Green’s function of the Lord–Shulman thermo-poroelasticity theory

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SUMMARY
The Lord–Shulman thermoelasticity theory combined with Biot equations of poroelasticity, describes wave dissipation due to fluid and heat flow. This theory avoids an unphysical behaviour of the thermoelastic waves present in the classical theory based on a parabolic heat equation, that is infinite velocity. A plane-wave analysis predicts four propagation modes: the classical $P$ and $S$ waves and two slow waves, namely, the Biot and thermal modes. We obtain the frequency-domain Green’s function in homogeneous media as the displacements-temperature solution of the thermo-poroelasticity equations. The numerical examples validate the presence of the wave modes predicted by the plane-wave analysis. The $S$ wave is not affected by heat diffusion, whereas the $P$ wave shows an anelastic behaviour, and the slow modes present a diffusive behaviour depending on the viscosity, frequency and thermoelasticity properties. In heterogeneous media, the $P$ wave undergoes mesoscopic attenuation through energy conversion to the slow modes. The Green’s function is useful to study the physics in thermoelastic media and test numerical algorithms.

Key words: Heat flow; Seismic attenuation; Wave propagation.

1 INTRODUCTION
The study of the geomechanical properties of non-isothermal media is an important subject in many fields. The theory of thermoelasticity couples elastic deformations with the temperature field, based on the thermodynamics of irreversible processes (Biot 1956a; Deresiewicz 1957). The classical theory of poroelasticity has been established by Biot (1956b, 1962) for two-phase fluid-saturated porous media based on the Lagrange equations and Hamilton principle, which predicts the existence of a slow $P$ wave (the Biot wave) besides the classical $P$ and $S$ waves. It has been verified experimentally (e.g. Plona 1980; Kelder & Smeulders 1997). The Biot wave is diffusive at low frequencies and has a lower velocity than that of the fast $P$ wave at high frequencies.

Thermo-poroelasticity is an extension of the classical Biot theory, which combines the equation of heat conduction with the poroelasticity equations to describe the interaction between the displacement and temperature fields in porous media (Noda 1990; Nield & Bejan 2006). Bear et al. (1992) and Sharma (2008) developed mathematical models for wave propagation. The theory, if applied to heterogeneous media, also describes the important mechanism of wave-induced mesoscopic attenuation due to the presence of the elastic and thermal slow modes, which behave wave-like at high frequencies (Carcione et al. 2019, 2020). The thermal mode is diffusive for low values of the thermal conductivity and becomes wave-like for high values of this property. Compared to the uncoupled case (isothermal case), the fast $P$ wave propagates at higher velocities and the $S$ wave is not affected by the thermal flow. The theory is relevant in geophysical and geothermal exploration (Treitel 1959; Armstrong 1984; Cermak et al. 1990; Fu 2012, 2017; Jacquey et al. 2015; Poletto et al. 2018), and earthquake seismology (e.g. Boschi 1973; Simmons et al. 2010; Ritsema et al. 2011). In this work, we obtain the Green’s function of thermo-poroelasticity, based on the Lord–Shulman theory.

The governing equation of thermoelasticity based on the parabolic-type equation of heat conduction has unphysical solutions such as discontinuities and infinite velocities as a function of frequency. As shown by Carcione et al. (2018), the heat equation can be generalized in analogy with the Maxwell model of viscoelasticity and the unphysical behaviour can be avoided by introducing relaxation terms into
the heat equation, usually termed the Lord–Shulman model with a relaxation term (Lord & Shulman 1967; Green & Lindsay 1972; Eslami et al. 2013). Based on the Lord–Shulman thermoelasticity equations, Rudgers (1990) analyzed the physics and Carcione et al. (2018, 2019) generalized this theory to the poroelastic case and provided further insight into the physics by numerical simulations. A plane-wave analysis of the thermo-poroelasticity equations predicts the presence of a slow thermal wave (T wave) besides the classical P and S waves and Biot slow P wave. The presence of a slow T mode has been verified experimentally in solid helium and NaF crystals (e.g. Ackerman et al. 1966; Jackson et al. 1970; McNelly et al. 1970).

The Green’s function plays an important role in many applications (e.g. Norris 1994; Fu & Bouchon 2004; Yang et al. 2007; Hu et al. 2009; Yu & Fu 2013; Wei & Fu 2019). There are many works, concerning classical thermoelasticity (e.g. Nowacki 1975; Kupradze et al. 1976; Tosaka 1985; Tosaka & Suh 1991), including Scarpetta (1990) who obtained the Green’s function for double-porosity media. Regarding the extended theory (hyperbolic heat equation), Wang et al. (2019) formulated a second-order tensor Green’s function for the Lord–Shulman thermoelasticity equations.

In this work, we obtain the Green’s function for the Lord–Shulman thermo-poroelasticity equations by means of elementary functions (Scarpetta 1990) and perform a plane-wave analysis to obtain the velocity and attenuation as a function of frequency. Then, we formulate the frequency-domain Green’s function and, finally, wavefield snapshots are computed to further illustrate the physics.

2 THEORY OF THERMO-POROELASTICITY

The classical theory of thermo-poroelasticity is based on a parabolic equation of heat conduction, leading to infinite phase velocities. The modified equations with a relaxation term (Carcione et al. 2018) are briefly introduced in this section, which are the basis to develop the frequency-domain Green’s function for wave propagation in non-isothermal porous media.

