i^{(3)} \dot{w}_j^{(3)} \end{aligned}$$

where

$$m_{ij}^{(l,2)} = \rho_2 \frac{1}{V_b} \int_{\Omega_2} \alpha_{ki}^{(l,2)} \alpha_{kj}^{(l,2)} dx, \quad l = 1, 3.$$

Next, using that $u_i^{(2)} = w_i / \phi + S_1 u_i^{(1)} + S_3 u_i^{(3)}$ in Eq. (5.8) gives

$$\begin{aligned} w_i^{(1)} &= w_i + S_3 \phi (u_i^{(3)} - u_i^{(1)}), \\ w_i^{(3)} &= w_i + S_1 \phi (u_i^{(1)} - u_i^{(3)}), \end{aligned} \quad (5.11)$$

and

$$\begin{aligned} w_i^{(1)} w_i^{(1)} &= w_i w_i + 2 S_3 \phi w_i (u_i^{(3)} - u_i^{(1)}) \\ &+ (S_3 \phi)^2 (u_i^{(3)} - u_i^{(1)}) (u_i^{(3)} - u_i^{(1)}), \\ w_i^{(3)} w_i^{(3)} &= w_i w_i + 2 S_1 \phi w_i (u_i^{(1)} - u_i^{(3)}) \\ &+ (S_1 \phi)^2 (u_i^{(3)} - u_i^{(1)}) (u_i^{(3)} - u_i^{(1)}), \quad (5.12) \\ w_i^{(1)} w_i^{(3)} &= w_i w_i + (S_3 - S_1) \phi w_i (u_i^{(3)} - u_i^{(1)}) \\ &- S_1 S_3 (\phi)^2 (u_i^{(3)} - u_i^{(1)}) (u_i^{(3)} - u_i^{(1)}). \end{aligned}$$

Using Eqs. (5.11) and (5.12) in Eq. (5.10), for the isotropic case the following expression for \mathcal{T}_2 is obtained:

$$\begin{aligned} \mathcal{T}_2 &= [\frac{1}{2} \rho_2 \phi + \frac{1}{2} (\phi)^2 ((S_3)^2 m^{(1,2)} + (S_1)^2 m^{(3,2)}) - S_3 \rho_2 \phi - \frac{1}{2} (S_1)^2 \rho_2 \phi] \dot{u}_i^{(1)} \dot{u}_i^{(1)} \\ &+ [\rho_2 + \phi (S_1 m^{(3,2)} - S_3 m^{(1,2)}) - S_1 \rho_2] \dot{u}_i^{(1)} \dot{w}_i + [\rho_2 \phi - (\phi)^2 ((S_3)^2 m^{(1,2)} + (S_1)^2 m^{(3,2)}) - S_1 S_3 \rho_2 \phi] \dot{u}_i^{(1)} \dot{u}_i^{(3)} \\ &+ \frac{1}{2} \left[m^{(1,2)} + m^{(3,2)} - \frac{\rho_2}{\phi} \right] \dot{w}_i \dot{w}_i + [\rho_w + \phi (S_3 m^{(1,2)} - S_1 m^{(3,2)}) - S_3 \rho_2] \dot{w}_i \dot{u}_i^{(3)} \\ &+ [\frac{1}{2} \rho_2 \phi + \frac{1}{2} (\phi)^2 ((S_3)^2 m^{(1,2)} + (S_1)^2 m^{(3,2)}) - S_1 \rho_2 \phi - \frac{1}{2} (S_3)^2 \rho_2 \phi] \dot{u}_i^{(3)} \dot{u}_i^{(3)}. \end{aligned} \quad (5.13)$$

The kinetic energy density \mathcal{T} in Ω is therefore

$$\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3. \quad (5.14)$$

Remark: The third term on the right-hand side of Eq. (5.13) represents dynamic interaction between the two solid phases.

B. Dissipation function

Here it is assumed that the dissipation function \mathcal{D} is a quadratic non-negative form in the variables $(\dot{u}_i^{(3)} - \dot{u}_i^{(1)})$ and \dot{w}_i . Then, if η denotes the fluid viscosity, in the statistically isotropic case the dissipation function has the form

$$\begin{aligned} \mathcal{D} &= \frac{1}{2} f_{11} (\dot{u}_i^{(3)} - \dot{u}_i^{(1)}) (\dot{u}_i^{(3)} - \dot{u}_i^{(1)}) + \frac{1}{2} f_{22} \dot{w}_i \dot{w}_i \\ &+ f_{12} (\dot{u}_i^{(3)} - \dot{u}_i^{(1)}) \dot{w}_i. \end{aligned} \quad (5.15)$$

Remark: Appendix B contains the derivation of formulas to compute the mass and dissipation coefficients in the ki-

netic energy density \mathcal{T} and the dissipation function \mathcal{D} so that for the case of uniform porosity the model presented in Ref. 4 is obtained. Thus, this new model is a generalization to the nonuniform porosity case of the previous models proposed in Refs. 6 and 4.

C. The differential equations of motion

Set $u = (u_j) = (u_i^{(1)}, w_i, u_i^{(3)})$, $1 \leq i \leq 3$, $1 \leq j \leq 9$. The Lagrangian formulation of the equations of motion for our system is

$$\frac{d}{dt} \left(\frac{\partial \mathcal{T}}{\partial \dot{u}_j} \right) + \frac{\partial \mathcal{D}}{\partial \dot{u}_j} = - \frac{\partial \mathcal{V}_d}{\partial u_j}, \quad 1 \leq j \leq 9. \quad (5.16)$$

Next, combine Eqs. (2.13), (5.6), (5.7), (5.13)–(5.16) to conclude that the equations of motion can be written in the form:

$$\begin{aligned}
& p_{11}\ddot{u}_i^{(1)} + p_{12}\ddot{w}_i + p_{13}\ddot{u}_i^{(3)} + f_{11}\dot{u}_i^{(1)} - f_{12}\dot{w}_i - f_{11}\dot{u}_i^{(3)} \\
&= \frac{\partial}{\partial x_j} [\sigma_{ij}^{(1)} - S_1 \phi p_f \delta_{ij}], \quad (5.17)
\end{aligned}$$

$$p_{12}\ddot{u}_i^{(1)} + p_{22}\ddot{w}_i + p_{23}\ddot{u}_i^{(3)} - f_{12}\dot{u}_i^{(1)} + f_{22}\dot{w}_i + f_{12}\dot{u}_i^{(3)} = -\frac{\partial p_f}{\partial x_i}, \quad (5.18)$$

$$\begin{aligned}
& p_{13}\ddot{u}_i^{(1)} + p_{23}\ddot{w}_i + p_{33}\ddot{u}_i^{(3)} - f_{11}\dot{u}_i^{(1)} + f_{12}\dot{w}_i + f_{11}\dot{u}_i^{(3)} \\
&= \frac{\partial}{\partial x_j} [\sigma_{ij}^{(3)} - S_3 \phi p_f \delta_{ij}], \quad i=1,2,3. \quad (5.19)
\end{aligned}$$

