Differential poroelasticity model for wave dissipation in self-similar rocks

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A B S T R A C T

The double-porosity theory of wave propagation in rocks considers a single scale based on soft and stiff pores, and wave attenuation occurs by local fluid flow between these pores. However, rocks exhibit more complicated fabric structures, with heterogeneities presented at multiple scales, having a self-similar fractal nature as reported. We develop a poroelasticity model to describe wave anelasticity in a fluid-saturated medium consisting of infinite components, i.e., an infinituple-porosity medium. Numerical modeling is achieved with infinite iterations, where at each iteration one component is added (a porous inclusion embedded in a porous host), analogous to the differential effective medium (DEM) theory of solid composites. By considering the self-similar structure, the properties of each component (inclusions) are scale-dependent. The example shows that the P-wave velocity dependence on frequency depends on the fractal dimension. Application of the model to sandy sea-bottom sediments shows a good agreement, with fractal dimension of 2.88.

1. Introduction

Many poroelasticity theories describe wave propagation in rocks, where the pores have different shapes and compressibilities, e.g., squirt-flow theories. In double-porosity media, a background medium and softer or stiffer porous inclusions are assumed, on the basis of a single heterogeneity scale. Due to the different compressibilities of host and inclusions, a P-wave induces a fluid pressure gradient and a local fluid flow (LFF). At low frequencies, the medium is relaxed since the fluid pressure has time to equilibrate at each cycle, whereas at high frequencies the dissimilar pressures (unrelaxation) and the consequent LFF causes a medium stiffening, with the P-wave velocity increasing with frequency. This implies the dispersion and dissipation due to P-wave energy conversion to Biot slow modes.

The poroelasticity theory has been generalized to the case of triple-porosity by Sun et al. and Zhang et al., where an inclusion of a triple-layer spherical patch (with outer shells and an inner core) is considered. Each of the three components has different mineral or fluid properties. Ba et al. presented a double double-porosity theory to analyze the overlapping effect of two types of heterogeneities, fabric and patchy-saturation. Although this theory considers four types of pores, it is not enough to characterize the complex structures of rocks, which generally undergo a multi-stage effect of tectonization, deposition and dissolution in geologic time, resulting in fractal pore and crack phases. Previous studies assumed uniform-to quadruple-porosity media, where a limited kind of pore phases were assumed as inclusions embedded within a host medium.

It is believed that a reasonable model has to be fractal, i.e., statistically self-similar. Studies were performed on the relationships between fractal geometry and thermal conductivity, elastic properties, permeability, and electrical conductivity. Pride and Mason suggested a relation between wave attenuation and fractal geometry based on numerical examples, but it remains not clear if a fractal geometry can be theoretically implemented to describe wave anelasticity.

The elastic moduli of solid composites can be estimated with the differential effective medium (DEM) theory, which usually incorporates only two components. It has been extended to multiple constituents by Han. Although the DEM was applied to fluid-saturated media by Berryman, wave-induced flow has not been incorporated, and consequently, wave dissipation has been neglected.

In analogy with the DEM theory for a solid, we present an extended approach for fluid-saturated media, where we add one component in each iteration. The complex moduli of an infinituple-porosity medium are then obtained after infinite iterations, and the differential equations of wave propagation are derived. To study the physics, we perform a plane-wave analysis, giving the wave anelastic properties of the medium as a function of the fractal dimension. Finally, the theory is applied to broad-band measurements in sea-bottom sediments.
2. Theory

2.1. Differential poroelasticity model

The Biot-Rayleigh theory \(^8\) for wave propagation in double-porosity rocks assumes one component (inclusion) embedded in a host medium. The governing equations for wave propagation are based on the Hamilton principle.