2.1 Equations of modified thermo-poroelasticity

The constitutive relations of thermo-poroelasticity for the stress components of the frame $\sigma_{ij}$ and pore-fluid pressure $p_f$ are (Carcione et al. 2019)

$$\begin{align*}
\sigma_{ij} &= \lambda \delta_{ij} u_{kk} + \mu (u_{ij} + u_{ji}) + \alpha M \delta_{ij} (a u_{kk} + w_{kk}) - \beta \delta_{ij} T
- p_f &= M (a u_{ij} + w_{ij}) - \frac{\beta_f}{\phi} T
\end{align*}$$

(1)

where $\lambda$ and $\mu$ are the Lamé constants of the drained matrix, $\delta_{ij}$ is the Kronecker-delta, $\phi$ is the porosity, $T$ is the increment of temperature over a reference $T_0$, $u_{ij}$ are the displacement components in the solid phase, $U_i$ are the displacement components of the fluid phase,

$$\begin{align*}
\alpha &= 1 - \frac{K_m}{K_s}, \quad K_m = \lambda + \frac{2}{3} \mu
M &= \frac{K_s}{1 - \phi - \frac{K_m}{K_s} + \frac{\phi}{K_f}}
w_i &= \phi (U_i - u_i), \quad \beta = \beta_s + \alpha \beta_f
\end{align*}$$

(2)

where $K_s$ and $K_f$ are the solid and fluid bulk moduli, respectively, and $\beta_s$ and $\beta_f$ are the coefficients of thermal stress for the solid and fluid phases, respectively. The dynamical equations are (Carcione et al. 2019)

$$\begin{align*}
\sigma_{ij,j} &= \rho \ddot{u}_i + \rho_f \ddot{w}_i
- p_f,_{ij} &= \rho_f \ddot{u}_i + q \ddot{w}_i + \tau \dot{u}_i
\gamma T_{ii} &= \rho C_v (\ddot{T} + \tau \ddot{T}) + \tau_0 \left[ \dot{u}_{ii} + \dot{w}_{ii} + \tau_0 (\ddot{u}_{ii} + \ddot{w}_{ii}) \right]
\end{align*}$$

(3)

where $\gamma$ is the coefficient of heat conduction, $C_v$ is the specific heat capacity, $q = \tau \rho_f / \phi$, with $\tau$ the tortuosity and $\tau_0$ is the relaxation time. The quantity $\rho = (1 - \phi) \rho_s + \phi \rho_f$ is the composite density with $\rho_s$ and $\rho_f$ the solid and fluid densities, respectively. The Darcy law defines the movement of viscous fluids of viscosity $\eta$ in the frame of permeability $\chi$, and $r = \eta / \chi$.

Thus, the non-isothermal wave equations for isotropic porous media saturated with a viscous fluid are

$$\begin{align*}
\rho \ddot{u}_i + \rho_f \ddot{w}_i &= (\lambda + \mu + \alpha^2 M) u_{i,j,j} + \mu u_{i,j,j} + \alpha M w_{i,j,j} - \beta T_{ii}
\rho_f \ddot{u}_i + q \ddot{w}_i + \tau \dot{u}_i &= M (a u_{i,j,j} + w_{i,j,j}) - \frac{\beta_f}{\phi} T_{ii}
\gamma T_{ii} &= \rho C_v (\ddot{T} + \tau_0 \ddot{T}) + \tau_0 \left[ \dot{u}_{ii} + \dot{w}_{ii} + \tau_0 (\ddot{u}_{ii} + \ddot{w}_{ii}) \right]
\end{align*}$$

(4)
2.2 Plane-wave analysis

To examine the characteristics of wave propagation in thermo-poroelastic media, we consider the following plane-wave expressions

\[
\begin{align*}
    u_i &= A_s e^{i(\frac{\omega}{v_c} x_j)}, \\
    w_j &= B_d e^{i(\frac{\omega}{v_c} x_i)}, \\
    T &= C e^{i(\frac{\omega}{v_c} x_j)}
\end{align*}
\]

where \( s_i \) and \( d_j \) are the polarizations for the motions of the solid and fluid particles, respectively, \( A, B \) and \( C \) are amplitude constants, \( \omega \) is the angular frequency, \( t \) is the time, \( v_c \) is the complex velocity, \( l_i \) denotes the propagation directions, \( x_j \) are the position components and \( i = \sqrt{-1} \).

Substituting eq. (5) into (4) yields a system of algebraic equations,

\[
\begin{align*}
(\lambda + \mu + \alpha^2 M) \left( \frac{\omega}{v_c} \right)^2 A_s l_i l_i + \alpha M \left( \frac{\omega}{v_c} \right)^2 B_d l_i l_i + \mu \frac{\omega}{v_c}^2 A_s l_i - \rho \omega^2 A_s l_i - \rho f \omega^2 B_d l_i - i \frac{\omega}{v_c} \beta C l_i &= 0 \\
M \left( \frac{\omega}{v_c} \right)^2 A_s l_i l_i + \frac{\omega}{v_c}^2 B_d l_i l_i - \rho_f \alpha \omega^2 s_i - q B \omega^2 d_i + r \omega^2 B_d l_i - \frac{\beta_f}{\omega} \frac{\omega}{v_c} \beta C l_i &= 0
\end{align*}
\]

(6)

For \( S \) waves, \( s_i = d_i \), \( l_i = 0 \), and the direction of propagation is perpendicular to the direction of displacement, and

\[
v^S_c = \sqrt{\frac{\mu}{\rho - \frac{\omega_f}{\omega}}}.
\]

(7)

We see that \( S \)-wave propagation is independent of the thermal properties in isotropic thermo-poroelastic media.