The mass coupling coefficients in Eqs. (5.17)–(5.19) are given by

$$\begin{aligned}
p_{11} &= \rho_2 \phi + (\phi)^2 ((S_3)^2 m^{(1,2)} + (S_1)^2 m^{(3,2)}) - 2S_3 \rho_2 \phi \\
&\quad - (S_1)^2 \rho_2 \phi + (\phi_1)^2 n^{(1,3)} + (\phi_3)^2 n^{(3,1)} - \phi_3 \rho_3, \\
p_{12} &= p_{21} = \rho_2 + \phi (S_1 m^{(3,2)} - S_3 m^{(1,2)}) - S_1 \rho_2, \\
p_{13} &= p_{31} = \rho_2 \phi - (\phi)^2 ((S_3)^2 m^{(1,2)} + (S_1)^2 m^{(3,2)}) \\
&\quad - S_1 S_3 \rho_2 \phi + \rho_1 \phi_1 - (\phi_1)^2 n^{(1,3)} + \rho_3 \phi_3 \\
&\quad - (\phi_3)^2 n^{(3,1)}, \quad (5.20) \\
p_{22} &= m^{(1,2)} + m^{(3,2)} - \frac{\rho_2}{\phi}, \\
p_{23} &= p_{32} = \rho_w + \phi (S_3 m^{(1,2)} - S_1 m^{(3,2)}) - S_3 \rho_2, \\
p_{33} &= \rho_2 \phi + (\phi)^2 ((S_3)^2 m^{(1,2)} + (S_1)^2 m^{(3,2)}) - 2S_1 \rho_2 \phi \\
&\quad - (S_3)^2 \rho_2 \phi + (\phi_1)^2 n^{(1,3)} + (\phi_3)^2 n^{(3,1)} - \phi_1 \rho_1.
\end{aligned}$$

The coefficients p_{ij} in Eq. (5.20) can be written in terms of the tortuosities a_{13} , a_{31} , a_{12} , and a_{32} defined in Refs. 4 and 6 as follows:

$$\begin{aligned}
n^{(1,3)} &= a_{13} \frac{\rho_1}{\phi_1}, \quad n^{(3,1)} = a_{31} \frac{\rho_3}{\phi_3}, \\
m^{(1,2)} &= a_{12} \frac{\rho_2}{\phi}, \quad m^{(3,2)} = a_{32} \frac{\rho_2}{\phi}, \quad (5.21) \\
a_{12} &= \frac{\phi_1 \rho}{\phi \rho_2} r_{12} + 1, \quad a_{32} = \frac{\phi_3 \rho'}{\phi \rho_2} r_{32} + 1, \\
a_{13} &= \frac{\phi_3 \rho'}{\phi_1 \rho_1} r_{13} + 1, \quad a_{31} = \frac{\phi_1 \rho}{\phi_3 \rho_3} r_{31} + 1,
\end{aligned}$$

where r_{ij} are the geometrical aspects of the boundaries separating the phases i and j (equal to $\frac{1}{2}$ for spheres) and

$$\rho = \frac{\phi \rho_2 + \phi_3 \rho_3}{\phi + \phi_3}, \quad \rho' = \frac{\phi \rho_2 + \phi_1 \rho_1}{\phi + \phi_1}.$$

Using the relations (5.21) in Eq. (5.20) yields

$$\begin{aligned}
p_{11} &= \rho_2 \phi (1 + (S_1)^2 a_{32} + (S_3)^2 a_{12} - 2S_3 - (S_1)^2) \\
&\quad + a_{13} \rho_1 \phi_1 + (a_{31} - 1) \rho_3 \phi_3, \\
p_{12} &= \rho_2 (S_3 (1 - a_{12}) + S_1 a_{32}),
\end{aligned}$$

$$\begin{aligned}
p_{13} &= \rho_2 \phi (1 - (S_1)^2 a_{32} - (S_3)^2 a_{12} - S_1 S_3) \\
&\quad + \rho_1 \phi_1 (1 - a_{13}) + \rho_3 \phi_3 (1 - a_{31}), \quad (5.22)
\end{aligned}$$

$$p_{22} = \frac{\rho_2}{\phi} (a_{12} + a_{32} - 1),$$

$$p_{23} = \rho_2 (S_1 (1 - a_{32}) + S_3 a_{12}),$$

$$\begin{aligned}
p_{33} &= \rho_2 \phi (1 + (S_1)^2 a_{32} + (S_3)^2 a_{12} - 2S_1 - (S_3)^2) \\
&\quad + a_{31} \rho_3 \phi_3 + (a_{13} - 1) \rho_1 \phi_1.
\end{aligned}$$

In Appendix B the mass coefficients p_{ij} are related to the mass coefficients of the previous formulations in Refs. 6 and 4.