In analogy with the DEM theory, where inclusion and host are solids, we consider spherical porous inclusions and host (fluid-saturated). The procedure is illustrated in Fig. 1. One component is considered as the initial host, and the composite is built by a number of incremental additions of inclusions. In each step, the phase estimated from the last addition is taken as the host, and inclusions are incrementally added by assuming a dilute concentration. The inclusions at different additions can be distinct. In this manner, an infinitesimal-porosity medium is obtained after infinite additions. The dry-composite moduli are determined by the differential scheme as in Berryman.\(^{36}\)

\[
dK_b(v_i) = \frac{(K_{b,i} - K_b)_L(v_i)}{1 - v_i} dV_i \tag{1a}
\]

\[
G_b(v_i) = \frac{(G_{b,i} - G_b)_L(v_i)}{1 - v_i} dV_i \tag{1b}
\]

where \(K_b(v_i) = K_{b,i}\) and \(G_b(v_i) = G_{b,i}\) are the dry-rock bulk and shear moduli of the initial host (inclusions), respectively, \(v_i\) is the volume fraction of inclusions, and \(L_1\) and \(L_2\) are the polarization factors for the bulk and shear moduli, respectively.\(^{37}\)

At each addition, the double-porosity model with the host absolute porosity \(\phi\) and the incremental porosity \(\phi_{0,i}\), \(\phi_{1,i}\), \(\phi_{2,i}\), \(\phi_{3,i}\), \(\phi_{4,i}\), \(\phi_{5,i}\), and \(\phi_{6,i}\) is applied to compute the complex moduli of the composite. Then, the differential equations for wave propagation are

\[
(N + NSd\phi)\nabla^2 \mathbf{u} + (A + N)d\phi \nabla \mathbf{e} + Q_0 \phi \nabla (\xi_H + \zeta d\phi) + R_0 \left( \frac{Q_H}{K_1} - \frac{Q_H}{K_2} - 1 \right) d\phi \nabla \mathbf{z}_H + Q_0 \phi \nabla (\xi_H - \phi \zeta) \nonumber \\
= \rho_0 \dot{\mathbf{u}} + (\rho_0 S - \rho_a - \rho_0 d\phi) \dot{\mathbf{u}} + \rho_0 (\phi_u \dot{\mathbf{U}}_H + \rho_0 \phi S d\phi \dot{\mathbf{U}}_H + \rho_0 d\phi \dot{\mathbf{U}}_1) + \frac{Q_H}{\kappa_H} \left( \dot{\mathbf{u}} - \mathbf{U}_H \right) + \frac{Q_H}{\kappa_H} \left( \dot{\mathbf{u}} - \mathbf{U}_1 \right) \tag{2a}
\]

\[Q_0 \nabla \mathbf{e} + R_0 \nabla (\xi_H + \zeta d\phi) + \frac{R_0}{\eta} \left( \frac{1}{\kappa_2} \nabla \mathbf{z}_H - \nabla \mathbf{e} \right) d\phi \nonumber \]

\[= \left( 1 + \frac{R_0}{\eta} \frac{\rho_H}{\rho_0} \frac{1}{\kappa_1} \right) d\phi \left( \rho_0 \dot{\mathbf{u}} + \rho_2 \dot{\mathbf{U}}_1 - \frac{\phi H}{\kappa_H} \left( \dot{\mathbf{u}} - \mathbf{U}_1 \right) \right) \tag{2b}
\]

\[Q_0 \nabla \mathbf{e} + R_0 \nabla (\xi_H - \phi \zeta) = \rho_0 \dot{\mathbf{u}} + \rho_2 \dot{\mathbf{U}}_1 - \frac{\phi H}{\kappa_H} \left( \dot{\mathbf{u}} - \mathbf{U}_1 \right) \tag{2c}
\]

\[(Q_0 e + R_0 (\xi_H + \zeta d\phi)) - (Q_0 e + R_0 (\xi_H - \phi \zeta)) = \frac{1}{2} \phi H \nabla (\xi_H - \phi \zeta) \tag{2d}
\]