For \( P \) waves, \( s_i = d_i \), \( l_i = 1 \), and the direction of propagation is parallel to the direction of displacement. The resulting dispersion relation can be solved by a cubic equation in \( v^P_c \) as

\[
a(v^P_c)^3 + b(v^P_c)^2 + cv^P_c + d = 0,
\]

(8)

where

\[
\begin{align*}
    a &= \rho C_v \phi N (\omega L - i \omega \rho) \\
    b &= i \rho r K - \omega (\phi r (\gamma \rho + \tau_0 K) + \phi \rho C_v H + T_0 \beta J) - i \omega^2 (\phi (\gamma L + \rho C_v H \tau_0) + T_0 \beta J \tau_0) \\
    c &= \omega (\phi (\rho C_v M + r \gamma F) + T_0 \beta G) + i \omega^2 (\phi (\gamma H + \rho C_v M E \tau_0) + T_0 \beta G \tau_0) \\
    d &= -i \omega^2 \gamma \phi ME
\end{align*}
\]

with

\[
\begin{align*}
    E &= \lambda + 2\mu, \\
    F &= E + \alpha^2 M, \\
    G &= \rho C_v \phi N (\omega L - i \omega \rho) \\
    H &= q F + \rho M - 2\alpha M \rho_f, \\
    J &= \beta_f (\rho - \rho_f) + \phi \beta (q - q_f) \; , \\
    K &= \rho C_v F + T_0 \beta^2 \\
    L &= q F - \rho^2 + M, \\
    N &= 1 + i \omega \tau_0
\end{align*}
\]

(9)

We see that the fast \( P \) waves are dissipative due to the coupling with the Biot and heat flow.

3 GREEN’S FUNCTION FOR THERMO-POROELASTICITY

We formulate the frequency-domain Green’s function for thermo-poroelasticity. We first apply a Fourier transform to eq. (4),

\[
\tilde{u} \left( \mathbf{x}, \omega \right) = \int_{-\infty}^{\infty} u \left( \mathbf{x}, t \right) e^{-i \omega t} dt,
\]

(11)

leading to the following differential equations in the frequency domain (omitting the hat for convenience):

\[
\begin{align*}
    a_1 u_{i,j} + \mu u_{i,j} + \rho \omega^2 u_i + \alpha M w_{j,i} + \rho f \omega^2 w_i - \beta T_j &= 0 \\
M (\alpha u_{i,j} + w_{j,i}) + \rho f \omega^2 u_i + a_2 w_i - \frac{\beta_f}{\phi} T_j &= 0 \\
\gamma T_{i,j} + \rho C_v a_3 T + \beta T_0 a_3 (u_{i,j} + w_{i,j}) &= 0
\end{align*}
\]

(12)

where \( a_1 = \lambda + \mu + \alpha^2 M, \; a_2 = q \omega^2 - i \omega \rho \) and \( a_3 = \tau_0 \omega^2 - i \omega \).
To derive the Green’s function for eq. (12), we introduce the following matrix differential operator based on eq. (12):

\[
A (\nabla) = \begin{bmatrix}
    a_1 \nabla \text{div} + \mu \Delta + \rho \omega^2 & \alpha M \nabla \text{div} + \rho_\gamma \omega^2 & -\beta \text{div} \\
    \alpha M \nabla \text{div} + \rho_\gamma \omega^2 & M \nabla \text{div} + a_2 & -\frac{\beta_\gamma}{\phi} \text{div} \\
    \beta T_{0a_3} \nabla & \beta T_{0a_3} \nabla & \gamma \Delta + \rho C_a a_3
\end{bmatrix},
\]

(13)

where \( \text{div} \) is the divergence symbol, and consider the system of non-homogeneous equations:

\[
\begin{bmatrix}
    a_1 \nabla \text{div} + \mu \Delta + \rho \omega^2 & \alpha M \nabla \text{div} + \rho_\gamma \omega^2 & \beta T_{0a_3} \nabla \\
    \alpha M \nabla \text{div} + \rho_\gamma \omega^2 & M \nabla \text{div} + a_2 & \beta T_{0a_3} \nabla \\
    -\beta \text{div} & -\frac{\beta_\gamma}{\phi} \text{div} & \gamma \Delta + \rho C_a a_3
\end{bmatrix}
\begin{bmatrix}
    \nabla \omega \\
    \nabla f \\
    \nabla T
\end{bmatrix} =
\begin{bmatrix}
    f_1 \\
    f_2 \\
    f_3
\end{bmatrix},
\]

(14)

or

\[
A^T (\nabla) \tilde{\alpha} = \tilde{b},
\]

(15)

where \( A^T (\nabla) \) is the transpose of matrix differential operator \( A(\nabla) \).

\[
A^T (\nabla) = \begin{bmatrix}
    a_1 \nabla \text{div} + \mu \Delta + \rho \omega^2 & \alpha M \nabla \text{div} + \rho_\gamma \omega^2 & \beta T_{0a_3} \nabla \\
    \alpha M \nabla \text{div} + \rho_\gamma \omega^2 & M \nabla \text{div} + a_2 & \beta T_{0a_3} \nabla \\
    -\beta \text{div} & -\frac{\beta_\gamma}{\phi} \text{div} & \gamma \Delta + \rho C_a a_3
\end{bmatrix},
\]

\[
\tilde{\alpha} = \begin{bmatrix}
    \nabla \omega \\
    \nabla f \\
    \nabla T
\end{bmatrix}, \text{ and } \tilde{b} = \begin{bmatrix}
    f_1 \\
    f_2 \\
    f_3
\end{bmatrix},
\]

\( \tilde{f}_1 \) and \( \tilde{f}_2 \) are two-component vector functions and \( f_3 \) is a scalar function.