VI. PLANE WAVE ANALYSIS

Let ω denote the angular temporal frequency and let us define the matrices $\mathcal{M} \in \mathbf{R}^{7 \times 7}$, $\mathcal{E} \in \mathbf{R}^{7 \times 7}$ be defined by

$$\mathcal{M} = \begin{bmatrix} m_{11} & 0 & 0 & m_{12} & m_{13} & 0 & 0 \\ 0 & q_1 & 0 & 0 & 0 & q_2 & 0 \\ 0 & 0 & q_1 & 0 & 0 & 0 & q_2 \\ m_{12} & 0 & 0 & m_{22} & m_{33} & 0 & 0 \\ m_{13} & 0 & 0 & m_{23} & m_{33} & 0 & 0 \\ 0 & q_2 & 0 & 0 & 0 & q_3 & 0 \\ 0 & 0 & q_2 & 0 & 0 & 0 & q_3 \end{bmatrix},$$

and

$$\mathcal{E} = \begin{bmatrix} H_1 + \frac{4}{3} \mu_1 & 0 & 0 & B_1 & B_3 + \frac{2}{3} \mu_{13} & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & \frac{1}{2} \mu_{13} & 0 \\ 0 & 0 & \mu_1 & 0 & 0 & 0 & \frac{1}{2} \mu_{13} \\ B_1 & 0 & 0 & K_{av} & B_2 & 0 & 0 \\ B_3 + \frac{2}{3} \mu_{13} & 0 & 0 & B_2 & H_3 + \frac{4}{3} \mu_3 & 0 & 0 \\ 0 & \frac{1}{2} \mu_{13} & 0 & 0 & 0 & \mu_3 & 0 \\ 0 & 0 & \frac{1}{2} \mu_{13} & 0 & 0 & 0 & \mu_3 \end{bmatrix},$$

where

$$\begin{aligned}
m_{11} &= p_{11} - i \frac{f_{11}}{\omega}, \quad m_{12} = p_{12} + i \frac{f_{12}}{\omega}, \quad m_{13} = p_{13} + i \frac{f_{11}}{\omega}, \\
m_{22} &= p_{22} - i \frac{f_{22}}{\omega}, \quad m_{23} = p_{23} + i \frac{f_{12}}{\omega}, \quad m_{33} = p_{33} + i \frac{f_{11}}{\omega}, \\
q_1 &= m_{11} - \frac{(m_{12})^2}{m_{22}}, \quad q_2 = m_{13} - \frac{m_{12} m_{23}}{m_{22}}, \\
q_3 &= m_{33} - \frac{(m_{23})^2}{m_{22}}.
\end{aligned}$$

Set

$$\mathcal{S} = \mathcal{E}^{-1} \mathcal{M}. \quad (6.1)$$

Then a generalization of the argument using plane waves given by Santos *et al.*¹⁴ shows that after finding the complex eigenvalues $1/(c_m)^2$, $m=1, \dots, 7$ of the matrix \mathcal{S} by solving

$\det(\mathcal{S} - (1/c^2)I) = 0$, the phase velocities α_m and the attenuation coefficients d_m (in dB) of the compressional and shear waves are obtained in terms of c_m from

$$\alpha_m = \frac{1}{\operatorname{Re}(c_m)}, \quad d_m = 2\pi 8.685\,889 \left| \frac{\operatorname{Im}(c_m)}{\operatorname{Re}(c_m)} \right|. \quad (6.2)$$

Three of the eigenvalues of the matrix \mathcal{S} , associated with the first, fourth, and fifth rows and columns are related with the compressional modes, while of the other four eigenvalues only two of them are different and they are related with the two shear modes of propagation. These wave modes are in agreement with those predicted previously in Refs. 6, 7, and 4.

The experimental observation of the additional (slow) waves was reported by Leclaire *et al.*⁷ The slow wave modes are important to explain attenuation and dispersion effects observed on the faster modes associated with scattering phenomena due to the presence of heterogeneities inside the composite poroelastic materials being analyzed.

VII. AN EXISTENCE AND UNIQUENESS RESULT

Let the positive definite mass matrix $\mathcal{P} \in \mathbf{R}^{9 \times 9}$ and the non-negative dissipation matrix $\mathcal{C} \in \mathbf{R}^{9 \times 9}$ be defined by

$$\mathcal{P} = \begin{bmatrix} p_{11}I & p_{12}I & p_{13}I \\ p_{12}I & p_{22}I & p_{23}I \\ p_{13}I & p_{23}I & p_{33}I \end{bmatrix},$$

$$\mathcal{C} = \begin{bmatrix} f_{11}I & -f_{12}I & -f_{11}I \\ -f_{12}I & f_{22}I & f_{12}I \\ -f_{11}I & f_{12}I & f_{11}I \end{bmatrix},$$

where I denotes the identity matrix in $\mathbf{R}^{3 \times 3}$. Also, let $\mathcal{L}(u)$ be the second-order differential operator defined by

$$\mathcal{L}(u) = \{ \nabla \cdot [\sigma_{ij}^{(1)}(u) - S_1 \phi p_f(u) \delta_{ij}], \\ - \nabla p_f(u), \nabla \cdot [\sigma_{ij}^{(3)}(u) - S_3 \phi p_f(u) \delta_{ij}] \}.$$

Then the equations of motion (5.17)–(5.19) can be stated in the form

$$\mathcal{P} \frac{\partial^2 u}{\partial t^2} + \mathcal{C} \frac{\partial u}{\partial t} - \mathcal{L}(u) = f(x, t),$$

$$(x, t) \in \Omega \times (0, T) \equiv \Omega \times J. \quad (7.1)$$

Let us consider the solution of Eq. (7.1) with initial conditions

$$u(x, 0) = u^0, \quad \frac{\partial u}{\partial t}(x, 0) = v^0, \quad x \in \Omega, \quad (7.2)$$

and boundary conditions

$$(\sigma_{ij}^{(1)}(u) - S_1 \phi p_f(u) \delta_{ij}) \nu_j = -g^{(1)}, \quad (x, t) \in \partial\Omega \times J, \quad (7.3)$$

$$(\sigma_{ij}^{(3)}(u)) - S_3 \phi p_f(u) \delta_{ij} \nu_j = -g^{(3)}, \quad (x, t) \in \partial\Omega \times J, \quad (7.4)$$

$$p_f(u) = g^{(2)}, \quad (x, t) \in \partial\Omega \times J. \quad (7.5)$$

To state a weak form of problem (7.1)–(7.5) some notation needs to be introduced. Let (\cdot, \cdot) and $\langle \cdot, \cdot \rangle$ denote the usual inner products in $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively. Also, for any real number s and $E = \Omega, \partial\Omega$ let $H^s(E)$ denote the usual Sobolev space with corresponding norm $\|\cdot\|_{s,E}$. Let

$$H(\operatorname{div}, \Omega) = \{ v \in [L^2(\Omega)]^3 : \nabla \cdot v \in L^2(\Omega) \},$$

$$V = [H^1(\Omega)]^3 \times H(\operatorname{div}, \Omega) \times [H^1(\Omega)]^3, \quad (7.6)$$

with the natural norm

$$\|v\|_V = [\|v^{(1)}\|_{1,\Omega}^2 + \|v^{(3)}\|_{1,\Omega}^2 + \|v^{(2)}\|_{H(\operatorname{div}, \Omega)}^2]^{1/2},$$

$$v = (v^{(1)}, v^{(2)}, v^{(3)}) \in V.$$

Set

$$\Lambda(u, v) = (\sigma_{ij}^{(1)}(u) - S_1 \phi p_f(u) \delta_{ij}, \epsilon_{ij}(v^{(1)})) + (\sigma_{ij}^{(3)}(u) \\ - S_3 \phi p_f(u) \delta_{ij}, \epsilon_{ij}(v^{(3)})) - (p_f(u), \nabla \cdot v^2). \quad (7.7)$$