where \(\mathbf{u}, \mathbf{U}_H, \text{ and } \mathbf{U}_1\) are the solid displacement, fluid displacement in the host medium and fluid displacement in the inclusions, respectively, and \(\xi_H\) and \(\xi_H - \phi \zeta\) are the corresponding displacement divergence fields. The scalar \(\zeta\) represents the variation of fluid content between the host and inclusions. \(\phi_H\) is the matrix porosity of the host. \(\kappa_H\) and \(\kappa_i\) are the permeabilities of the host and inclusions, respectively. \(r\) is the inclusion radius, and \(\eta\) is the fluid viscosity. \(K_f(K_i)\) is the grain (fluid) bulk modulus. \(\rho_f\) is the mineral (or grain) density. The derivation of equation (2) is given in Appendix A. The stiffness coefficients \(A, A_H, N, Q_0, Q_1, R_0, K_f, S\) and density coefficients \(\rho_{00}, \rho_{01}, \rho_{02}, \rho_{11}, \rho_{22}\) are given in Appendix B.

The P- and S-wave complex wave numbers \((K_p, K_s)\) are obtained by a plane wave analysis in equation (2). Then, the complex bulk and shear moduli are

\[
K_{sat} = \left( (1 - \phi - d\phi) \rho_f + (\phi + d\phi) \rho_i \right) \left( \frac{\omega}{K_p} \right)^2 - \frac{4}{3} G_{sat}, \tag{3a}
\]

\[
G_{sat} = \left( (1 - \phi - d\phi) \rho_f + (\phi + d\phi) \rho_i \right) \left( \frac{\omega}{K_s} \right)^2, \tag{3b}
\]

where \(\rho_f\) is the fluid density and \(\omega\) is the angular frequency.

The Gassmann equation \(^{26}\) is applied to compute the dry-rock complex moduli at the end of each addition, used as the moduli of the new host for the next addition:

\[
\frac{K_{sat} - K_{sat}}{K_f - K_b} = \frac{K_f}{K_f - K_b} + \left( \frac{1}{(1 - \phi - d\phi)} \right) \left( K_f - K_b \right), \tag{4a}
\]

\[G_{sat} = G_{sat}, \tag{4b}\]

Using equations (1)–(4) at each addition, the process consists of infinite steps, where at each step the host is assumed homogeneous and
anelastic. The infinituple-porosity model is realized if the inclusion phases are distinct at each addition. At the end, the wave velocity and attenuation are computed from the last complex wave number.

2.2. Double-porosity medium

A discretization of the process actually leads to finite additions. In the particular case that all the steps assume the same type of inclusion, a double-porosity medium is obtained. Let us consider an example, where the bulk modulus of the grain is 38 GPa, the initial host bulk/shear modulus is 17GPa/15 GPa, the inclusion bulk/shear modulus is 1.7 GPa/ 1.5 GPa, the grain density is 2650 kg/m³, the fluid modulus is 2.5 GPa, the fluid density is 1040 kg/m³, the fluid viscosity is 0.001 Pa·s; the initial host porosity/permeability is 0.15/0.01 D, the inclusion porosity/permeability is 0.08/1 D, the total inclusion volume ratio is 0.1, and the inclusion radius is 0.01 m. The results for n = 1, 25, 50, and 100 additions are shown in Fig. 2. The low-frequency limit of the P-wave velocity is given in Fig. 2a, where the dry-rock moduli are determined by the classical DEM model, and Gassmann equation for fluid substitution is used to obtain the properties of the saturated medium.