Applying the divergence operator to the first and second terms of eq. (14), we obtain

\[
B (\Delta) \tilde{\beta} = \tilde{c},
\]

(16)

where

\[
B (\Delta) = \begin{bmatrix}
    a_0 \Delta + \rho \omega^2 & \alpha M \Delta + \rho_\gamma \omega^2 & \beta T_{0a_3} \Delta \\
    \alpha M \Delta + \rho_\gamma \omega^2 & M \Delta + a_2 & \beta T_{0a_3} \Delta \\
    -\beta & -\frac{\beta_\gamma}{\phi} & \gamma \Delta + \rho C_a a_3
\end{bmatrix}, \text{ with } a_0 = a_1 + \mu,
\]

\[
\tilde{\beta} = \begin{bmatrix}
    \text{div} \omega \\
    \text{div} f \\
    \nabla T
\end{bmatrix}, \text{ and } \tilde{c} = \begin{bmatrix}
    \text{div} \tilde{f}_1 \\
    \text{div} \tilde{f}_2 \\
    f_3
\end{bmatrix},
\]

By introducing the operator

\[
\Lambda_1 (\Delta) = \frac{1}{\gamma M a_0} \det B (\Delta) = \prod_{i=1}^{3} (\Delta + \lambda_i^2),
\]

the following equation

\[
\Lambda_1 (\Delta) \tilde{\beta} = \tilde{g},
\]

(17)

can be obtained from eq. (16), where

\[
g_j = \frac{1}{\gamma M a_0} \sum_{i=1}^{3} B_{ij} (\Delta) c_i,
\]

(18)

with \( B_{ij} (\Delta) \) the cofactor of the element \( B_{ij} (\Delta) \) of matrix \( B(\Delta) \).

From the first and second terms of eq. (14), we have

\[
\begin{cases}
    (\mu \Delta + \rho \omega^2) \nabla \omega + \rho_\gamma \omega^2 \nabla f_1 = \tilde{f}_1 - a_1 \nabla \beta_1 - \alpha M \nabla \beta_2 - \beta T_{0a_3} \nabla \beta_3 \\
    \rho_\gamma \omega^2 \nabla \omega + a_2 \nabla f_1 = \tilde{f}_2 - M (\alpha \nabla \beta_1 + \nabla \beta_2) - \beta T_{0a_3} \nabla \beta_3
\end{cases}
\]

(19)
Then, we apply the divergence operator $\Lambda_1(\Delta)$ to eq. (19) to yield

$$C(\Delta) \vec{\gamma} = \vec{d},$$

(20)

where

$$C(\Delta) = \begin{bmatrix} \mu \Delta + \rho \omega^2 & \rho_f \omega^2 \\ \rho_f \omega^2 & a_2 \end{bmatrix},$$

$$\vec{\gamma} = \begin{bmatrix} \Lambda_1(\Delta) \vec{u} \\ \Lambda_1(\Delta) \vec{w} \end{bmatrix},$$

and

$$\vec{d} = \begin{bmatrix} \Lambda_1 \vec{f}_1 - a_1 \nabla g_1 - \alpha M \nabla g_2 - \beta T_0 a_3 \nabla g_3 \\ \Lambda_1 \vec{f}_2 - M (\alpha \nabla g_1 + \nabla g_2) - \beta T_0 a_3 \nabla g_3 \end{bmatrix}.$$  

By introducing the operator

$$\Lambda_2(\Delta) = \frac{\Lambda_1(\Delta)}{\mu a_2} \text{det}(\Delta) = \Lambda_1(\Delta) \left( \Delta + \lambda_2^2 \right),$$

(21)

where

$$\lambda_2^2 = \frac{\rho \omega^2}{\mu} - \frac{\rho_f \omega^4}{a_2 \mu},$$

the following equation

$$\frac{1}{\mu a_2} \text{det}(\Delta) \begin{bmatrix} \Lambda_1(\Delta) \vec{u} \\ \Lambda_1(\Delta) \vec{w} \end{bmatrix} = \begin{bmatrix} \Lambda_2(\Delta) & 0 \\ 0 & \Lambda_2(\Delta) \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{w} \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix},$$

(22)

can be obtained from eqs (17), (20) and (21), where

$$h_j = \frac{1}{\mu a_2} \sum_{i=1}^{2} C_{ij}^* (\Delta) d_i,$$

(23)

with $C_{ij}^*(\Delta)$ being the cofactor of the element $C_{ij}(\Delta)$ of matrix $C(\Delta)$.

On the basis of eqs (17) and (22), we have

$$\Lambda(\Delta) \vec{\alpha} = \vec{m},$$

(24)

where

$$\Lambda(\Delta) \vec{\alpha} = \begin{bmatrix} \Lambda_2(\Delta) & 0 & 0 \\ 0 & \Lambda_2(\Delta) & 0 \\ 0 & 0 & \Lambda_1(\Delta) \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{w} \\ T \end{bmatrix},$$

and $\vec{m} = \begin{bmatrix} h_1 \\ h_2 \\ g_3 \end{bmatrix}$.