Next recall that the strain energy density \mathcal{W} in Eq. (3.1) is a positive definite quadratic form in the variables $\epsilon_{ij}(u^{(1)})$, $\epsilon_{ij}(u^{(3)})$, and ζ , which implies that

$$\mathcal{W}(u) \geq M \left[\sum_{ij} ((\epsilon_{ij}(u^{(1)}))^2 + (\epsilon_{ij}(u^{(3)}))^2) + \zeta^2 \right], \quad (7.8)$$

and consequently applying Korn's second inequality^{15,16} it follows that

$$\Lambda(u, u) \geq M \int_{\Omega} \left[\sum_{ij} ((\epsilon_{ij}(u^{(1)}))^2 + (\epsilon_{ij}(u^{(3)}))^2) \right. \\ \left. + (\nabla \cdot w)^2 \right] d\Omega \\ \geq M_1 [\|u^{(1)}\|_{1,\Omega}^2 + \|u^{(3)}\|_{1,\Omega}^2 + \|w\|_{H(\operatorname{div}, \Omega)}^2] \\ - M_2 \|u\|_{0,\Omega}^2, \quad \forall u \in V. \quad (7.9)$$

In Eqs. (7.8) and (7.9) M , M_1 , and M_2 denote positive constants depending only on the upper and lower bounds of the coefficients of our differential problem and the domain Ω .

Next, the weak form of problem (7.1)–(7.5) is obtained as usual by multiplying Eq. (7.1) by $v \in V$ and integrating the result over Ω , using integration by parts in the $(\mathcal{L}(u), v)$ -term and applying the boundary conditions (7.3)–(7.5). Thus a variational form for our problem can be formulated as follows: find the map $u: J \rightarrow V$ such that

$$\left(\mathcal{P} \frac{\partial^2 u}{\partial t^2}, v \right) + \left(\mathcal{C} \frac{\partial u}{\partial t}, v \right) + \Lambda(u, v) + \langle g^{(1)}, v^{(1)} \rangle + \langle g^{(3)}, v^{(3)} \rangle \\ + \langle v^{(2)} \cdot \nu, g^{(2)} \rangle = (f, v), \quad v \in V, \quad t \in J. \quad (7.10)$$

Set

$$\begin{aligned}
Q_r^2 = & \left\| \frac{\partial^r f}{\partial t^r} \right\|_{L^2(J, [L^2(\Omega)]^9)} + \left\| \frac{\partial^r g^{(1)}}{\partial t^r} \right\|_{L^\infty(J, [H^{-1/2}(\partial\Omega)]^3)} \\
& + \left\| \frac{\partial^{r+1} g^{(1)}}{\partial t^{r+1}} \right\|_{L^2(J, [H^{-1/2}(\partial\Omega)]^3)} \\
& + \left\| \frac{\partial^r g^{(3)}}{\partial t^r} \right\|_{L^\infty(J, [H^{-1/2}(\partial\Omega)]^3)} \\
& + \left\| \frac{\partial^{r+1} g^{(3)}}{\partial t^{r+1}} \right\|_{L^2(J, [H^{-1/2}(\partial\Omega)]^3)} \\
& + \left\| \frac{\partial^r g^{(2)}}{\partial t^r} \right\|_{L^\infty(J, H^{1/2}(\partial\Omega))} + \left\| \frac{\partial^{r+1} g^{(2)}}{\partial t^{r+1}} \right\|_{L^2(J, H^{1/2}(\partial\Omega))},
\end{aligned} \tag{7.11}$$

$$P^2 = \|u^0\|_{2,\Omega}^2 + \|v^0\|_{1,\Omega}^2 + \|f(x,0)\|_{0,\Omega}^2 + 1. \tag{7.12}$$

Let us state a theorem about the existence, uniqueness, and regularity of the solution u of problem (7.1)–(7.5). The proof is similar to that given by Santos *et al.*¹⁷ for the case in which the porous solid matrix consists of only one solid phase and is omitted.

Theorem 1: *Let $f, g^{(1)}, g^{(2)}, g^{(3)}, u^0, v^0$ be given and such that $Q_0 < \infty, Q_1 < \infty, P < \infty$. Then there exists a unique solution $u(x,t)$ of (7.1)–(7.5) such that $u, \partial u/\partial t \in L^\infty(J, V)$ and $\partial^2 u/\partial t^2 \in L^\infty(J, [L^2(\Omega)]^9)$.*

VIII. EXAMPLES

A. Shaley sandstones

The theory can be applied to various composite media, such as shaley sandstones,¹ permafrost,^{3,18} gas-hydrate bearing sediments,⁴ and frozen foods.⁵ Let us consider a shaley sandstone, and denote the sand fraction by S_1 and the clay fraction by S_3 . As stated previously, the theory predicts three compressional waves (P waves) and two shear waves (S waves), whose phase velocities can be obtained by solving the eigensystem resulting from the equation of motion as indicated in Sec. VI [cf. Eq. (6.2)]; see also Ref. 1. In this example, we consider the faster P and S waves.

The bulk and shear moduli of the sand and clay (dry) matrices versus porosity ϕ are obtained from a relationship proposed by Krief *et al.*¹⁹ using formulas (A6) and (A7) in Appendix A. We consider the data set published by Han *et al.*²⁰ obtained at a confining pressure of 40 MPa. Han and his co-workers provide ultrasonic measurements of P- and S-wave velocities for 75 sandstone samples with porosities ranging from 2% to 30% and clay content from 0 to 50%. One feature of this data set is that a small amount of clay significantly softens the rock moduli, leading to reduced velocities. Table I shows the properties of the different constituents. The friction coefficients and permeabilities are calculated by using the equations given in Appendix B. Moreover, the mass coefficients are $r_{12}=r_{32}=r_{13}=r_{31}=1/2$. The predictions of the theory against the measurements obtained by Han *et al.*²⁰ are shown in Figs. 1(a) and

TABLE I. Material properties of the clay-bearing sandstone.