Use of one addition (n = 1) leads to an underestimation of the P-wave velocity (both at the low and high frequency limits) and attenuation, similar to the results of the classical DEM theory.6,24,25 where a great deviation is observed between the results of 1-addition and multi-addition. With the increase of n, the 25-, 50-, and 100-addition results of P-wave velocity and attenuation approach to the same limits at low- and high-frequency ends. Fig. 2a shows that the P-wave velocities at low frequency agree with Gassmann equation. The convergence with increasing n and the consistency with Gassmann equation validate the theory. The dispersion and attenuation caused by the LFF occurs at a narrower band with increasing n, indicating a single-scale inclusion cannot explain the broadband dispersion and attenuation of field observations. The dispersion due to the Biot global fluid flow (GFF) is weak in the ultrasonic band in Fig. 2a, and correspondingly, the attenuation is also weak (Fig. 2b).

We also consider the intermediate results of 5, 10, 15, and 20 additions in the 25-addition case, shown as dashed curves in Fig. 2. With the increasing addition of soft inclusions, the P-wave velocity decreases at the low frequency limit. In contrast, the P-wave velocity at high frequencies significantly increases when n increases from 5 to 15, and then slightly decreases when n increases from 15 to 20. The LFF P-wave attenuation peak increases with each iteration, and correspondingly the relaxation frequency slightly shifts to the low frequencies. A continuous addition of inclusions results in a stronger stiffening effect of the rock and more dissipation.

3. Wave propagation in fractal rocks

3.1. Scale-dependent infinituple-porosity theory

If the inclusion at each iteration is different, the resulting model is scale-dependent. Therefore, \( d\varphi = \varphi'(r)dr \). For an infinituple-porosity model, the governing equations of wave propagation are expressed by an integral scheme:

\[
\mathcal{Q}_H \nabla^2 \mathbf{u} + (\mathbf{A} + \mathbf{N}) \nabla e + \mathcal{Q}_H \varphi \nabla \left( \zeta_H + \int_0^\infty \varphi'(r)dr \right) = \mathcal{Q}_H \mathbf{u} + \mathcal{Q}_H \mathbf{u}_H - \frac{\varphi \eta}{k_1} (\mathbf{u} - \mathbf{U}_H), \tag{5a}
\]

\[
\mathcal{Q}_H \nabla^2 \mathbf{u} + \mathcal{Q}_H \nabla (\zeta_H - \varphi \xi) = \mathcal{Q}_H \mathbf{u} + \mathcal{Q}_H \mathbf{u}_H - \frac{\varphi \eta}{k_2} (\mathbf{u} - \mathbf{U}_H), \tag{5b}
\]

\[
\mathcal{Q}_H \nabla^2 \mathbf{u} + \mathcal{Q}_H \nabla (\zeta_H - \varphi \xi) = \frac{1}{3} \varphi \eta \left( \frac{\rho_H \zeta_1 + \eta \eta}{\zeta_H} \right), \tag{5c}
\]

\[
\begin{align*}
\left( \mathcal{Q}_H \nabla + \mathcal{Q}_H \nabla \right) \left( \zeta_H + \int_0^\infty \varphi'(r)dr \right) + \mathcal{Q}_H \mathbf{u} + \mathcal{Q}_H \mathbf{u}_H - \frac{\varphi \eta}{k_1} (\mathbf{u} - \mathbf{U}_H) \end{align*}, \tag{5d}
\]

where \( \rho_H \) and \( k_1 \) are relevant with the inclusions at the scale r. \( \eta_0 \), \( \eta_1 \), the stiffnesses \( \mathbf{A}, \mathbf{N}, \mathbf{Q}_H, \mathbf{Q}_H, \mathbf{R}_H \), and the density coefficients \( \rho_{00}, \rho_{01}, \rho_{02}, \rho_{11}, \rho_{22} \), can be determined as in the discretization procedure illustrated in Section 2.1.