From eqs (18) and (23), $\vec{m}$ can be expanded as follows:

$$\vec{m} = \begin{bmatrix} \frac{1}{\mu a_2} \sum_{i=1}^{2} C_{11i}^* (\Delta) d_i \\ \frac{1}{\mu a_2} \sum_{i=1}^{2} C_{12i}^* (\Delta) d_i \\ \frac{1}{\gamma M a_0} \sum_{i=1}^{3} B_{ij}^* (\Delta) c_i \end{bmatrix}. $$

(25)

We separate eq. (25) into the following operator product form

$$\vec{m} = \begin{bmatrix} C_{11} & C_{12}^* \\ \mu a_2 & \mu a_2 \\ C_{12} & C_{12}^* \\ \mu a_2 & \mu a_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Lambda_1 \vec{f}_1 - a_1 \nabla g_1 - \alpha M \nabla g_2 - \beta T_0 a_3 \nabla g_3 \\ \Lambda_1 \vec{f}_2 - M (\alpha \nabla g_1 + \nabla g_2) - \beta T_0 a_3 \nabla g_3 \end{bmatrix}. $$

(26)

Using to eq. (18), eq. (26) can be expressed as
Table 1. Material properties.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grain bulk modulus, $K_s$</td>
<td>35 GPa</td>
</tr>
<tr>
<td>Density, $\rho_s$</td>
<td>2650 kg m$^{-3}$</td>
</tr>
<tr>
<td>Frame bulk modulus, $K_m$</td>
<td>1.7 GPa</td>
</tr>
<tr>
<td>Shear modulus, $\mu_m$</td>
<td>1.885 GPa</td>
</tr>
<tr>
<td>Porosity, $\phi$</td>
<td>0.3</td>
</tr>
<tr>
<td>Permeability, $\chi$</td>
<td>1 darcy</td>
</tr>
<tr>
<td>Tortuosity, $\tau$</td>
<td>2</td>
</tr>
<tr>
<td>Water density, $\rho_f$</td>
<td>1000 kg m$^{-3}$</td>
</tr>
<tr>
<td>Viscosity, $\eta$</td>
<td>0.001 Pa s</td>
</tr>
<tr>
<td>Bulk modulus, $K_f$</td>
<td>2.4 GPa</td>
</tr>
<tr>
<td>Thermoelasticity coefficient, $\beta_f$</td>
<td>40 000 kg (m · s$^2$ · deg K)$^{-1}$</td>
</tr>
<tr>
<td>Bulk specific heat capacity, $C_e$</td>
<td>0.38 m$^2$ (s$^2$ · deg K)$^{-1}$</td>
</tr>
<tr>
<td>Thermoelasticity coefficient, $\beta$</td>
<td>120 000 kg (m · s$^2$ · deg K)$^{-1}$</td>
</tr>
<tr>
<td>Absolute temperature, $T_0$</td>
<td>300 °K</td>
</tr>
</tbody>
</table>

Figure 1. Phase velocity and attenuation factor of the $P$ wave (a and b), slow $P$ wave (c and d) and $T$ wave (e and f) as a function of frequency for different thermal conductivities.
Figure 2. Phase velocity and attenuation factor of the $P$ wave (a and b), slow $P$ wave (c and d) and $T$ wave (e and f) as a function of frequency for different relaxation times.

Figure 3. Phase velocity and attenuation factor of the $S$ wave (a and b) as a function of frequency.
Figure 4. (a) Particle-velocity $u_z$ at 50 ms with $\eta = 0$ due to a heat source and (b) temperature $T$ at 50 ms with $\eta = 0$ due to a vertical elastic force for $\gamma = 5 \text{ m} \cdot \text{kg/(s}^3 \cdot \text{deg K)}^{-1}$ with $t_0 = 1.5 \times 10^{-8} \text{ s}$.

$$n_{i1} (\Delta) = -\frac{1}{\gamma M a_0} \left( a_i B_{i1}^* (\Delta) + \alpha M B_{i2}^* (\Delta) + \beta T \alpha_3 B_{i3}^* (\Delta) \right),$$

$$n_{i2} (\Delta) = -\frac{1}{\gamma M a_0} \left( M (\alpha B_{11}^* (\Delta) + B_{12}^* (\Delta)) + \beta T \alpha_3 B_{13}^* (\Delta) \right) \text{ and } n_{i3} (\Delta) = \frac{1}{\gamma M a_0} B_{i3}^* (\Delta).$$
Figure 5. (a) Particle-velocity $v_z$ at 50 ms with $\eta = 0$ due to a heat source and (b) temperature $T$ at 50 ms with $\eta = 0$ due to a vertical elastic force for $\gamma = 4.5 \times 10^6$ m kg s$^{-3}$ deg K$^{-1}$ with $t_0 = 1.5 \times 10^{-3}$ s.

we have

$$\vec{m} = \begin{bmatrix} C_{11} & C_{21} & 0 \\ \mu a_2 & \mu a_2 & 0 \\ \mu a_2 & \mu a_2 & I \end{bmatrix} \begin{bmatrix} \Lambda_1 + n_{11} \nabla \text{div} \\ n_{21} \nabla \text{div} \\ n_{31} \nabla \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix},$$ (28)

By combining eqs (15) and (28), we obtain the following operator,

$$L^T \vec{f} = \vec{m},$$ (29)

because

$$L^T \vec{b} = \vec{m}.$$
\[
\lambda \tilde{\alpha} = L^T A^T \alpha,
\]

thus

\[
\mathbf{AL} = \Lambda.
\]