Material	Property	Value
Solid grain	Bulk modulus, K_{s1}	39 GPa
	Shear modulus, μ_{s1}	39 GPa
	Density, ρ_1	2650 kg/m ³
	Average radius, R_{s1}	50 μ m
Clay	Bulk modulus, K_{s3}	20 GPa
	Shear modulus, μ_{s3}	10. GPa
	Density, ρ_3	2650 kg/m ³
	Average radius, R_{s3}	1 μ m
Fluid	Bulk modulus, K_f	2.4 GPa
	Density, ρ_2	1000 kg/m ³
	Viscosity, η	1.798 cP

(b), where $A=2$ and $a=0.5$ [see Eqs. (A6) and (A7)]. To fit the experimental data a frequency of 5 kHz was assumed. Strictly speaking, this is not correct since the data have been acquired at ultrasonic frequencies of the order of hundreds of kilohertz. However, it is well known that Biot-type dissipation mechanisms alone do not account for the level of attenuation observed in rocks. A correct description of this phenomenon would require the generalization of the different stiffness moduli to relaxation functions.¹⁰ However, this fact reflects the robustness of the model for this particular example. The figure shows the compressional and shear velocities versus porosity, where each curve corresponds to a dif-

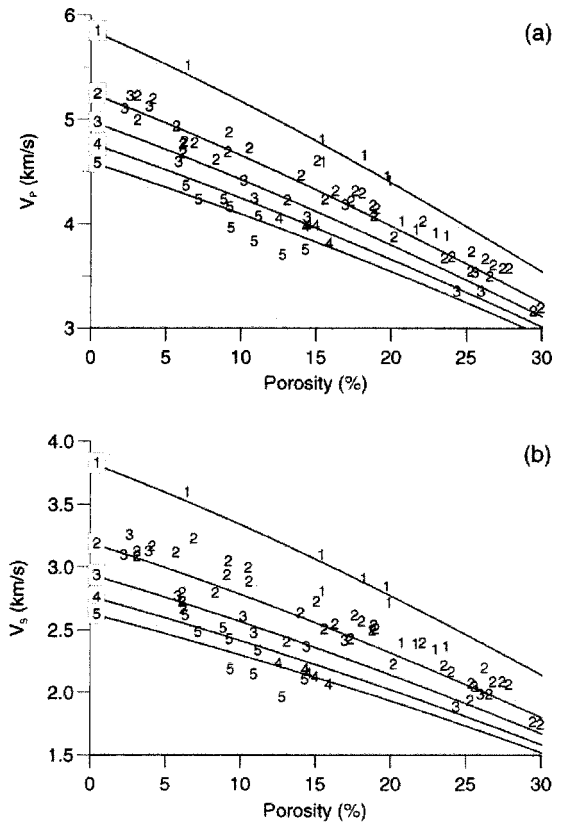


FIG. 1. Velocities of the faster compressional (a) and shear (b) waves vs porosity ϕ for different values of clay content S_3 , indicated by the numbers inside the boxes (1: $S_3=0\%$, 2: $S_3=10\%$, 3: $S_3=20\%$, 4: $S_3=30\%$ and 5: $S_3=40\%$). The experimental data, represented with numbers, correspond to the data set published by Han *et al.* (Ref. 20). In this case, 1, 2, 3, 4, and 5 correspond to S_3 values in the ranges $[S_3, S_3+5\%]$, $S_3=0, \dots, 40\%$. The frequency is 5 kHz.

ferent value of the clay content S_3 . The root-mean-square deviation computed for all samples, apart from five outliers for P waves and seven outliers for S waves, is 93 m/s for the P-wave velocity and 100 m/s for the S-wave velocity.

B. Permafrost

To illustrate the use of the variable-porosity differential equations, let us consider wave propagation through an interface separating two sandstones with different ice content. Basically, the model consists of two homogeneous half spaces separated by a plane boundary. The upper medium is the sandstone used in the examples of Carcione and Seriani,¹⁸ with no ice in the pores. It has a porosity of 20% when the medium is unfrozen. The lower media has 90% ice content in the pores. In this case the bulk and shear moduli of the sandstone and ice matrices are computed using a percolation model as indicated in Appendix A. We omit the properties of the sandstone (including the ice and water properties) since they are given in Ref. 18 [the properties correspond to those of Figs. 2(b) and 3(c) of that paper].

The time stepping method is a Runge–Kutta fourth-order algorithm, and the spatial derivatives are calculated with the Fourier method by using the fast Fourier transform.¹³ This spatial approximation is infinitely accurate for band-limited periodic functions with cutoff spatial wave numbers which are smaller than the cutoff wave numbers of the mesh. Since the presence of quasistatic modes makes the differential equations stiff, a time-splitting integration algorithm is used to solve the stiff part analytically. Due to the splitting algorithm, the modeling is second-order accurate in the time discretization. The method is illustrated in detail in Carcione and Helle²¹ for a two-phase medium and in Carcione and Seriani¹⁸ for a three-phase medium.

A 357×357 mesh is used, with square cells and a grid spacing of 14 m (the model has a dimension of approximately 5×5 km). The source is a vertical force with a dominant frequency of 12 Hz, applied at 380 m above the interface. The time step required by the Runge–Kutta algorithm is 0.5 ms. Snapshots of the wave field at 0.6 s are shown in Fig. 2. The faster P and S waves and planar head waves can be seen in the snapshots. Strong converted waves, interpreted as slow waves, can be observed in the lower medium. The high amplitudes of the slow waves at low frequencies can be due to the very high permeability of the ice frame ($5 \times 10^{-4} \text{ m}^2$).