3.2. Example of self-similar porous rocks

Natural or synthetic rocks can be assumed fractal and statistically self-similar.22-24,39 By considering a self-similar medium, the volume fraction of inclusions is dependent on its scale, based on the porosity-scale relation presented by Rieu and Perrier:40

\[
v = 1 - \left( \frac{r_{\min}}{r_{\max}} \right)^{D_h - D}, \tag{6}
\]

where \( D_h \) is the Euclidean dimension, and \( D_f \) is the fractal dimension satisfying \( 2 < D_f < 3 \), which represents the degree of filling of pore spaces or fabric grains in a self-similar rock. \( r_{\min} \) and \( r_{\max} \) are the minimum and maximum inclusion radii of the self-similarity range, respectively. The P-wave velocity and attenuation as functions of frequency are shown in Fig. 3. The rock is assumed fractal in the scale ranges of \([0.00001, 0.05]\) and \([0.000001, 0.05]\) m. The total inclusion volume ratio is equal to 0.1, and the volume fraction of each addition is determined from Equation (6). The other rock properties are the same as in Fig. 2.

A comparison between the curves with different fractal parameters and the same scale range shows that when \( D_f \) increases, the P-wave velocity decreases, and the frequency range of the dispersion/attenuation is broader. A larger \( D_f \) indicates that the inclusions are more uniformly distributed (finely textured). For a self-similar rock, the dispersion and attenuation due to LFF are significant in the full frequency range, and their dependence depends on the fractal dimension \( D_f \). Fig. 3a also shows that at the same \( D_f \), the P-wave velocity decreases with a larger scale range of the radius.

3.3. Example. Marine sediments

The fractal model is applied to broadbroad data recorded on sandy sea-bottoms (summarized by41). These sediments are from 20 different sites around the world and the measurements span the frequency range of 50 Hz–400 kHz. The corresponding porosity and permeability are in the range of 0.36–0.47 and 0.65 × 10⁻¹¹–10⁻¹² m², respectively. Moreover, the media are composed primarily of fine sand or sand-silt mixture (low frequency experiment)22,42 and coarse-to-medium sand (high frequency experiment).44,45 The grain bulk modulus \( K_s \) is 32 GPa,
due to the presence of silt, and the grain density $\rho_s$ is 2650 kg/m$^3$.

The rock properties are consistent with theoretical considerations and/or experiments.$^{46,47}$ Let us assume that the host is a sand with a dry-rock bulk modulus of 0.5 GPa, a dry-rock shear modulus of 0.03 GPa, a porosity of 0.395 and a permeability of $0.8 \times 10^{-11}$ m$^2$.

Grain/particle aggregates with different degrees of compaction are considered as inclusions, with a dry-rock bulk modulus of 0.22 GPa, a dry-rock shear modulus of 0.028 GPa, a porosity of 0.5 and a permeability of $1 \times 10^{-11}$ m$^2$. Moreover, the fluid (brine) bulk modulus, density and viscosity are 2.14 GPa, 1040 kg/m$^3$ and 0.001 Pa s, respectively.

It is assumed that the average porosity is 0.4 and it is fractal in the scale range of [0.003, 30] m. We use the fractal dimension $D_f$ as a fitting parameter and the results are obtained by a least-square regression method.

The data compared to the predictions of the fractal and classical Biot models are shown in Fig. 4. Both models agree with the observed P-wave velocities and attenuation at low frequencies ($<1$ kHz), but the Biot model fails to match the measurements at frequencies higher than 1 kHz, where the GFF peak of Biot is not the dominant mechanism for anelasticity in this kind of loose media (sand/clay). It is clear that the significant contribution is given by the LFF at multiple scales. Due to this discrepancy with the Biot model, Chotiros and Isakson$^{48}$ and Kimura$^4$ incorporated a squirt-flow mechanism. However, this model cannot be applied to simulate wave anelasticity in the full frequency range. On the other hand, the fractal model provides a reasonable description for $D_f=2.88$. The fractal dimension agrees with previous reported values in the range 2.49–2.89 for rocks.$^{22,23}$