We also introduce the matrix

\[
\Phi = \begin{bmatrix} 0 & 0 & 0 \\ \Phi_2 & 0 & 0 \\ 0 & \Phi_2 & \Phi_1 \end{bmatrix},
\]

where \(\Phi_1 = 3 \sum_{j=1}^{3} \eta_{ij} \gamma^{(j)}(\tilde{x}) \) and \(\Phi_2 = 4 \sum_{j=1}^{4} \eta_{ij} \gamma^{(ij)}(\tilde{x})\), with

\[
\begin{align*}
\gamma^{(j)}(\tilde{x}) &= \frac{1}{\Delta} H_0^{(1)} \left( \lambda_j \rho \tilde{x} \right), \\
\eta_{ij} &= \prod_{i=1, \neq j}^{3} (\lambda_i^2 - \lambda_j^2)^{-1}, \quad (j = 1, 2, 3; k = 1, 2, 3, 4) \\
\eta_{2k} &= \prod_{i=1, \neq k}^{4} (\lambda_i^2 - \lambda_k^2)^{-1},
\end{align*}
\]

to constitute the Green’s function of the Helmholtz equation \((\lambda + \lambda_j^2) \gamma^{(j)}(\tilde{x}) = -\delta(\tilde{x})\). According to Scarpetta et al. (2014) (Lemma 1), we have

\[
\mathbf{AL} \Phi = \Lambda \Phi = \mathbf{I}.
\]

Thus, the Green’s function becomes

\[
\mathbf{V} = \mathbf{L} \Phi,
\]

where

\[
\mathbf{L} = \begin{bmatrix} \Lambda_1 + n_{11} \nabla \text{div} & n_{12} \nabla \text{div} & n_{13} \nabla \\ n_{21} \nabla \text{div} & \Lambda_1 + n_{22} \nabla \text{div} & n_{23} \nabla \\ n_{31} \nabla \text{div} & n_{32} \nabla \text{div} & n_{33} \nabla \end{bmatrix} \\
\begin{bmatrix} C_{11}^{*} & C_{12}^{*} & 0 \\ \frac{\mu a_2}{\mu a_2} & \frac{\mu a_2}{\mu a_2} & 0 \\ \frac{\mu a_2}{\mu a_2} & \frac{\mu a_2}{\mu a_2} & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

leading to

\[
\mathbf{V} = \begin{bmatrix} (\Lambda_1 + n_{11} \nabla \text{div}) C_{11}^{*} \frac{\mu a_2}{\mu a_2} \Phi_1 + n_{12} \nabla \text{div} C_{12}^{*} \frac{\mu a_2}{\mu a_2} \Phi_2 + n_{13} \nabla \text{div} C_{13}^{*} \Phi_1 \\ n_{21} \nabla \text{div} C_{21}^{*} \frac{\mu a_2}{\mu a_2} \Phi_1 + (\Lambda_1 + n_{22} \nabla \text{div}) C_{22}^{*} \frac{\mu a_2}{\mu a_2} \Phi_2 + n_{23} \nabla \text{div} C_{23}^{*} \Phi_1 \\ n_{31} \nabla \text{div} C_{31}^{*} \frac{\mu a_2}{\mu a_2} \Phi_1 + n_{32} \nabla \text{div} C_{32}^{*} \frac{\mu a_2}{\mu a_2} \Phi_2 + n_{33} \nabla \text{div} C_{33}^{*} \Phi_1 \\ n_{31} \nabla \text{div} C_{31}^{*} \frac{\mu a_2}{\mu a_2} \Phi_1 + n_{32} \nabla \text{div} C_{32}^{*} \frac{\mu a_2}{\mu a_2} \Phi_2 + n_{33} \nabla \text{div} C_{33}^{*} \Phi_1 \end{bmatrix}.
\]

4 NUMERICAL EXAMPLES

We consider the thermo-poroelastic model with material properties (Table 1) given by Carcione et al. (2019), although the value \(C_v\) is actually low for rocks. A plane-wave analysis and a simulation for wavefield snapshots are conducted in this section to illustrate the dispersion and attenuation properties of thermo-poroelastic waves. Wave velocity and attenuation are calculated as a function of frequency for two sets of thermal properties: (1) \(\tau_0 = 1.5 \times 10^{-8} \text{ s}\) with \(\gamma = 1, 5 \text{ and } 10 \text{ m} \cdot \text{kg} (s^3 \cdot \text{deg} K)^{-1}\) and (2) \(\gamma = 10 \text{ m} \cdot \text{kg} (s^3 \cdot \text{deg} K)^{-1}\) with \(\tau_0 = 1.5 \times 10^{-8}, 5.5 \times 10^{-8} \text{ and } 9.5 \times 10^{-8} \text{ s}\). The phase velocity \(v_p = 1/(\text{Re}(1/v_c))\) and the attenuation factor \(A_p = -4\pi v_p \text{Im}(1/v_c)\) can be obtained from the complex velocity \(v_c\) as a function of frequency (see eqs 7 and 8).