IX. CONCLUSIONS

A theory was developed to study the processes of deformation and wave propagation in porous media composed of three interacting phases (two solids and one fluid), for the case of spatially variable porosity. The model, based on first principles, can be generalized to the case of multiple solid constituents. Appropriate constitutive relations were established, and equivalence between the elastic moduli and those corresponding to the uniform porosity case, given in previous formulations, was found, which can also be related to known petrophysical models. Using the classical Lagrangian approach, the differential equations of motion were obtained

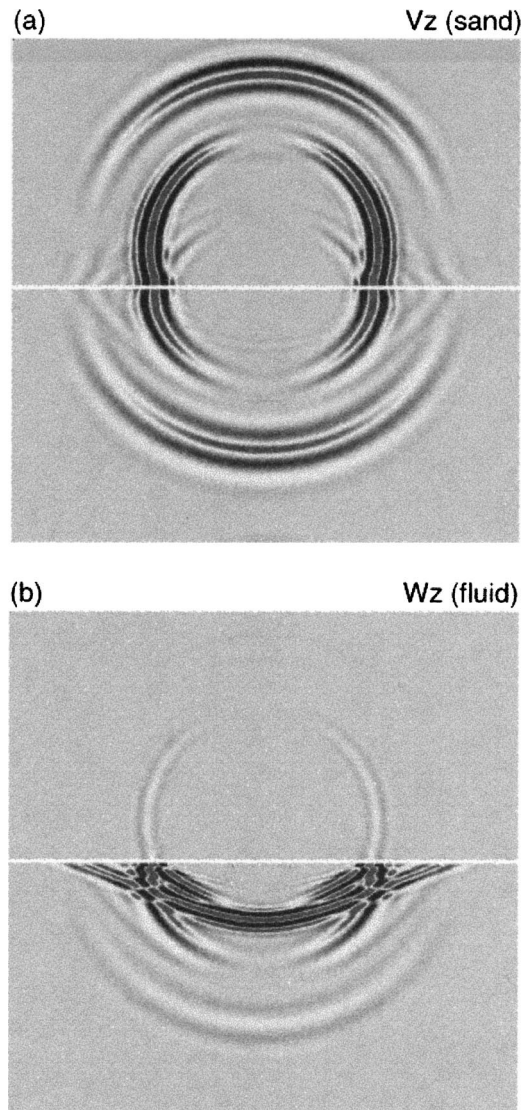


FIG. 2. Snapshots of the particle velocities of the frame (a) and particle velocity of the fluid relative to the solid phases (b) at 0.6 s. The upper medium is unfrozen and the lower medium has an ice content of 90%. The source is a vertical force in the frame with a central frequency of 12 Hz. Its location is 380 m above the interface. The ratio maximum amplitude in (a) to maximum amplitude in (b) is 547.

and the problem of existence and uniqueness of the solution under appropriate initial and boundary conditions was analyzed. It was shown that five wave modes can propagate in this composite medium (three compressional and two shear waves).

The model was applied to the study of two geophysical problems. First, the phase velocities of the faster waves in a shaly sandstone were computed for different values of water saturation and clay content. The predictions of our model agree very well with the observations. The second application consists of the numerical simulation of the wave fields generated by a point source within a frozen sandstone with variable ice content. The simulation reveals strong wave-mode conversions, indicating that the model can be useful to study the freezing conditions of porous media.

In future works simulations in heterogeneous media will be performed and the effects that the slow waves (modes)

have on the faster modes will be analyzed. The present paper has been written to obtain (and justify) the differential equations for such media.

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APPENDIX A: A FORM OF COMPUTING THE COEFFICIENTS IN THE CONSTITUTIVE RELATIONS

This section presents a form of evaluating the coefficients in the stress–strain relations (4.6)–(4.8), which combined with Eq. (4.9) yield the values of the coefficients for this new variable porosity formulation.

Let $K_{s1,m}$, $K_{s3,m}$, $\mu_{s1,m}$, and $\mu_{s3,m}$ denote the bulk and shear modulus of the two solid (dry) frames, respectively. In Appendices A1 and A2 it is indicated how to determine these moduli for the two examples presented in this paper. Also, let K_{s1} , μ_{s1} , K_{s3} , μ_{s3} denote the bulk and shear moduli of the grains in the two solid phases, respectively, and let K_f denote the bulk modulus of the fluid phase. For the coefficients μ_1 , μ_3 , and μ_{13} the formulas given in Ref. 4 were used:

$$\begin{aligned}\mu_j &= [(1-g_j)\phi_j]^2 \mu_{av} + \mu_{sj,m}, \quad g_j = \frac{\mu_{sj,m}}{\phi_j \mu_{sj}}, \quad j=1,3, \\ \mu_{13} &= (1-g_1)(1-g_3)\phi_1\phi_3\mu_{av}, \\ \mu_{av} &= \left[\frac{(1-g_1)\phi_1}{\mu_{s1}} + \frac{\phi}{2\omega\eta} + \frac{(1-g_3)\phi_3}{\mu_{s3}} \right]^{-1},\end{aligned}\quad (\text{A1})$$

where g_1 and g_3 are the so-called shear consolidation coefficients of the solid frames 1 and 3.⁶ The symbol ω in the definition of μ_{av} above denotes the angular frequency.

Also, since $K_2 = \phi^2 K_{av}$, [cf. Eq. (4.5)], to determine K_2 the following expression for K_{av} given in Ref. 4 is used:

$$K_{av} = \left[(1-c_1) \frac{\phi_1}{K_{s1}} + \frac{\phi}{K_f} + (1-c_3) \frac{\phi_3}{K_{s3}} \right]^{-1}. \quad (\text{A2})$$

The remaining elastic coefficients are given by⁴

$$\begin{aligned}K_j &= [(1-c_j)\phi_j]^2 K_{av} + K_{sj,m}, \quad c_j = \frac{K_{sj,m}}{\phi_j K_{sj}}, \quad j=1,3, \\ C_{12} &= (1-c_1)\phi_1\phi K_{av}, \\ C_{13} &= (1-c_1)(1-c_3)\phi_1\phi_3 K_{av}, \\ C_{23} &= (1-c_3)\phi\phi_3 K_{av},\end{aligned}\quad (\text{A3})$$

where c_1 , c_3 are the bulk consolidation coefficients of the solid frames 1 and 3.