4. Conclusions

The differential poroelasticity model is presented to describe wave propagation and dissipation in fluid-saturated rocks, which consists of infinite components. The method to obtain the differential equations of motion follows the classical DEM theory for composites made of solid phases, to obtain the dry-rock moduli iteratively. Then, the double-porosity theory is used to incorporate the wave-induced fluid-flow loss mechanism at each iteration. Since the inclusions in different additions...
can be distinct, an infinituple-porosity model is realized. By assuming that the inclusion properties are scale-dependent and fractal, a self-similar medium is realized. The example shows that wave anelasticity can be resolved. The model is applied to broadband measurements of acoustic waves in marine sediments. The comparisons between the theoretical predictions and the data yields a fractal dimension of 2.88, in agreement with previous results. The model of this work is derived by considering one fractal dimension. It can be generalized to a more complete model (i.e. with more fractal dimensions) in a future study, so that the natural rocks with more complex geometries can be characterized.

Appendix A. Derivation process of Equation (2)

For a double-porosity medium, the Biot-Rayleigh theory\(^8\) assumes one component (inclusion) embedded in a host medium. In the differential scheme, the host absolute porosity is \(\phi\), while the incremental absolute porosity for inclusion addition is \(d\phi\). The governing equations for wave propagation are

\[
\begin{align*}
\bar{N} \Delta^2 \vec{u} + (\bar{A} + \bar{N}) \nabla \cdot \bar{Q}_1 &+ \bar{R}_1 \nabla \cdot (\xi + \zeta \partial \phi) = \bar{\rho}_{10} \vec{U}_1 + \bar{\rho}_{12} (\vec{u} - \vec{U}_1), \\
\bar{Q}_1 \Delta \vec{e} + \bar{R}_1 \nabla \cdot (\xi + \zeta \partial \phi) &+ \bar{\rho}_{10} \bar{U}_1 + \bar{\rho}_{12} \bar{U}_1 + \bar{b}_1 (\vec{u} - \vec{U}_1), \\
\bar{Q}_1 \Delta \vec{e} + \bar{R}_1 \nabla \cdot (\xi + \zeta \partial \phi) &+ \bar{\rho}_{10} \bar{U}_1 + \bar{\rho}_{12} \bar{U}_1 + \bar{b}_2 (\vec{u} - \vec{U}_1), \\
(\bar{Q}_1 \Delta \vec{e} + \bar{R}_1 \nabla \cdot (\xi + \zeta \partial \phi)) &\partial \phi - \phi (\bar{Q}_1 \Delta \vec{e} + \bar{R}_1 \nabla \cdot (\xi + \zeta \partial \phi)) = \frac{1}{3} \phi^2 \phi \partial \phi \left( \frac{\partial H}{\partial \phi} \right) + \frac{\bar{b}_1}{K_i} \phi \quad (A1d)
\end{align*}
\]

The stiffness coefficients \(\bar{A}, \bar{N}, \bar{Q}_1, \bar{Q}_2, \bar{R}_1, \bar{R}_2\), density coefficients \(\bar{\rho}_{10}, \bar{\rho}_{12}, \bar{\rho}_{11}, \bar{\rho}_{22}\) and dissipation coefficients \(\bar{b}_1, \bar{b}_2\) are dependent on \(\phi\) and \(d\phi\), which are expressed as