Fig. 1 shows the phase velocity and attenuation of thermo-poroelastic waves as a function of frequency for the first set of thermal properties. We see that the thermo-poroelastic dispersion is of a dual-peak structure in both the phase velocity and quality factor. In general, the velocity dispersion with frequencies is relevant to intrinsic attenuation. The dual-peak feature of dispersion is strongly associated with classical Biot loss and heat diffusion. The wave-like \(T\) mode has characteristics similar to the Biot slow diffusive wave, since the poroelasticity and thermoelasticity have similar constitutive relations. Both the slow \(P\) and \(T\) waves are highly attenuated toward low frequencies. We note that the second peak in both the phase velocity and attenuation becomes strong with increasing thermal conductivities. In conclusion, the velocity and attenuation dispersions for \(P\), slow \(P\) and \(T\) waves increase with increasing thermal conductivities.
Fig. 2 shows the phase velocity and attenuation with frequencies for P, slow P and T waves at different relaxation times. We see that the second peak in both the phase velocity and attenuation for all the modes is significantly affected by relaxation times, becoming weak with increasing relaxation times. Nevertheless, the wave-like T mode is prone to be observed in either high thermal conductivities or small relaxation times within the high-frequency band. Fig. 3 shows the phase velocity and attenuation as a function of frequency for S wave, obtained from eq. (7). We see that the S wave has only one attenuation peak due to the Biot loss, which is independent of heat diffusion. We have to stress that the physics of thermo-poroelastic dispersion is rather complex and could be addressed by incorporating laboratory measurements in the future.

Snapshots are calculated by the Green’s function (eq. 32) for two sets of thermal properties (1) $\gamma = 5 \text{ m kg} (\text{s}^3 \cdot \text{deg K})^{-1}$ with $\tau_0 = 1.5 \times 10^{-4} \text{s}$ and (2) $\gamma = 4.5 \times 10^6 \text{ m} \cdot \text{kg} (\text{s}^3 \cdot \text{deg K})$ with $\tau_0 = 1.5 \times 10^{-2} \text{s}$. The source, located at the center of the mesh, is a Ricker waveform with the central frequency of 150 Hz. The wavefields generated by a heat source and a vertical elastic force at 50 ms with $\eta = 0$ are calculated for two sets of thermal properties, with the results shown in Figs 4 and 5, respectively. We observe that the T wave is diffuse and weak. As expected from Figs 1 and 2, it is hard to see the mode in Fig. 4 with small thermal conductivities, which, however, becomes obvious in Fig. 5 with large thermal conductivities. Different dilatational waves can be seen in Figs 4 and 5, where the heat source generates significant compressional waves, but there is no shear wave. The S wave can be generated by the vertical elastic force, but this mode is not present the temperature field.

5 CONCLUSIONS

We have derived the frequency-domain Green’s function of the Lord–Shulman thermo-poroelasticity theory, where the displacement-temperature fields are obtained for isotropic porous media. Our numerical results clearly show the presence of a slow thermal P wave (the T wave) besides the classical P and S waves and the diffusive/wave-like Biot P mode. The S wave is not affected by the thermal effects, but the three P waves exhibit a lossy behaviour under certain conditions, which depends on the viscosity, frequency and thermoelastic constants. In the seismic frequency band, the T and Biot waves are highly attenuated and are coupled with the fast P wave inducing energy dissipation at high frequencies.

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REFERENCES


**APPENDIX: LIST OF SYMBOLS**

\[
\begin{align*}
\sigma_{ij} & \quad \text{the stress components of the frame} \\
p_f & \quad \text{pore-fluid pressure} \\
\lambda, \mu & \quad \text{the Lamé constants of the drained matrix} \\
\delta_{ij} & \quad \text{Kronecker-delta} \\
\phi & \quad \text{the porosity} \\
T_0 & \quad \text{the reference absolute temperature} \\
T & \quad \text{the increment of temperature over } T_0 \\
u_i & \quad \text{the displacement components in the solid phase} \\
U_i & \quad \text{the displacement components of the fluid phase} \\
w_i & \quad \text{the displacement components of the fluid relative to the solid frame} \\
K_s & \quad \text{the solid bulk moduli} \\
K_f & \quad \text{the fluid bulk moduli} \\
\beta_s & \quad \text{the coefficients of thermal stress for the solid phase} \\
\beta_f & \quad \text{the coefficients of thermal stress for the fluid phase} \\
\gamma & \quad \text{the coefficient of heat conduction} \\
C_e & \quad \text{the specific heat capacity} \\
\tau & \quad \text{the tortuosity} \\
\tau_0 & \quad \text{the relaxation time} \\
\rho_s & \quad \text{the solid density} \\
\rho_f & \quad \text{the fluid density} \\
\rho & \quad \text{the composite density} \\
\eta & \quad \text{Viscosity} \\
\chi & \quad \text{Permeability} \\
s_i & \quad \text{the polarizations for the motions of the solid particle} \\
d_i & \quad \text{the polarizations for the motions of the fluid particle} \\
\Delta & \quad \text{the divergence symbol} \\
A, B, C & \quad \text{amplitude constants} \\
\omega & \quad \text{angular frequency} \\
t & \quad \text{Time} \\
v_c & \quad \text{complex velocity} \\
l_j & \quad \text{the propagation directions} \\
x_j & \quad \text{position components} \\
\Delta & \quad \text{Laplacian operator}
\end{align*}
\]
Key words

Authors are requested to choose key words from the list below to describe their work. The key words will be printed underneath the summary and are useful for readers and researchers. Key words should be separated by a semi-colon and listed in the order that they appear in this list. An article should contain no more than six key words.