These elastic moduli for constant porosity can be rewritten in terms of a set of coefficients analogous to those given by Gassmann²² as follows:

$$\begin{aligned}K_j &= K_{Gj} - 2\alpha_j S_j \phi K_{av} + (S_j \phi)^2 K_{av}, \quad j=1,3, \\ C_{12} &= \phi K_{av} (\alpha_1 - S_1 \phi), \quad C_{23} = \phi K_{av} (\alpha_3 - S_3 \phi), \\ C_{13} &= K_{av} (\alpha_1 - S_1 \phi) (\alpha_3 - S_3 \phi),\end{aligned}\quad (\text{A4})$$

where

$$K_{Gj} = K_{sj,m} + (\alpha_j)^2 K_{av}, \quad \alpha_j = S_j - \frac{K_{sj,m}}{K_{sj}}, \quad j=1,3. \quad (\text{A5})$$

The moduli K_{G1} and K_{G3} are analogous to Gassmann's modulus (a calculation shows that $K_{Gj} = H_j$, H_j , $j=1,3$), while the coefficients α_1 and α_3 correspond to the effective stress coefficients in the classic Biot theory.^{10,13} The bulk and shear moduli $K_{s1,m}$, $K_{s3,m}$, $\mu_{s1,m}$ and $\mu_{s3,m}$ in Eqs. (A1) and (A5) can be determined in several fashions. In some cases, they can be obtained from the measurements of compressional and shear wave velocities on the empty rock or alternatively, they can be estimated using known petrophysical models. The procedure used in this paper to determine those moduli for the cases of shaley sandstones and frozen porous media is indicated in the following.

1. The case of shaley sandstones

The porosity dependence of the sand and clay (dry) matrices is consistent with the concept of critical porosity, since the moduli should vanish above a certain value of the porosity (usually from 0.4 to 0.5). This dependence is determined by the empirical coefficient A in Eq. (A6). In some rocks there is an abrupt change of rock matrix properties with the addition of a small amount of clay, attributed to softening of cements, clay swelling, and surface effects.²³ That is, the wave velocities decrease significantly when the clay content increases from 0 to a few percentages. In order to model this effect, the shear modulus of the sand matrix is multiplied by a factor depending on the empirical coefficient a in Eq. (A7) (this factor tends to 1 when $a \rightarrow \infty$). If Ω_1 represents the sand and Ω_3 the clay minerals, then the bulk and shear moduli of the sand and clay (dry) matrices are assumed to satisfy

$$K_{sj,m} = S_j K_{sj} (1-\phi)^{1+A/(1-\phi)}, \quad j=1,3, \quad (\text{A6})$$

$$\mu_{s1,m} = \exp\{-[(1-S_3)S_3]^a\} K_{s1,m} \mu_{s1} / K_{s1}, \quad (\text{A7})$$

$$\mu_{s3,m} = K_{s3,m} \mu_{s3} / K_{s3}.$$

2. The case of frozen porous media

Following Refs. 6 and 4 it is assumed that $K_{s1,m}$ is known, and that the other modulus may be computed using a percolation-type model with critical exponent 3.8.²⁴ As explained by Leclaire *et al.*,⁶ the percolation theory is used here to describe the transition of a system from the continuous to the discontinuous state, which is governed by a power law independent of the system material. Hence, $\mu_{s1,m}$, $\mu_{s3,m}$ and $K_{s3,m}$ are obtained using

$$\mu_{sj,m} = [\mu_{sj,m}^{(\max)} - \mu_{sj,m}^0] \left[\frac{\phi_3}{1-\phi_1} \right]^{3.8} + \mu_{sj,m}^0, \quad j=1,3, \quad (\text{A8})$$

$$K_{s3,m} = [K_{s3,m}^{(\max)} - K_{s3,m}^0] \left[\frac{\phi_3}{1-\phi_1} \right]^{3.8} + K_{s3,m}^0,$$

where $\mu_{s1,m}^{(\max)}$, $\mu_{s3,m}^{(\max)}$, and $K_{s3,m}^{(\max)}$ are computed using Kuster and Toksöz's model,²⁵ taking the known values of K_{s1} , μ_{s1} , K_{s3} , μ_{s3} for the background medium with inclusions of air, with properties K_a , μ_a . For the solid matrix Ω_1 the concentration of inclusions is $c = 1 - \phi_1$ and for the ice matrix Ω_3

we consider that the water is totally frozen, i.e., $c = \phi_1$. The moduli $\mu_{s1,m}^0$, $\mu_{s3,m}^0$, and $K_{s3,m}^0$ are appropriate reference values. Here it is assumed that⁴

$$K_{s3,m}^0 = \mu_{s3,m}^0 = 0. \quad (\text{A9})$$

For variable temperatures, the porosity (or water proportion) may be computed as function of temperature using the expressions given in Refs. 6 and 4.

APPENDIX B: IDENTIFICATION OF THE MASS AND DISSIPATIVE COEFFICIENTS FROM PREVIOUS FORMULATIONS

In this section it is shown that under the assumption of uniform porosity from Eqs. (5.17) to (5.19) the equations of motion in the form presented in Refs. 6 and 4 are obtained.

First, use Eqs. (2.7) and (5.18) in Eq. (5.17) to obtain

$$\begin{aligned} \frac{\partial}{\partial x_j} \sigma_{ij}^{(1)} = & (p_{11} - 2S_1 \phi p_{12} + (S_1 \phi)^2 p_{22}) \ddot{u}_i^{(1)} + \phi (p_{12} \\ & - S_1 \phi p_{22}) \ddot{u}_i^{(2)} + (p_{13} - S_1 \phi p_{23} - S_3 \phi p_{12} \\ & + S_1 S_3 (\phi)^2 p_{22}) \ddot{u}_i^{(3)} - \phi (f_{12} + S_1 \phi f_{22}) \\ & \times (\dot{u}_i^{(2)} - \dot{u}_i^{(1)}) - (f_{11} + (S_1 - S_3) \phi f_{12} \\ & - S_1 S_3 (\phi)^2 f_{22}) (\dot{u}_i^{(3)} - \dot{u}_i^{(1)}). \end{aligned} \quad (\text{B1})$$