\[
\bar{A} = (1 - \phi - d\phi) K_i - \frac{2}{3} \bar{N} - \frac{\beta \phi (1 - \phi - d\phi - K_s/K_i) K_i^2}{\beta (1 - \phi - d\phi - K_s/K_i) + K_i/K_i (\beta \phi + d\phi)} \\
\bar{N} = G_i, \\
\bar{Q}_1 = \frac{\beta (1 - \phi - d\phi - K_s/K_i) \phi K_i}{\beta (1 - \phi - d\phi - K_s/K_i) + K_i/K_i (\beta \phi + d\phi)}, \\
\bar{Q}_2 = \frac{(1 - \phi - d\phi - K_s/K_i) \phi K_i}{1 - \phi - d\phi - K_s/K_i + K_i/K_i (\beta \phi + d\phi)}, \\
\bar{R}_1 = \frac{(\beta \phi + d\phi) \phi K_i}{\beta (1 - \phi - d\phi - K_s/K_i) + K_i/K_i (\beta \phi + d\phi)}, \\
\bar{R}_2 = \frac{\beta \phi + d\phi) \phi K_i}{1 - \phi - d\phi - K_s/K_i + K_i/K_i (\beta \phi + d\phi)}, \\
\beta = \frac{q_1}{q_0} \left[ 1 - \frac{(1 - \phi) K_i/K_i}{1 - (1 - \phi) K_i/K_i} \right], \\
\bar{\rho}_{10} = (1 - \phi - d\phi) \rho_i - \bar{\rho}_{11} - \bar{\rho}_{12}, \\
\bar{\rho}_{11} = \phi \rho_1 - \bar{\rho}_{11}, \\
\bar{\rho}_{12} = \rho_1 d\phi - \bar{\rho}_{12}, \\
\bar{\rho}_{11} = \alpha_1 \rho_2, \\
\bar{\rho}_{12} = \alpha_1 \rho_2 d\phi.
\]

Declaration of competing interest

The authors declare that to their knowledge they have no personal interest or relations with academic or private Institutions that may have influenced the contents of this paper.

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\[ \tilde{b}_1 = \frac{\rho \phi_f \eta}{\kappa_H}, \]  
\[ \tilde{b}_2 = \frac{\rho_f \beta \phi_f}{\kappa_f}, \]

where \( \alpha_H \) and \( \alpha_I \) are tortuosities of the host and inclusions, respectively.

We introduce two coefficients \( B_1 \) and \( B_2 \), which are

\[ B_1 = \beta(1 - \phi) - \frac{2}{3} N - K_f / K_H \phi_f \]  
\[ B_2 = (1 - \phi_f) - \frac{2}{3} N - Q_f (K_f / K_H + K_f / K_f \phi_f) \]

Equation (A-1a) multiplied by Equations (A-3a) and (A-3b), and neglecting the higher order terms of \( d \phi \), gives Equation (2-a). Similarly, Equation (2-b \( \sim \) d) can be derived.

**Appendix B. Poroelasticity coefficients**

The stiffness and density coefficients are

\[ A = (1 - \phi)K - \frac{2}{3} N - K_z / K_H \phi_f. \]  
\[ A_f = (1 - \phi)K_f - \frac{2}{3} N - Q_f (K_f / K_H + K_f / K_f \phi_f) \]  
\[ N = \bar{N} \]  
\[ Q_H = \frac{K_z(1 - \phi - K_z / K_z)}{1 - \phi - K_z / K_z + K_z / K_z \phi_f} \]  
\[ Q_f = \frac{K_f(1 - \phi - K_f / K_f)}{1 - \phi - K_f / K_f + K_f / K_f \phi_f} \]  
\[ R_H = \frac{K_z \phi_f}{1 - \phi - K_z / K_z + K_z / K_z \phi_f} \]  
\[ R_f = \frac{K_f \beta \phi_f}{1 - \phi - K_z / K_z + K_z / K_z \phi_f} \]  
\[ S = R_H / \phi_f (1 / \beta / K_f - 1 / K_f) + R_f / \beta / \phi_f (1 / K_f - 1 / K_f), \]  
\[ \rho_{h0} = (1 - \phi) \rho_h - \rho \phi_f \phi, \]  
\[ \rho_{f0} = (1 - a_H) \rho_f, \]  
\[ \rho_{h1} = \alpha_H \rho_f, \]  
\[ \rho_{f1} = \alpha_f \rho_f, \]  
\[ \rho_{h2} = \alpha_H \rho_f. \]

**Appendix C. Supplementary data**

Supplementary data to this article can be found online at https://doi.org/10.1016/j.ijrmms.2020.104281.

**References**

[References not visible in the image]