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- Composition and structure of the continental crust
- Composition and structure of the core
- Composition and structure of the mantle
- Composition and structure of the oceanic crust
- Composition of the planets
- Creep and deformation
- Defects
- Elasticity and anelasticity
- Electrical properties
- Equations of state
- Fault zone rheology
- Fracture and flow
- Friction
- High-pressure behaviour
- Magnetic properties
- Microstructure
- Permeability and porosity
- Phase transitions
- Plasticity, diffusion, and creep

**GEODESY and GRAVITY**
- Ultra-high pressure metamorphism
- Ultra-high temperature metamorphism

**GEOMAGNETISM and ELECTROMAGNETISM**
- Archaeomagnetism
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- Electromagnetic theory
- Environmental magnetism
- Geomagnetic excursions
- Geomagnetic induction
- Ground penetrating radar
- Magnetic anomalies: modelling and interpretation
- Magnetic fabrics and anisotropy
- Magnetic field variations through time
- Magnetic mineralogy and petrology
- Magnetotellurics
- Marine electromagnetics
- Marine magnetics and palaeomagnetics
- Non-linear electromagnetics
- Palaeointensity
- Palaeomagnetic secular variation
- Palaeomagnetism
- Rapid time variations
- Remagnetization
- Reversals: process, time scale, magnetostratigraphy
- Rock and mineral magnetism
- Satellite magnetics

**GEOPHYSICAL METHODS**
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- Image processing

**GEOPHYSICAL METHODS**
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- Joint inversion
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- Non-linear differential equations
- Numerical approximations and analysis
- Numerical modelling
- Numerical solutions
- Persistence, memory, correlations, clustering
- Probabilistic forecasting
- Probability distributions
- Self-organization
- Spatial analysis
- Statistical methods
- Thermobarometry
- Time-series analysis
- Tomography
- Waveform inversion
- Wavelet transform

**GENERAL SUBJECTS**
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- Gas and hydride systems
- Geomechanics
- Geomorphology
- Glaciology
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- Hydrogeophysics
- Hydrology
- Hydrothermal systems
- Infrasound
- Instrumental noise
- Ionosphere/atmosphere interactions
- Ionosphere/magnetosphere interactions
- Mantle processes
- Ocean drilling
- Structure of the Earth
- Thermochronology
- Tsunamis
- Ultra-high pressure metamorphism
- Ultra-high temperature metamorphism

**GEOGRAPHIC LOCATION**
- Africa
- Antarctica
- Arctic region
- Asia
- Atlantic Ocean
- Australia
- Europe
- Indian Ocean
- Japan
- New Zealand
- North America
- Pacific Ocean
- South America

**PLANETS**
- Planetary interiors
- Planetary volcanism

**PLATE MOTIONS**
- Plate motions
- Plate tectonics

**PLANETS**
- Planetary interiors
- Planetary volcanism

**SEISMOLOGY**
- Acoustic properties
- Body waves
- Coda waves
- Computational seismology
- Controlled source seismology
- Crustal imaging
- Earthquake dynamics
- Earthquake early warning
- Earthquake ground motions
- Earthquake hazards
- Earthquake interaction, forecasting, and prediction
- Earthquake monitoring and test-ban treaty verification
- Earthquake source observations
- Guided waves
- Induced seismicity
- Interface waves
- Palaeoseismology
- Rheology and friction of fault zones
- Rotational seismology
- Seismic anisotropy
- Seismic attenuation
- Seismic instruments
- Seismic interferometry
- Seismicity and tectonics
- Seismic noise
- Seismic tomography
- Site effects
- Statistical seismology
- Surface waves and free oscillations
- Theoretical seismology
Tsunami warning
Volcano seismology
Wave propagation
Wave scattering and diffraction

TECTONOPHYSICS
Backarc basin processes
Continental margins: convergent
Continental margins: divergent
Continental margins: transform
Continental neotectonics
Continental tectonics: compressional
Continental tectonics: extensional
Continental tectonics: strike-slip and transform
Cratons
Crustal structure
Diapirism
Dynamics: convection currents, and mantle plumes
Dynamics: gravity and tectonics
Dynamics and mechanics of faulting
Dynamics of lithosphere and mantle
Folds and folding
Fractures, faults, and high strain deformation zones
Heat generation and transport
Hotspots
Impact phenomena
Intra-plate processes
Kinematics of crustal and mantle deformation
Large igneous provinces
Lithospheric flexure
Mechanics, theory, and modelling
Microstructures
Mid-ocean ridge processes
Neotectonics
Obduction tectonics
Oceanic hotspots and intraplate volcanism
Oceanic plateaus and microcontinents
Oceanic transform and fracture zone processes
Paleoseismology
Planetary tectonics
Rheology: crust and lithosphere
Rheology: mantle
Rheology and friction of fault zones
Sedimentary basin processes
Subduction zone processes
Submarine landslides
Submarine tectonics and volcanism
Tectonics and climatic interactions
Tectonics and landscape evolution
Transform faults
Volcanic arc processes

VOLCANOLOGY
Atmospheric effects (volcano)
Calderas
Effusive volcanism
Eruption mechanisms and flow emplacement
Experimental volcanism
Explosive volcanism
Lava rheology and morphology
Magma chamber processes
Magma genesis and partial melting
Magma migration and fragmentation
Mud volcano
Physics and chemistry of magma bodies
Physics of magma and magma bodies
Planetary volcanism
Pluton emplacement
Remote sensing of volcanoes
Subaqueous volcano
Tephrochronology
Volcanic gases
Volcanic hazards and risks
Vulcaniclastic deposits
Volcano/climate interactions
Volcano monitoring
Volcano seismology

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