Next, setting $\sigma = -\phi p_f$, it follows from Eq. (5.18) that

$$\begin{aligned} \frac{\partial}{\partial x_i} \sigma = & \phi (p_{12} - S_1 \phi p_{22}) \ddot{u}_i^{(1)} + (\phi)^2 p_{22} \ddot{u}_i^{(2)} + \phi (p_{23} \\ & - S_3 \phi p_{22}) \ddot{u}_i^{(3)} + \phi (f_{12} + S_1 \phi f_{22}) (\dot{u}_i^{(2)} - \dot{u}_i^{(1)}) \\ & + \phi (S_3 \phi f_{22} - f_{12}) (\dot{u}_i^{(2)} - \dot{u}_i^{(3)}). \end{aligned} \quad (\text{B2})$$

Also, using Eqs. (2.7) and (5.18) in Eq. (5.19) yields

$$\begin{aligned} \frac{\partial}{\partial x_j} \sigma_{ij}^{(3)} = & (p_{13} - S_1 \phi p_{23} - S_3 \phi p_{12} + S_1 S_3 (\phi)^2 p_{22}) \ddot{u}_i^{(1)} \\ & + \phi ((p_{23} - S_3 \phi p_{22}) \ddot{u}_i^{(2)} + (p_{33} - 2S_3 \phi p_{23} \\ & + (S_3 \phi)^2 p_{22}) \ddot{u}_i^{(3)} - \phi (S_3 \phi f_{22} - f_{12}) \\ & \times (\dot{u}_i^{(2)} - \dot{u}_i^{(3)}) + (f_{11} + (S_1 - S_3) \phi f_{12} \\ & - S_1 S_3 (\phi)^2 f_{22}) (\dot{u}_i^{(3)} - \dot{u}_i^{(1)}). \end{aligned} \quad (\text{B3})$$

Using the expressions given in Eq. (5.22) a calculation shows the following equivalence between the mass coefficients defined in Ref. 4 and the new coefficients p_{ij} :

$$\begin{aligned} \rho_{11} & \equiv p_{11} - 2S_1 \phi p_{12} + (S_1 \phi)^2 p_{22} \\ & = \rho_1 \phi_1 a_{13} + \rho_2 \phi_2 (a_{12} - 1) + \rho_3 \phi_3 (a_{31} - 1), \\ \rho_{12} & \equiv \phi (p_{12} - S_1 \phi p_{22}) = -\rho_2 \phi (a_{12} - 1), \\ \rho_{13} & \equiv p_{13} - S_1 \phi p_{23} - S_3 \phi p_{12} + S_1 S_3 (\phi)^2 p_{22} \\ & = -\rho_1 \phi_1 (a_{13} - 1) - \rho_3 \phi_3 (a_{31} - 1), \end{aligned} \quad (\text{B4})$$

$$\begin{aligned} \rho_{22} & \equiv (\phi)^2 p_{22} = \rho_2 \phi_2 (a_{12} + a_{32} - 1), \\ \rho_{23} & \equiv \phi (p_{23} - S_3 \phi p_{22}) = -\rho_2 \phi (a_{32} - 1), \\ \rho_{33} & \equiv p_{33} - 2S_3 \phi p_{23} + (S_3 \phi)^2 p_{22} \\ & = \rho_1 \phi_1 (a_{13} - 1) + \rho_2 \phi_2 (a_{32} - 1) + \rho_3 \phi_3 a_{31}. \end{aligned}$$

The expressions for the mass-coupling coefficients ρ_{ij} in Eq. (B4) coincide with those given by Carcione and Tinivella.⁴

Next let us give a procedure to choose the dissipation coefficients f_{11} , f_{22} , and f_{12} . For the case of frozen porous media, following Ref. 4, the dissipation coefficients b_{12} , b_{23} , and b_{13} are defined as follows:

$$b_{12} = (\phi)^2 \frac{\eta}{\kappa_1}, \quad b_{23} = (\phi)^2 \frac{\eta}{\kappa_3}, \quad (\text{B5})$$

b_{13} = friction coefficient between the ice

and the solid frames,

where η denotes the fluid viscosity and the permeability coefficients κ_1 , κ_3 are defined in terms of the absolute permeabilities $\kappa_{1,0}$, $\kappa_{3,0}$ of the two solid frames by (see also Ref. 6)

$$\kappa_1 = \kappa_{1,0} \frac{(\phi)^3}{(1 - \phi_1)^3}, \quad \kappa_3 = \kappa_{3,0} \frac{(1 - \phi_1)^2}{\phi_3^2} \left(\frac{\phi}{\phi_1} \right)^3. \quad (\text{B6})$$

For the case of shaley sandstones, following Ref. 1 the coefficient b_{13} can be assumed to be zero and the friction coefficients b_{12} and b_{23} are taken to be of the form:

$$\begin{aligned} b_{12} & = 45 \eta R_{s1}^{-2} \phi^{-1} (1 - \phi) \phi_1, \\ b_{23} & = 45 \eta R_{s3}^{-2} \phi^{-1} (1 - \phi) \phi_3, \end{aligned} \quad (\text{B7})$$

where R_{s1} , R_{s3} denote the average radii of the sand and clay particles, respectively. These expressions are given in Appendix B3 of Carcione *et al.*¹ (but in that paper the viscous drag coefficients are respectively denoted by b_{11} and b_{33}).

It follows from Eqs. (B1) to (B3) that to recover the uniform porosity formulation in Ref. 4 the coefficients f_{11} , f_{22} , and f_{12} must be taken to satisfy the following nonsingular system of equations:

$$\begin{aligned} f_{11} + (S_1 - S_3) \phi f_{12} - S_1 S_3 (\phi)^2 f_{22} & = b_{13}, \\ \phi (S_3 \phi f_{22} - f_{12}) & = b_{23}, \\ \phi (f_{12} + S_1 \phi f_{22}) & = b_{12}. \end{aligned} \quad (\text{B8})$$

The coefficients f_{11} , f_{12} , and f_{22} are determined by Eq. (B8) with the coefficient b_{13} left as a free parameter chosen so that the condition

$$f_{11} f_{22} - f_{12}^2 \geq 0 \quad (\text{B9})$$

is satisfied, which is needed in order to have a non-negative dissipation function \mathcal{D} in the variables $(\dot{u}_i^{(3)} - \dot{u}_i^{(1)})$, and \dot{w}_i . Since the coefficient b_{13} takes into account friction between the two solid phases, a proper model based, for example, in Coulomb's friction theory may be used, but this problem is beyond the scope of this work. For simplicity in all the numerical examples presented in this article the coefficient b_{13} was set to be zero.

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